Vol 16 No 3, March 2007
1009-1963/2007/16(03)/0599-06
Chinese Physics
© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

A series of non-Noether conservative quantities and Mei symmetries of nonconservative systems*

Liu Hong-Ji(刘鸿基)[a], Fu Jing-Li(傅景礼)[b,d]†, and Tang Yi-Fa(唐贻发)[c]

[a] *Institute of Mathematical Physics, Shangqiu Teacher's College, Shangqiu* 476000, *China*
[b] *Department of Physics, Zhejiang Sci-Tech University, Hangzhou* 310018, *China*
[c] *State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences, Beijing* 100080, *China*
[d] *Shanghai University, Shanghai Institute of Applied Mathematics and Mechanics, Shanghai* 200072, *China*

(Received 14 June 2005; revised manuscript received 20 July 2006)

In this paper Mei symmetry is introduced for a nonconservative system. The necessary and sufficient condition for a Mei symmetry to be also a Lie symmetry is derived. It is proved that the Mei symmetry leads to a non-Noether conservative quantity via a Lie symmetry, and deduces a Lutzky conservative quantity via a Lie point symmetry.

Keywords: Mei symmetry, non-Noether conservative quantity, Lutzky conservative quantity, nonconservative system

PACC: 0320

1. Introduction

It is well known that symmetry principles are most important issues in mathematics and physics, and there are close relationships between symmetries and conservative quantities. Noether first proposed a famous symmetry theory, which deals with an invariance of Hamiltonian action under the group of infinitesimal transformations.[1] Djukić and Vujanović further investigated Noether symmetries and conservative quantities of nonconservative holonomic systems by including generalized velocities.[2] Lie first contributed another symmetry theory, which is an invariance of differential equations under the groups of infinitesimal transformations,[3] but the theory had not applied to mechanics until Lutzky's contribution in 1979,[4] in recent years, there are well results on study of Lie symmetries.[5−14] Recently, Mei presented a formal invariance theory[15,16] which is called the Mei symmetry. The researchers proved that the Noether conservative quantities can be obtained from the Mei symmetry.[15−17] In addition, Lutzky, along with other methods, found non-Noether conservative quantities from Lie point symmetries, the velocity-dependent symmetries for Lagrangian systems, and some other generalizations.[18−22] Recently, Mei *et al* studied a formal invariance and a Lutzky conservative quantity of Lagrange system,[23] Fu and Chen presented non-Noether conservative quantities of nonconservative systems, [24] and further extended the theory to Lagrangian electromechanical systems.[25]

The purpose of this paper is to develop Mei symmetry theory for nonconservative mechanical systems. To do this, firstly we show that the non-Noether conservative quantities are derived from Lie symmetries, and the Lutzky conservative quantities from Lie point symmetries for nonconservative systems, secondly introduce the Mei symmetry for such systems, and establish the necessary and sufficient conditions on which Mei symmetries is also Lie symmetries, thirdly derive the non-Noether conserved quantities and Lutzky conserved quantities from the Mei symmetries, finally give an example to illustrate these results.

2. Lie symmetries of nonconservative systems

Consider a nonconservative system

$$E_s(L) = Q_s \ (s = 1, \ldots, n)\,, \tag{1}$$

*Project supported by the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences and the National Natural Science Foundation of China (Grant Nos 10672143, 10471145 and 10372053) and the Natural Science Foundation of Henan Province Government of China(Grant Nos 0511022200 and 0311011400).
†Corresponding author. E-mail: sqfujingli@163.com

where L denotes Lagrangian, Q_s denotes non-potential generalized force, and $E_s = (\mathrm{d}/\mathrm{d}t)(\partial/\partial\dot{q}_s) - (\partial/\partial q_s)$ is Eulerian operator. From Eq.(1), one can obtain all generalized accelerations of the system as

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \ldots, n)\,. \tag{2}$$

Introduce the following infinitesimal transformations with respect to time and generalized coordinates:

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\,, \tag{3}$$

where ε is a small parameter, and ξ_0, ξ_s are infinitesimal generators.

If Eq.(1) keeps invariant under the infinitesimal transformation Eq.(3), the infinitesimal generators $\xi_0(t, \boldsymbol{q}, \boldsymbol{p})$, $\xi_s(t, \boldsymbol{q}, \boldsymbol{p})$ satisfy

$$X^{(2)}\{E_s(L) - Q_s\} = 0 \quad (s = 1, \ldots, n)\,. \tag{4}$$

The invariance is the Lie symmetry associated with nonconservative system Eq.(1). Equation (2) is invariant under the infinitesimal transformation Eq.(3), which leads to the determining equations of Lie symmetry of the system as

$$\ddot{\xi}_s - \dot{q}_s\ddot{\xi}_0 - 2\dot{\xi}_0\alpha_s = X^{(1)}(\alpha_s) \quad (s = 1, \ldots, n)\,. \tag{5}$$

In Eqs.(4) and (5), the generator vector of the infinitesimal transformation Eq.(3) is

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s}\,, \tag{6}$$

with its first order expansion

$$X^{(1)} = X^{(0)} + (\dot{\eta}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial\dot{q}_s}, \tag{7}$$

and the second order expansion

$$X^{(2)} = X^{(1)} + \Big[\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) - \ddot{q}_s\dot{\xi}_0\Big]\frac{\partial}{\partial\ddot{q}_s}\,, \tag{8}$$

where for any function $y(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, its temporal derivative is defined by

$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial y}{\partial t} + \dot{q}_l\frac{\partial y}{\partial q_l} + \alpha_l\frac{\partial y}{\partial\dot{q}_l} \quad (l = 1, \ldots, j)\,. \tag{9}$$

3. Lie symmetries and non-Noether conserved quantities of nonconservative systems

Now, we present the non-Noether conserved quantities associated with the Lie symmetry for nonconservative systems.

Theorem 1 If the infinitesimal generators $\xi_0 = \xi_0(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, $\xi_s = \eta_s(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ are Lie symmetric for Eqs.(1) or (2), and a gauge function G exists such that the quantity

$$\frac{\partial\alpha_s}{\partial\dot{q}_s} + \dot{G} = \psi \quad (s = 1, \ldots, n) \tag{10}$$

is an invariant, then there is a conserved quantity given by

$$\phi = \frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_s}{\partial q_s} + \frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) + X^{(1)}(G) - \dot{\xi}_0 = \mathrm{const}\,. \tag{11}$$

Proof In fact, ϕ is a conserved quantity if and only if $\dot{\phi} = 0$. Notice that

$$\dot{\phi} = \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_s}{\partial q_s} + \frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\Big] + \frac{\mathrm{d}}{\mathrm{d}t}X^{(1)}(G) - \ddot{\xi}_0. \tag{12}$$

For any function $f = f(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial t} = \frac{\partial\dot{f}}{\partial t} - \frac{\partial\alpha_k}{\partial t}\frac{\partial f}{\partial\dot{q}_k} \quad (k = 1, \ldots, n)\,, \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_s} = \frac{\partial\dot{f}}{\partial q_s} - \frac{\partial\alpha_k}{\partial q_s}\frac{\partial f}{\partial\dot{q}_k} \quad (s, k = 1, \ldots, n)\,, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial\dot{q}_s} = \frac{\partial\dot{f}}{\partial\dot{q}_s} - \frac{\partial f}{\partial q_s} - \frac{\partial\alpha_k}{\partial\dot{q}_s}\frac{\partial f}{\partial\dot{q}_k} \quad (s, k = 1, \ldots, n), \tag{15}$$

$$X^{(1)}\Big(\frac{\partial f}{\partial\dot{q}_s}\Big) = \frac{\partial}{\partial\dot{q}_s}X^{(1)}(f) - \frac{\partial\xi_0}{\partial\dot{q}_s}\frac{\partial f}{\partial t} - \frac{\partial\xi_k}{\partial\dot{q}_s}\frac{\partial f}{\partial q_k} - \frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial f}{\partial\dot{q}_k} \quad (s, k = 1, \ldots, n)\,. \tag{16}$$

Given Eq.(5), ξ_0, ξ_s can define the symmetry Eq.(8) for $\ddot{q}_s = \alpha_s$. So if $X^{(1)}$ has a symmetry, then

$$\dot{X}^{(1)}(f) = X^{(1)}(\dot{f}) + \dot{\xi}_0\dot{f}. \tag{17}$$

From Eqs.(13), (14) and (15), one can obtain

$$\begin{aligned}\dot{\phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_s}{\partial q_s} + \frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\Big] \\ &= \ddot{\xi}_0 + \dot{\xi}_0\frac{\partial\alpha_s}{\partial\dot{q}_s} + \frac{\partial}{\partial\dot{q}_s}(\ddot{\xi}_s - 2\alpha_s\dot{\xi}_0 - \dot{q}_s\ddot{\xi}_0) \\ &\quad - \frac{\partial\alpha_k}{\partial t}\frac{\partial\xi_0}{\partial\dot{q}_k} - \frac{\partial\alpha_k}{\partial q_s}\frac{\partial\xi_s}{\partial\dot{q}_k} \\ &\quad - \frac{\partial\alpha_k}{\partial\dot{q}_s}\frac{\partial}{\partial\dot{q}_k}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\,.\end{aligned} \tag{18}$$

Furthermore, Eqs.(5) and (16), one can get

$$X^{(1)}\Big(\frac{\partial\alpha_s}{\partial\dot{q}_s}\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_s}{\partial q_s} + \frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\Big]$$

$$-\ddot{\xi}_0-\frac{\partial\alpha_s}{\partial\dot{q}_s}\quad(s=1,\ldots,n)\,. \tag{19}$$

Substituting Eqs.(17) and (19) into Eq.(12) we get

$$\dot{\phi}=X^{(1)}\Big(\frac{\partial g_s}{\partial\dot{q}_s}+\dot{G}\Big)=X^{(1)}(\psi)=0\,. \tag{20}$$

Thus ϕ should be a conserved quantity.

Equation (10) is the structural equation of the Lie symmetry, and Eq.(11) expresses a non-Noether conserved quantity associated with the Lie symmetry for nonconservative systems. It should point out that the Theorem 1 is a generalized conclusion on the Lie symmetry of nonconservative systems and several specific cases on the Lie symmetry of the systems can be deduced.

(1) When $\psi=0$, $G=\ln\lambda$ in Theorem 1, it will lead to the generalized Hojman theorem of the system as follows.[26]

If the infinitesimal generators $\xi_0=\xi_0(\boldsymbol{q},\dot{\boldsymbol{q}},t)$, $\xi_s=\eta_s(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ have Lie symmetry for Eqs.(1) or (2), and a gauge function λ can be found such that the quantity

$$\frac{\partial\alpha_s}{\partial\dot{q}_s}+\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda=0\quad(s=1,\ldots,n), \tag{21}$$

then a constant of motion is given by

$$\begin{aligned}\phi=&\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_s}{\partial q_s}+\frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\\&+X^{(1)}(\ln\lambda)-\dot{\xi}_0=\mathrm{const}\,.\end{aligned} \tag{22}$$

(2) The theorem associated with Lie point symmetry can be located via Theorem 1:

If the infinitesimal generators $\xi_0=\xi_0(t,\boldsymbol{q})$, $\xi_s=\xi_s(t,\boldsymbol{q})$ $(\partial\xi_0/\partial q_s=\partial\dot{\xi}_0/\partial\dot{q}_s,\,\partial\xi_s/\partial q_s=\partial\dot{\xi}_s/\partial\dot{q}_s)$ have Lie point symmetry for Eqs.(1) or (2), and a gauge function G exists to satisfy Eq.(10), then the system possesses a Lutzky conserved quantity

$$\phi=2\Big(\frac{\partial\xi_s}{\partial q_s}-\dot{q}_s\frac{\partial\xi_0}{\partial q_s}\Big)-n\dot{\xi}_0+X^{(1)}(G)=\mathrm{const}. \tag{23}$$

Specifically, If the infinitesimal generators $\xi_0=0$, $\xi_s=\xi_s(t,\boldsymbol{q})$ have Lie point symmetry for Eqs.(1) or (2), then there is a Lutzky conserved quantity

$$\phi=X^{(1)}(G)+2\frac{\partial\xi_s}{\partial q_s}=\mathrm{const}\,. \tag{24}$$

Actually, for a Lie point symmetry of Eqs.(1) or (2), the Lutzky conserved quantity also takes other forms. For an example, if the nonconservative forces of the systems satisfy

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial}{\partial\dot{q}_k}\Big(\frac{M_{\mathrm{ks}}}{D}Q_s\Big)\,, \tag{25}$$

where M_{ks} and D are defined in Ref.[20], then Eq.(1) possesses a Lutzky conserved quantity[21]

$$\begin{aligned}\phi=&2\Big(\frac{\partial\xi_s}{\partial q_s}-\dot{q}_s\frac{\partial\xi_0}{\partial q_s}\Big)-n\dot{\xi}_0\\&+X^{(1)}(\ln D)-X^{(1)}(f)=\mathrm{const}\,.\end{aligned} \tag{26}$$

(3) If the infinitesimal generators $\xi_0=0$, $\xi_s=\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ for Eqs.(1) or (2) satisfy determining equation

$$\ddot{\xi}_s-\xi_s\frac{\partial\alpha_s}{\partial q_s}-\dot{\xi}_s\frac{\partial\alpha_s}{\partial\dot{q}_s}=0\,, \tag{27}$$

and a gauge function G can be found to satisfy Eq.(10), then the system possesses a non-Noether conserved quantity[27]

$$\phi=X^{(1)}(G)+\frac{\partial\xi_s}{\partial q_s}+\frac{\partial\dot{\xi}_s}{\partial\dot{q}_s}=\mathrm{const}\,, \tag{28}$$

we can lead to the Hojman theorem when $G=\ln\lambda$, and $\psi=0$ in Eq.(10). In that case, Eq.(28) yields the Hojman conserved quantity[28]

$$\phi=\frac{1}{\lambda}\frac{\partial(\lambda\xi_s)}{\partial q_s}+\frac{1}{\lambda}\frac{\partial(\lambda\dot{\xi}_s)}{\partial\dot{q}_s}=\mathrm{const}\,. \tag{29}$$

(4) If the infinitesimal generators $\xi_s=0$, $\xi_0=\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ for nonconservative system Eqs.(1) or (2) satisfy the determining equation

$$\dot{q}_s\ddot{\xi}_0+2\alpha_s\dot{\xi}_0+\xi_0\frac{\partial\alpha_s}{\partial t}-\dot{q}_s\dot{\xi}_0\frac{\partial\alpha_s}{\partial\dot{q}_s}=0\,, \tag{30}$$

and a gauge function G exists to satisfy Eq.(10), then the system possesses a non-Noether conserved quantity

$$\begin{aligned}\phi=&\frac{\partial\xi_0}{\partial t}-\frac{\partial(\dot{q}_s\dot{\xi}_0)}{\partial\dot{q}_s}+X^{(1)}(G)-\dot{\xi}_0\\=&\xi_0\frac{\partial G}{\partial t}-\dot{q}_s\dot{\xi}_0\frac{\partial G}{\partial\dot{q}_s}+\frac{\partial\xi_0}{\partial t}-\dot{q}_s\frac{\partial\dot{\xi}}{\partial\dot{q}_s}-(n+1)\dot{\xi}_0\\=&\mathrm{const}\,.\end{aligned} \tag{31}$$

4. On definition and criterion of Mei symmetries of nonconservative systems

Based on Mei symmetry theory of the systems,[15,16] under the infinitesimal transformation Eq.(3), the Lagrangian L and non-potential generalized force Q_s become L^* and Q_s^* respectively,

$$L^*=L(t^*,\boldsymbol{q}^*,\dot{\boldsymbol{q}}^*)=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$$

$$+\varepsilon\Big[\xi_0\frac{\partial L}{\partial t}+\xi_s\frac{\partial L}{\partial q_s}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial L}{\partial\dot{q}_s}\Big]+O(\varepsilon^2)$$
$$=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon X^{(1)}(L)+O(\varepsilon^2)\,, \tag{32}$$

and

$$Q_s^*=Q_s(t^*,\boldsymbol{q}^*,\dot{\boldsymbol{q}}^*)=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$$
$$+\varepsilon\Big[\xi_0\frac{\partial Q_s}{\partial t}+\xi_s\frac{\partial Q_s}{\partial q_s}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial Q_s}{\partial\dot{q}_s}\Big]+O(\varepsilon^2)$$
$$=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon X^{(1)}(Q_s)+O(\varepsilon^2)\,. \tag{33}$$

Definition For a nonconservative system, if the form of Eq.(1) is an invariant after the replacement of the Lagrangian L and the non-potential generalized force Q_s by the transformed Lagrangian $L^*$and the transformed non-potential generalized force Q_s^*, then the invariance is defined as a Mei symmetry.

According to the definition,

$$E_s(L^*)=Q_s^*\ (s=1,\ldots,n) \tag{34}$$

is an invariant. Substituting Eqs.(32) and (33) into Eq.(34), using Eq.(1), and neglecting ε^2 and the higher order infinitesimal terms in the resulting equation, one obtains

$$E_s\{X^{(1)}(L)\}=X^{(1)}(Q_s)\ (s=1,\ldots,n). \tag{35}$$

Now, one has

Criterion The symmetry of a nonconservative system is Mei symmetry if its generators ξ_0 and ξ_s satisfy Eq.(35).

Equation (35) is a criterion associated with the Mei symmetries of a nonconservative system.

5. Necessary and sufficient condition for a Mei symmetry to be a Lie symmetry

The Mei symmetries of nonconservative systems can lead to Lie symmetries under certain condition. We give the proposition as follows:

Proposition 1 For a nonconservative system, the necessary and sufficient condition on which Mei symmetry is a Lie symmetry (velocity-dependent symmetry) is that the symmetry satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\xi_0}{\partial\dot{q}_s}\frac{\partial L}{\partial t}\Big)-\frac{\partial\xi_0}{\partial q_s}\frac{\partial L}{\partial t}+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\xi_k}{\partial\dot{q}_s}\frac{\partial L}{\partial q_k}\Big)-\frac{\partial\xi_k}{\partial q_s}\frac{\partial L}{\partial q_k}+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\dot{\xi}_k}{\partial\dot{q}_s}\frac{\partial L}{\partial\dot{q}_k}\Big)-\frac{\partial\dot{\xi}_k}{\partial q_s}\frac{\partial L}{\partial\dot{q}_k}$$
$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\dot{\xi}_0}{\partial\dot{q}_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}\Big)+\frac{\partial\dot{\xi}_0}{\partial q_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}-\ddot{\xi}_0\frac{\partial L}{\partial\dot{q}_s}+X^{(0)}(Q_s)=0\,. \tag{36}$$

Proof Direct calculation gives

$$E_s\{X^{(1)}(L)\}-X^{(1)}(Q_s)-X^{(2)}\{E_s(L)-Q_s\}=\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\xi_0}{\partial\dot{q}_s}\frac{\partial L}{\partial t}\Big)-\frac{\partial\xi_0}{\partial q_s}\frac{\partial L}{\partial t}$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\xi_k}{\partial\dot{q}_s}\frac{\partial L}{\partial q_k}\Big)-\frac{\partial\xi_k}{\partial q_s}\frac{\partial L}{\partial q_k}+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\dot{\xi}_k}{\partial\dot{q}_s}\frac{\partial L}{\partial\dot{q}_k}\Big)-\frac{\partial\dot{\xi}_k}{\partial q_s}\frac{\partial L}{\partial\dot{q}_k}-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\dot{\xi}_0}{\partial\dot{q}_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}\Big)$$
$$+\frac{\partial\dot{\xi}_0}{\partial q_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}-\ddot{\xi}_0\frac{\partial L}{\partial\dot{q}_s}-X^{(0)}(Q_s). \tag{37}$$

Substituting Eqs.(35) and (36) into Eq.(37) we get the determining equation (4). Hence the Mei symmetry is a Lie symmetry.

It should be pointed out that Eq.(36) is a generalized form of necessary and sufficient condition on which Mei symmetry is a Lie symmetry for nonconservative system. Using this condition, we can obtain the following results.

(1) For a nonconservative system, the necessary and sufficient condition on which Mei symmetry is a Lie point symmetry is that the symmetry satisfies

$$-\frac{\partial\xi_0}{\partial q_s}\Big(\frac{\partial L}{\partial t}+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}\Big)\Big)+\frac{\partial\xi_k}{\partial q_s}Q_s$$

$$-\ddot{\xi}_0\frac{\partial L}{\partial\dot{q}_s}+X^{(0)}(Q_s)=0\,, \tag{38}$$

where the infinitesimal generators $\xi_0=\xi_0(t,\boldsymbol{q})$, $\xi_s=\xi_s(t,\boldsymbol{q})$ have Lie point symmetry. Substituting the relation

$$\begin{aligned}&\frac{\partial\dot{\xi}_s}{\partial\dot{q}_s}=\frac{\partial\xi_s}{\partial q_s},\qquad \frac{\partial\dot{\xi}_0}{\partial\dot{q}_s}=\frac{\partial\xi_0}{\partial q_s},\\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s}=\frac{\partial\dot{\xi}_s}{\partial q_s},\quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial q_s}=\frac{\partial\dot{\xi}_0}{\partial q_s}\end{aligned} \tag{39}$$

as well as Eqs.(35) and (38) into Eq.(37) we get the determining equation (4). Hence the Mei symmetry is a Lie point symmetry.

Particularly, if the symmetry is such a Lie point symmetry that $\xi_0=0$, $\xi_s=\xi_s(t,\boldsymbol{q})$, then Eq.(38) becomes

$$\frac{\partial\xi_k}{\partial q_s}Q_k+\xi_s\frac{\partial Q_s}{\partial q_s}=0. \tag{40}$$

(2) For a nonconservative system, if the infinitesimal generators $\xi_0=0$, $\xi_s=\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, the necessary and sufficient condition on which Mei symmetry is a Lie symmetry is that the symmetry satisfies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\xi_k}{\partial\dot{q}_s}\frac{\partial L}{\partial q_k}\Big)-\frac{\partial\xi_k}{\partial q_s}\frac{\partial L}{\partial q_k}+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\dot{\xi}_k}{\partial\dot{q}_s}\frac{\partial L}{\partial\dot{q}_k}\Big)\\ &-\frac{\partial\dot{\xi}_k}{\partial q_s}\frac{\partial L}{\partial\dot{q}_k}+X^{(0)}(Q_s)=0\,.\end{aligned} \tag{41}$$

(3) For a nonconservative system, if the infinitesimal generators $\xi_s=0$, $\xi_0=\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, then the necessary and sufficient condition on which Mei symmetry is a Lie symmetry is that the symmetry satisfies

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\xi_0}{\partial\dot{q}_s}\frac{\partial L}{\partial t}\Big)-\frac{\partial\xi_0}{\partial q_s}\frac{\partial L}{\partial t}-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial\dot{\xi}_0}{\partial\dot{q}_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}\Big)\\ &+\frac{\partial\dot{\xi}_0}{\partial q_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}-\ddot{\xi}_0\frac{\partial L}{\partial\dot{q}_s}+X^{(0)}(Q_s)=0\,.\end{aligned} \tag{42}$$

6. Non-Noether conserved quantities deduced from Mei symmetries

In this section, we present that the Mei symmetries can lead to non-Noether conserved quantities for nonconservative systems.

Proposition 2 For a nonconservative system under the infinitesimal transformation Eq.(3), the Mei symmetry of the system leads to a non-Noether conserved quantity Eq.(11), if the infinitesimal generators ξ_0 and ξ_s satisfy Eq.(36) and there is a gauge function G defined by Eq.(10).

We now point out that for a nonconservative system under the infinitesimal transformations ξ_0 and ξ_s, and there is a gauge function G defined by Eq.(10), by using proposition 2, we can obtain a series of the non-Noether conserved quantities as follows:

(1) If the infinitesimal generators $\xi_0=\xi_0(t,\boldsymbol{q})$, $\xi_s=\xi_s(t,\boldsymbol{q})$ satisfy Eq.(38), the Mei symmetry of the system will lead to a Lutzky conserved quantity (23).

Further, if the infinitesimal generators $\xi_0=0$, $\xi_s=\xi_s(t,\boldsymbol{q})$ satisfy Eq.(40), the Mei symmetry of the system will lead to a Lutzky conserved quantity (24).

(2) If the infinitesimal generators $\xi_0=0$, $\xi_s=\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfy Eq.(41), the Mei symmetry of the system will lead to a non-Noether conserved quantity (28).

(3) If the infinitesimal generators $\xi_s=0$, $\xi_0=\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfy Eq.(42), the Mei symmetries of the system will lead to a non-Noether conserved quantity (31).

Lagrangian system can be treated as a special case of nonconservative systems in which the generalized non-potential forces are zero, and gauge function $G=\ln D(D=\det(\partial^2L/\partial\dot{q}_s\partial\dot{q}_k))$. According to above discussions, one can obtain the analogue of the necessary and sufficient conditions on which the Mei symmetry is a Lie symmetry, and a series of the non-Noether conserved quantities and Lutzky conserved quantities of Lie symmetries and Mei symmetries for the systems.

7. An example

Consider a system with Lagrangian $L=\frac{1}{2}\mathrm{e}^{\gamma t}\dot{q}^2$, subjected to a nonconservative force $Q'=\mathrm{e}^{\gamma t}k\dot{q}^2$. Its equation of motion is

$$\ddot{q}_s=-\gamma\dot{q}+k\dot{q}^2. \tag{43}$$

The determining equation of the Lie symmetry for Eq.(43) under the infinitesimal transformation $\xi=\xi(t,q)$, $\xi_0=\xi_0(t,q)$ is

$$\begin{aligned}&\xi_{tt}+2\dot{q}\xi_{tq}-\gamma\dot{q}\xi_q+k\dot{q}^2\xi_q+\dot{q}^2\xi_{qq}-\dot{q}\xi_{0tt}\\ &-2\dot{q}^2\xi_{0tq}+\gamma\dot{q}^2\xi_{0q}-k\dot{q}^3\xi_{0q}-\dot{q}^3\xi_{0qq}\\ &+2\gamma\dot{q}\xi_{0t}-2k\dot{q}^2\xi_{0t}+2\gamma\dot{q}^2\xi_{0q}-2k\dot{q}^3\xi_{0q}\\ &+\gamma\xi_t+\gamma\dot{q}\xi_q-\gamma\dot{q}\xi_{0t}-\gamma\dot{q}^2\xi_{0q}-2k\dot{q}\xi_t\\ &-2k\dot{q}^2\xi_q+2k\dot{q}^2\xi_{0t}+2k\dot{q}^3\xi_{0q}=0.\end{aligned} \tag{44}$$

Equation (44) results in

$$\xi_{tt}+\gamma\xi_t=0, \tag{45a}$$

$$2\xi_{tq} - \xi_{0tt} + \gamma\xi_{0t} - 2k\xi_t = 0, \tag{45b}$$

$$\xi_{qq} - 2\xi_{0tq} + 2\gamma\xi_{0q} - k\xi_q = 0, \tag{45c}$$

$$k\xi_{0q} + \xi_{0qq} = 0. \tag{45d}$$

Equation (45) has the solution

$$\xi = -\frac{c_7}{\gamma}\mathrm{e}^{kq-\gamma t} + c_2\mathrm{e}^{kq} + c_1,$$

$$\xi_0 = -\frac{c_5}{k}\mathrm{e}^{-kq+\gamma t} + c_4\mathrm{e}^{\gamma t} + c_3. \tag{46}$$

It can be proved that ξ solved from Eq.(46) makes Eq.(38) hold true. Therefore, the Lie point transformation Eq.(46) is also the Mei symmetry.

Substituting of Eq.(43) into Eq.(10) we get

$$-\gamma + 2k\dot{q} + \frac{\mathrm{d}}{\mathrm{d}t}\ln G = 0, \tag{47}$$

hence

$$G = \mathrm{e}^{\gamma t - 2kq}. \tag{48}$$

Substituting of Eqs.(46) and (48) into Eq.(23) we get the Lutzky conserved quantity

$$\begin{aligned}\phi = &-2\frac{k}{\gamma}c_7\mathrm{e}^{kq-\gamma t} + 2kc_2\mathrm{e}^{kq} - 2\dot{q}c_5\mathrm{e}^{-kq+\gamma t}\\ &- c_5\dot{q}\mathrm{e}^{-kq+\gamma t} + \frac{\gamma}{k}c_5\mathrm{e}^{-kq+\gamma t}\\ &- c_4\gamma\mathrm{e}^{\gamma t} + X^{(1)}(\gamma t - 2kq)\\ = &-3\dot{q}c_5\mathrm{e}^{-kq+\gamma t} + \gamma c_3 - 2kc_1 = \mathrm{const}\,.\end{aligned} \tag{49}$$

Therefore, the Mei symmetry of the nonconservative system Eq.(43) possesses the non-Noether conserved quantity (49).

8. Conclusions

This paper deals with the conserved quantities arising from nonconservative systems. The velocity-dependent Lie symmetry is introduced to obtain conserved quantities. Based on the definition of the Mei symmetries, necessary and sufficient conditions are presented, under which the Mei symmetries is the Lie symmetry. Therefore, conserved quantities can also be located via the Mei symmetries.

References

[1] Noether A E 1918 *Nachr. Akad. Wiss. Göttingen Math. Phys.* **KI II** 235
[2] Djukić Dj S and Vujanović B 1975 *Acta Mech.* **23** 17
[3] Lie S 1894 *Theorie der Transformationsgruppen* (Leipzig: Teubner)
[4] Lutzky M 1979 *J. Phys.* A: *Math. Gen.* **12** 973
[5] Chen X W, Liu C M and Li Y M 2006 *Chin. Phys.* **15** 470
[6] Chen X W and Li Y M 2003 *Chin. Phys.* **12** 1349
[7] Zhang Y, Shang M and Mei F X 2000 *Chin. Phys.* **9** 401
[8] Fang J H, Chen P S, Zhang J and Li H 2003 *Acta Phys. Sin.* **52** 2945 (in Chinese)
[9] Liu R W and Chen L Q 2004 *Chin. Phys.* **13** 1615
[10] Zhang H B, Chen L Q and Liu R W 2005 *Chin. Phys.* **14** 1053
[11] Lou Z M 2006 *Chin. Phys.* **15** 891
[12] Wu H B 2005 *Chin. Phys.* **14** 452
[13] Xu X J, Mei F X and Zhang Y F 2006 *Chin. Phys.* **15** 19
[14] Luo S K, Huang F J and Lu Y B 2004 *Chin. Phys.* **13** 2182
[15] Mei F X 2000 *Journal of Beijing Institute of Technology* **9** 120
[16] Mei F X 2001 *Chin. Phys.* **10** 177
[17] Fu J L and Chen L Q 2004 *Mech. Res. Commun.* **31** 9
[18] Lutzky M 1979 *Phys. Lett.* A **72** 86
[19] Lutzky M 1995 *J. Phys.* A: *Math. Gen.* **28** 637
[20] Lutzky M 1998 *Int. Non-Linear Mech.* **33** 393
[21] Fu J L and Chen L Q 2004 *Phys. Lett.* A **331** 138
[22] Qiao Y F, Zhang S H and Li R J 2004 *Chin. Phys.* **13** 292
[23] Mei F X and Xu X J 2005 *Chin. Phys.* **14** 449
[24] Fu J L and Chen L Q 2003 *Phys. Lett.* A **317** 255
[25] Fu J L, Chen L Q and Liu R W 2004 *Chin. Phys.* **13** 1784
[26] Zhang H B, Chen L Q, Liu R W and Gu S L 2005 *Acta Phys. Sin.* **54** 2489 (in Chinese)
[27] Fu J L, Chen L Q and Yang X D 2004 *Chin. Phys.* **13** 287
[28] Hojman S A 1992 *J. Phys.* A: *Math. Gen.* **25** L291