Vol 16 No 3, March 2007
1009-1963/2007/16(03)/0605-06
Chinese Physics
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Multi-component Dirac equation hierarchy and its multi-component integrable couplings system*

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(Received 13 April 2006; revised manuscript received 12 September 2006)

A general scheme for generating a multi-component integrable equation hierarchy is proposed. A simple 3M-dimensional loop algebra $\tilde{X}$ is produced. By taking advantage of $\tilde{X}$, a new isospectral problem is established and then by making use of the Tu scheme the multi-component Dirac equation hierarchy is obtained. Finally, an expanding loop algebra $\tilde{F}_M$ of the loop algebra $\tilde{X}$ is presented. Based on the $\tilde{F}_M$, the multi-component integrable coupling system of the multi-component Dirac equation hierarchy is investigated. The method in this paper can be applied to other nonlinear evolution equation hierarchies.

Keywords: loop algebra, zero curvature equation, multi-component Dirac equation hierarchy, multi-component integrable couplings system

PACC: 0340K, 0220, 0365G

1. Introduction

In the soliton theory, two important subjects are to search for the new Lax integrable equations and their integrable coupling system. Various efficient techniques have been proposed and many integrable hierarchies of soliton equations such as the Ablowitz–Kaup–Nell–Segur (AKNS) system, the Kortweg–de Vries (KdV) system, the Kaup–Nell (KN) system, the Schrödinger system, the coupled Burgers system and so on[1−38] have been obtained. In recent years, the study of nonlinear integrable multi-component potential functions has become a hot spot. Many efficient approaches have been developed in Refs.[15–17,29,34]. Ma *et al* have made use of the generalized Tu scheme to arrive at the multi-component AKNS hierarchy and other interesting results. Tsuchida *et al* used a generalized Lax matrix to extend an nonlinear Schrödinger equation to a multi-component Schrödinger equation in Ref.[15]. In Ref.[30], Guo *et al* proposed a systematic approach to generating multi-component integrable hierarchies. However, there are some clues to deduce integrable hierarchies. A simple and efficient method of generating multi-component integrable hierarchies was proposed in Refs.[17,31]. Another new simple method was presented by Tom in Ref.[37]. Constructing a simple loop algebra becomes a key step in this method, but the method of constructing loop algebra is easy and general. In this paper, we introduce a new isospectral problem by using a new loop algebra $\tilde{X}$. By taking advantage of the Tu scheme, the multi-component Dirac equation hierarchy is given. In addition, an expanding loop algebra $\widetilde{F}_M$ of the loop algebra $\tilde{X}$ is presented, which is devoted to deducing the integrable couplings of the multi-component Dirac equation hierarchy. The method in this paper can be applied to other nonlinear evolution equation hierarchies.

2. A new type of Lie algebra

Above all, we define a few notations.

Definition 1 Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_M)^{\mathrm{T}}$ and $\beta = (\beta_1, \beta_2, ..., \beta_M)^{\mathrm{T}}$ be two column vectors, define their vector product $\alpha * \beta$ as

$$\alpha * \beta = \beta * \alpha = (\alpha_1\beta_1, \alpha_2\beta_2, ..., \alpha_M\beta_M)^{\mathrm{T}}, \quad (1)$$

and introduce the diagonal matrix $\tilde{\alpha} = \mathrm{diag}\ (\alpha_1,$

*Project supported by the Liuhui Applied Mathematics Center of Nankai University and Tianjin University, China (Grant No T34), Educational Committee of Liaoning Province, China (Grant No 2004C057) and Science Funds of Shanghai Education Committee, China (Grant No 06AZ081).

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$\alpha_2, ..., \alpha_M)$, then we will have

$$\alpha * \beta = \tilde{\alpha}\beta. \tag{2}$$

Definition 2 Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_M)^{\mathrm{T}}$ and $A = (0, ..., 0, a_i, 0, ..., 0)_{M\times N}$, we define their product as

$$\alpha \cdot A = A \cdot \alpha = (0, ..., 0, \alpha * a_i, 0, ..., 0),$$

where

$$a_i = (a_{i1}, a_{i2}, ..., a_{iM})^{\mathrm{T}},$$

$a_{ij} \in$ R or C is real or complex number and unrelated to the independent variables x, t.

Definition 3 Let $x^1 = (x_1^1, 0, 0)$, $x^2 = (0, x_2^2, 0)$ and $x^3 = (0, 0, x_3^3)$ be linear real or complex $M \times 3$ matrices and independent of x and t. Let X represent a linear space generated by $\{x^1, x^2, x^3\}$, i.e.

$$X = \{w_a | w_a = a_1 \cdot x^1 + a_2 \cdot x^2 + a_3 \cdot x^3\},$$
$$a_i = (a_{i1}, a_{i2}, ..., a_{iM})^{\mathrm{T}}, \quad (i = 1, 2, 3) \tag{3}$$

represent real or complex vectors, $x_i^i = (x_{i1}, x_{i2}, ..., x_{iM})^{\mathrm{T}}$, $x_{ij} \in$ R or C.

Define a commutation operation $[w_a, w_b]$ in X as

$$\begin{aligned}[w_a, w_b] \equiv w_a \wedge w_b &= (a_1 * b_2 - a_2 * b_1) \cdot x^1 \wedge x^2 \\ &+ (a_1 * b_3 - a_3 * b_1) \cdot x^1 \wedge x^3 \\ &+ (a_2 * b_3 - a_3 * b_2) \cdot x^2 \wedge x^3, \end{aligned} \tag{4}$$

$$x^1 \wedge x^2 = -x^2 \wedge x^1 = x^3, \quad x^2 \wedge x^3 = -x^3 \wedge x^2 = -x^1,$$
$$x^1 \wedge x^3 = -x^3 \wedge x^1 = x^2, \ \ x^i \wedge x^i = 0, \ \ i = 1, 2, 3. \tag{5}$$

It is easy to verify that for $\forall\ w_a, w_b, w_c \in X$,

$$[[w_a, w_b], w_c] + [[w_b, w_c], w_a] + [[w_c, w_a], w_b] = 0. \tag{6}$$

holds.

Therefore, Eq.(3) along with Eqs.(4) and (5) becomes a Lie algebra.

Specially, take

$$x^1 = (I_M, 0, 0), \ \ x^2 = (0, I_M, 0), \ \ x^3 = (0, 0, I_M), \tag{7}$$

where $I_M = \underbrace{(1, 1, ..., 1)^{\mathrm{T}}}_{M}$, then Eq.(4) will become

$$\begin{aligned}[w_a, w_b] = (&a_3 * b_2 - a_2 * b_3, a_1 * b_3 \\ &- a_3 * b_1, a_1 * b_2 - a_2 * b_1). \end{aligned} \tag{8}$$

Definition 4 Set

$$x^i(n) = x^i \otimes \lambda^n \equiv x^i \lambda^n, \quad i = 1, 2, 3. \tag{9a}$$
$$[x^1(m), x^2(n)] = -[x^2(n), x^1(m)] = x^3(m + n),$$
$$[x^2(m), x^3(n)] = -[x^3(n), x^2(m)] = -x^1(m + n),$$
$$[x^1(m), x^3(n)] = -[x^3(n), x^1(m)] = x^2(m + n),$$
$$\deg x^i(n) = n, \quad i = 1, 2, 3. \tag{9b}$$

Then, X with the expression (9) consists of a loop algebra $\tilde{X}$.

Let us consider the following isospectral problems:

$$\begin{cases} \phi_x = [U, \phi], \\ \phi_t = [V, \phi], \quad U, V, \phi \in \tilde{X}, \end{cases} \tag{10}$$

whose compatibility leads to

$$\phi_{xt} = [U_t, \phi] + [U, [V, \phi]] = \phi_{tx} = [V_x, \phi] + [V, [U, \phi]],$$
$$[U_t, \phi] - [V_x, \phi] + [U, [V, \phi]] + [V, [\phi, U]] = 0. \tag{11}$$

By employing Eq.(6), Eq.(11) is expressed as

$$[U_t, \phi] - [V_x, \phi] + [[U, V], \phi] = 0. \tag{12}$$

Due to ϕ being arbitrary, Eq.(12) holds if the following condition exists:

$$U_t - V_x + [U, V] = 0, \tag{13}$$

which is just the zero curvature equation. This indicates that the soliton equations derived from the Lax pair (10) are Lax integrable.

3. The multi-component Dirac equation hierarchy

Consider an isospectral problem

$$\begin{aligned} &\phi_x = [U, \phi], \quad \lambda_t = 0, \\ &U = I_M \cdot x^3(1) + q \cdot x^1(0) + r \cdot x^2(0), \end{aligned} \tag{14}$$

where

$$q = (q_1, q_2, ..., q_M)^{\mathrm{T}}, \quad r = (r_1, r_2, ..., r_M)^{\mathrm{T}}.$$

Set

$$\begin{aligned} V = \sum_{i=0}^{\infty} &(a(0, i) \cdot x^1(-i) \\ &+ b(0, i) \cdot x^2(-i) + c(0, i) \cdot x^3(-i)) \end{aligned}$$

where

$$a(0, i) = (a_{i1}^{(0)}, a_{i2}^{(0)}, ..., a_{iM}^{(0)})^{\mathrm{T}},$$
$$b(0, i) = (b_{i1}^{(0)}, b_{i2}^{(0)}, ..., b_{iM}^{(0)})^{\mathrm{T}}$$

and $c(0, i) = (c_{i1}^{(0)}, c_{i2}^{(0)}, ..., c_{iM}^{(0)})^{\mathrm{T}}$.

Solving the adjoint equation

$$V_x = [U, V] \tag{15}$$

leads to the recurrence relations

$$\begin{cases} a_x(0,i) = b(0,i+1) - r*c(0,i), \\ b_x(0,i) = -a(0,i+1) + q*c(0,i), \\ c_x(0,i) = q*b(0,i) - r*a(0,i), \\ a(0,0) = b(0,0) = 0 = \underbrace{(0,0,...,0)^{\mathrm{T}}}_{M}, \quad c(0,0) = 2(\beta_1,\beta_2,...,\beta_M)^{\mathrm{T}} = 2\beta = \text{const}, \\ a(0,1) = 2\beta*q, \quad b(0,1) = 2\beta*r, \quad c(0,1) = 2(\alpha_1,\alpha_2,...,\alpha_M)^{\mathrm{T}} = 2\alpha = \text{const}, \\ b(0,2) = 2\beta*q + 2\alpha*r, \quad a(0,2) = -2\beta*r_x + 2\alpha*q, \quad c(0,2) = \beta(q^2+r^2),... \end{cases} \tag{16}$$

Denoting

$$V_+^{(n)} = \sum_{i=0}^{n}(a(0,i)\cdot x^1(n-i) + b(0,i)\cdot x^2(n-i) + c(0,i)\cdot x^3(n-i)), \quad V_-^{(n)} = \lambda^n V - V_+^{(n)},$$

Equation (15) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]. \tag{17}$$

It is easy to verify that the terms on the left-hand side in Eq.(17) are of degree ≥ 0, while the terms on the right-hand side are of degree ≤ 0. Thus, we have

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = b(0,n+1)\cdot x^1(0) - a(0,n+1)\cdot x^2(0).$$

Taking an arbitrary modified term for $V_+^{(n)}$ as $\Delta_n = 0$. Noticing $V^{(n)} = V_+^{(n)} + \Delta_n$, the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{18}$$

gives rise to

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} b(0,n+1) \\ -a(0,n+1) \end{pmatrix} = \begin{pmatrix} 0 & 2I_M \\ -2I_M & 0 \end{pmatrix} \begin{pmatrix} \dfrac{a(0,n+1)}{2} \\ \dfrac{b(0,n+1)}{2} \end{pmatrix} = J \begin{pmatrix} \dfrac{a(0,n+1)}{2} \\ \dfrac{b(0,n+1)}{2} \end{pmatrix}, \tag{19}$$

where J is Hamiltonian operator.

From the recurrence relations (16), the following recurrence relation exists:

$$\begin{aligned} \begin{pmatrix} \dfrac{a(0,n+1)}{2} \\ \dfrac{b(0,n+1)}{2} \end{pmatrix} &= \begin{pmatrix} -q*\partial^{-1}r* & -\partial + q*\partial^{-1}q* \\ \partial - r*\partial^{-1}r* & r*\partial^{-1}q* \end{pmatrix} \begin{pmatrix} \dfrac{a(0,n)}{2} \\ \dfrac{b(0,n)}{2} \end{pmatrix} \\ &= L \begin{pmatrix} \dfrac{a(0,n)}{2} \\ \dfrac{b(0,n)}{2} \end{pmatrix} = \begin{pmatrix} -\tilde{q}\partial^{-1}\tilde{r} & -\partial + \tilde{q}\partial^{-1}\tilde{q} \\ \partial - \tilde{r}\partial^{-1}\tilde{r} & \tilde{r}\partial^{-1}\tilde{q} \end{pmatrix} = \tilde{L} \begin{pmatrix} \dfrac{a(0,n)}{2} \\ \dfrac{b(0,n)}{2} \end{pmatrix}, \end{aligned}$$

where $\tilde{q}$ and $\tilde{r}$ denote q^* and r^*, respectively. Therefore, the system (19) can be written as

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = \tilde{J}\tilde{L}^{n-1} \begin{pmatrix} \beta q \\ \beta r \end{pmatrix}. \tag{20}$$

When $M = 1$, the system (20) is the standard Dirac equation hierarchy. When $M > 1$, the system (20) is the multi-component Dirac equation hierarchy. The system (20) has the following particular cases:

Case 1 When $n = 1$, the system (20) becomes:

$$\begin{cases} q_t = 2\tilde{\beta}q_x + 2\tilde{\alpha}r, \\ r_t = 2\tilde{\beta}r_x - 2\tilde{\alpha}q. \end{cases} \tag{21}$$

Case 2 When $n = 2$, the system becomes

$$\begin{cases} q_t = 2\tilde{\alpha}q_x - 2\tilde{\beta}r_{xx} + \tilde{\beta}\tilde{r}q^2 + \tilde{\beta}r^3, \\ r_t = -2\tilde{\alpha}r_x - 2\tilde{\beta}q_{xx} + \tilde{\beta}\tilde{q}r^2 + \tilde{\beta}q^3. \end{cases} \tag{22}$$

4. The multi-component integrable couplings system

In order to find the coupling system of some known hierarchy derived from the loop $\tilde{A}_1$, Guo *et al*[30] set forward a simple method, the main idea is to construct an expanding loop algebra $\tilde{G}_M$ of the loop algebra $\tilde{A}_1$, which has two subalgebras $\tilde{G}_1$ and $\tilde{G}_2$ satisfying following conditions:

$$\begin{aligned}&\text{(i)}\quad \tilde{G}_M=\tilde{G}_1\oplus\tilde{G}_2,\quad \tilde{G}_1\cong\tilde{A}_1;\\&\text{(ii)}\quad [\tilde{G}_1,\tilde{G}_2]\subset\tilde{G}_2;\end{aligned}\tag{23}$$

where the symbols $\oplus$ and $\cong$ denote direct sum and isomorphic relation, respectively.

Following the main idea in Ref.[30], let F_M denote a linear space spanned by a linear independent vector $\{x^1,x^2,x^3,x^4,x^5\}$ (where $x^1,\ldots$and x^5 are independent of x,t). Define a commutative operation among x^i, $i=1,2,\ldots,5$ as follows:

$$\begin{aligned}&[x^1,x^2]=-[x^2,x^1]=x^3,\ \ [x^2,x^3]=-[x^3,x^2]=-x^1,\\&[x^1,x^3]=-[x^3,x^1]=x^2,\ \ [x^1,x^4]=-[x^4,x^1]=x^5,\\&[x^1,x^5]=-[x^5,x^1]=\frac{1}{2}x^4,\ \ [x^2,x^5]=-[x^5,x^2]=-\frac{1}{2}x^5,\\&[x^3,x^4]=-[x^4,x^3]=\frac{1}{2}x^5,\ \ [x^2,x^4]=-[x^4,x^2]=\frac{1}{2}x^4,\\&[x^3,x^5]=-[x^5,x^3]=-\frac{1}{2}x^4,\ [x^4,x^5]=-[x^5,x^4]=0,\\&[x^i,x^i]=0,\quad i=1,2,...,5,\end{aligned}\tag{24}$$

then F_M will become a Lie algebra. In particular, taking

$$\begin{aligned}&x^1=(I_M,0,0,0,0),\quad x^2=(0,I_M,0,0,0),\\&x^3=(0,0,I_M,0,0),\quad x^4=(0,0,0,I_M,0),\\&x^5=(0,0,0,0,I_M),\end{aligned}$$

we obtain

$$\begin{aligned}w_a&=a_1\cdot x^1+a_2\cdot x^2+a_3\cdot x^3+a_4\cdot x^4+a_5\cdot x^5\\&=(a_1,a_2,a_3,a_4,a_5),\ \forall w_a\in F_M.\end{aligned}\tag{25}$$

The corresponding loop algebra is given by

$$\tilde{F}_M=\{(x^1(n),x^2(n),x^3(n),x^4(n),x^5(n))\},$$

$$x^i(n)=x^i\otimes\lambda^n\equiv x^i\lambda^n,\quad i=1,2,...,5,\tag{26}$$

$$\begin{aligned}&[x^1(m),x^2(n)]=-[x^2(n),x^1(m)]=x^3(m+n),\\&[x^2(m),x^3(n)]=-[x^3(n),x^2(m)]=-x^1(m+n),\\&[x^1(m),x^3(n)]=-[x^3(n),x^1(m)]=x^2(m+n),\\&[x^1(m),x^4(n)]=-[x^4(n),x^1(m)]=\frac{1}{2}x^5(m+n),\\&[x^1(m),x^5(n)]=-[x^5(n),x^1(m)]=\frac{1}{2}x^4(m+n),\\&[x^2(m),x^5(n)]=-[x^5(n),x^2(m)]=-\frac{1}{2}x^5(m+n),\\&[x^3(m),x^4(n)]=-[x^4(n),x^3(m)]=\frac{1}{2}x^5(m+n),\\&[x^2(m),x^4(n)]=-[x^4(n),x^2(m)]=\frac{1}{2}x^4(m+n),\\&[x^3(m),x^5(n)]=-[x^5(n),x^3(m)]=-\frac{1}{2}x^4(m+n),\\&[x^4(m),x^5(n)]=-[x^5(n),x^4(m)]=0,\\&[x^i(m),x^i(n)]=0,\quad i=1,2,...,5,\\&\deg x^i(n)=n,\quad i=1,2,...,5.\end{aligned}\tag{27}$$

Let $\tilde{F}_{M(1)}$ and $\tilde{F}_{M(2)}$ be two subalgebras of the loop algebra $\tilde{F}_M$, which are expressed as respectively

$$\begin{aligned}\tilde{F}_{M(1)}&=\{(x^1(n),x^2(n),x^3(n),0,0)\},\\\tilde{F}_{M(2)}&=\{(0,0,0,x^4(n),x^5(n))\},\end{aligned}$$

then we will find

$$\begin{aligned}&\text{(iii)}\quad \tilde{F}_M=\tilde{F}_{M(1)}\oplus\tilde{F}_{M(2)},\tilde{F}_{M(1)}\cong\tilde{X};\\&\text{(iv)}\quad [\tilde{F}_{M(1)},\tilde{F}_{M(2)}]\subset\tilde{F}_{M(2)}.\end{aligned}$$

According to the loop algebra $\tilde{F}_M$ and the relations (iii) and (iv), we consider the following isospectral problem:

$$\begin{aligned}&\phi_x=[U,\phi],\quad \lambda_t=0,\\&U=I_M\cdot x^3(1)+u_1\cdot x^1(0)\\&\qquad+u_2\cdot x^2(0)+u_3\cdot x^4(0)+u_4\cdot x^5(0),\end{aligned}\tag{28}$$

where

$$\begin{aligned}u_1&=(u_1^{(1)},u_2^{(2)},...,u_M^{(M)})^{\mathrm{T}},\\u_2&=(u_1^{(1)},u_2^{(2)},...,u_M^{(M)})^{\mathrm{T}},...\end{aligned}$$

Denoting

$$V=\sum_{i=0}^{\infty}(a(0,i)\cdot x^1(-i)+b(0,i)\cdot x^2(-i)+c(0,i)\cdot x^3(-i)+d(0,i)\cdot x^4(i)+f(0,i)\cdot x^5(-i)),$$

where $a(0,i)=(a_{i1}^{(0)},a_{i2}^{(0)},...,a_{iM}^{(0)})^{\mathrm{T}}$, $b(0,i)=(b_{i1}^{(0)},b_{i2}^{(0)},...,b_{iM}^{(0)})^{\mathrm{T}}$, $c(0,i)=(c_{i1}^{(0)},c_{i2}^{(0)},...,c_{iM}^{(0)})^{\mathrm{T}},...$ and solving the equation similar to Eq.(15) yield

$$\begin{cases} a_x(0,i)=b(0,i+1)-u_2*c(0,i),\\ b_x(0,i)=-a(0,i+1)+u_1*c(0,i),\\ c_x(0,i)=u_1*b(0,i)-u_2*a(0,i),\\ d_x(0,i)=-\frac{1}{2}(f(0,i+1)-u_1*f(0,n)-u_2*d(0,i)+u_3*b(0,i)+u_4*(a(0,i)-c(0,i)),\\ f_x(0,i)=\frac{1}{2}(d(0,i+1)+u_1*d(0,i)-u_2*f(0,i)-u_3*(a(0,i)+c(0,i))+u_4*b(0,i)),\\ a(0,0)=b(0,0)=d(0,0)=f(0,0)=0=\underbrace{(0,0,...,0)^{\mathrm{T}}}_{M},\quad c(0,0)=2(\beta_1,\beta_2,...,\beta_M)^{\mathrm{T}}=2\beta=\mathrm{const},\\ a(0,1)=2\beta*u_1,\quad b(0,1)=2\beta*u_2,\quad c(0,1)=2(\alpha_1,\alpha_2,...,\alpha_M)^{\mathrm{T}}=2\alpha=\mathrm{const},\\ d(0,1)=2\beta u_3,\quad f(0,1)=2\beta u_4,... \end{cases} \tag{29}$$

Similar to Eq.(17), the equation

$$-V_x^{(n)}+[U,V^{(n)}]=b(0,n+1)\cdot x^1(0)-a(0,n+1)\cdot x^2(0)-\frac{1}{2}f(0,n+1)\cdot x^4(0)+\frac{1}{2}d(0,n+1)\cdot x^5(0)$$

is obtained.

By setting $V^{(n)}=V_+^{(n)}+\Delta_n$, where $\Delta_n=0$, the zero curvature equation (18) gives rise to

$$u_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}_t=\begin{pmatrix}b(0,n+1)\\-a(0,n+1)\\-\frac{1}{2}f(0,n+1)\\\frac{1}{2}d(0,n+1)\end{pmatrix}=\begin{pmatrix}0&2I_M&0&0\\-2I_M&0&0&0\\0&0&0&-I_M\\0&0&I_M&0\end{pmatrix}\begin{pmatrix}\frac{1}{2}a(0,n+1)\\\frac{1}{2}b(0,n+1)\\\frac{1}{2}d(0,n+1)\\\frac{1}{2}f(0,n+1)\end{pmatrix}=J\begin{pmatrix}\frac{1}{2}a(0,n+1)\\\frac{1}{2}b(0,n+1)\\\frac{1}{2}d(0,n+1)\\\frac{1}{2}f(0,n+1)\end{pmatrix}. \tag{30}$$

In terms of the expression (29), a recurrence operator L is given by

$$\begin{pmatrix}\frac{1}{2}a(0,n+1)\\\frac{1}{2}b(0,n+1)\\\frac{1}{2}d(0,n+1)\\\frac{1}{2}f(0,n+1)\end{pmatrix}=\begin{pmatrix}-u_1*\partial^{-1}u_2*&-\partial+u_1*\partial^{-1}u_1*&0&0\\\partial-u_2*\partial^{-1}u_2*&u_2*\partial^{-1}u_1*&0&0\\u_3*-u_3*\partial^{-1}u_2&u_4*+u_3*\partial^{-1}u_1*&-u_1*&2\partial+u_2*\\-u_4*-u_3*\partial^{-1}u_2*&-u_3*+u_3*\partial^{-1}u_1*&-2\partial+u_1*+u_2*&0\end{pmatrix}\begin{pmatrix}\frac{1}{2}a(0,n)\\\frac{1}{2}b(0,n)\\\frac{1}{2}d(0,n)\\\frac{1}{2}f(0,n)\end{pmatrix}$$

$$=L\begin{pmatrix}\frac{1}{2}a(0,n)\\\frac{1}{2}b(0,n)\\\frac{1}{2}d(0,n)\\\frac{1}{2}f(0,n)\end{pmatrix}=\begin{pmatrix}-\tilde{u}_1\partial^{-1}\tilde{u}_2&-\partial+\tilde{u}_1\partial^{-1}\tilde{u}_1&0&0\\\partial-\tilde{u}_2\partial^{-1}\tilde{u}_2&\tilde{u}_2\partial^{-1}\tilde{u}_1&0&0\\\tilde{u}_3-\tilde{u}_3\partial^{-1}\tilde{u}_2&\tilde{u}_4+\tilde{u}_3\partial^{-1}\tilde{u}_1&-\tilde{u}_1&2\partial+\tilde{u}_2\\-\tilde{u}_4-\tilde{u}_3\partial^{-1}\tilde{u}_2&-\tilde{u}_3+u_3\partial^{-1}\tilde{u}_1&-2\partial+\tilde{u}_1+\tilde{u}_2&0\end{pmatrix}$$

$$\times\begin{pmatrix}\frac{1}{2}a(0,n)\\\frac{1}{2}b(0,n)\\\frac{1}{2}d(0,n)\\\frac{1}{2}f(0,n)\end{pmatrix}=\tilde{L}\begin{pmatrix}\frac{1}{2}a(0,n)\\\frac{1}{2}b(0,n)\\\frac{1}{2}d(0,n)\\\frac{1}{2}f(0,n)\end{pmatrix},$$

where $\widetilde{u_1},\widetilde{u_2},\widetilde{u_3}$ and $\widetilde{u_4}$ denote u_1^*,u_2^*,u_3^* and u_4^*. As a result, the system (31) can be written as

$$u_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}_t=\tilde{J}\tilde{L}^{n-1}\begin{pmatrix}\beta u_1\\\beta u_2\\\beta u_3\\\beta u_4\end{pmatrix}. \tag{31}$$

According to the definition of integrable couplings in Refs.[31,32], we conclude that the system (31) is a multi-component integrable couplings of the multi-component Dirac equation hierarchy.

Obviously, taking $u_1=q$, $u_2=r$, $u_3=u_4=0$, the system (31) reduces to the system (20). Therefore, we regard Eq.(31) as an expanding integrable model of the system (20).

The system (31) has the following particular cases:

Case 1 When $n = 1$, $u_1 = q$, and $u_2 = r$, the system becomes

$$\begin{cases} q_t = 2\tilde{\beta}q_x + 2\tilde{\alpha}r, \\ r_t = 2\tilde{\beta}r_x - 2\tilde{\alpha}q, \\ u_{3t} = 2\tilde{\beta}u_{3x} - \tilde{\alpha}u_4, \\ u_{4t} = 2\tilde{\beta}u_{4x} + \tilde{\alpha}u_3, \end{cases} \tag{32}$$

which is a coupling system of the system (21).

Case 2 When $n = 2$, $u_1 = q$, and $u_2 = r$, the system becomes

$$\begin{cases} q_t = 2\tilde{\alpha}q_x - 2\tilde{\beta}r_{xx} + \tilde{\beta}\tilde{r}q^2 + \beta r^3, \\ r_t = -2\tilde{\alpha}r_x - 2\tilde{\beta}q_{xx} + \tilde{\beta}\tilde{q}r^2 + \beta q^3, \\ u_{3t} = 4\tilde{\beta}u_{4xx} + 2\tilde{\alpha}u_{3x} + 2\tilde{\beta}\tilde{q}u_{3x} - \tilde{\beta}\tilde{r}u_{4x} + \tilde{\beta}\tilde{u}_3 q_x - \tilde{\beta}\tilde{u}_4 r_x - \frac{1}{2}\tilde{\beta}\tilde{u}_4(q^2 + r^2), \\ u_{4t} = -4\tilde{\beta}u_{3xx} + 2\tilde{\alpha}u_{4x} - 2\tilde{\beta}\tilde{q}u_{4x} - \tilde{\beta}\tilde{r}u_{3x} - \tilde{\beta}\tilde{u}_3 r_x - \tilde{\beta}\tilde{u}_4 q_x + \frac{1}{2}\tilde{\beta}\tilde{u}_3(q^2 + r^2), \end{cases} \tag{33}$$

which is a coupling system of the system (21).

When $M = 1$, the system (31) is the integrable couplings of the standard Dirac equation hierarchy.

When $M > 1$, the system (31) is the integrable couplings of the multi-component Dirac equation hierarchy.

Remark The loop algebra presented in this paper can be used to other known integrable hierarchies of soliton equations for generating the multi-component systems. But there exist two open problems, i.e. how to improve our method and work out the multi-component integrable systems such that they are Liouville integrable. Further, how do we find out Hamiltonian structures and conserved laws of such multi-component integrable hierarchies? These problems are worthwhile being studied in the future.

Acknowledgments

Xia T C would like to express his sincere thanks to Professor Zhang Y F and Guo F K for valuable discussions.

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