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The Liouville integrable coupling system of the m-AKNS hierarchy and its Hamiltonian structure

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In this paper a type of 9-dimensional vector loop algebra $\tilde{F}$ is constructed, which is devoted to establish an isospectral problem. It follows that a Liouville integrable coupling system of the m-AKNS hierarchy is obtained by employing the Tu scheme, whose Hamiltonian structure is worked out by making use of constructed quadratic identity. The method given in the paper can be used to obtain many other integrable couplings and their Hamiltonian structures.

Keywords: loop algebra, integrable coupling, Hamiltonian structure, quadratic identity
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1. Introduction

A host of new integrable systems were obtained by employing the Tu scheme[1,2] under the frame of the zero curvature equation. Ma, Hu and Fan further developed the Tu scheme and worked out many interesting results.[3−7] Recently, the concept and related theory on integrable couplings have been proposed due to soliton theory[8,9]. In Refs.[10–19], a simple method for obtaining the integrable coupling was presented. However, we find that some integrable couplings can be worked out by use of vector loop algebra, such as the result in Ref.[10], whose Hamiltonian structure cannot be produced by the trace identity.[2] In this paper, we deal with the problem by means of the quadratic identity.[20] In addition, by making use of a 9-dimensional vector loop algebra based on the ideas in Ref.[20], a Liouville integrable coupling system of the m-AKNS hierarchy is worked out, specially its Hamiltonian structure is obtained.

2. A type of 9-dimensional vector loop algebra and the Liouville integrable coupling system of the m-AKNS hierarchy as well as its Hamiltonian structure

Set

$$\begin{aligned}
f_1 &= (1,0,0,0,0,0,0,0,0)^{\mathrm{T}},\\
f_2 &= (0,1,0,0,0,0,0,0,0)^{\mathrm{T}},\\
f_3 &= (0,0,1,0,0,0,0,0,0)^{\mathrm{T}},\\
f_4 &= (0,0,0,1,0,0,0,0,0)^{\mathrm{T}},\\
f_5 &= (0,0,0,0,1,0,0,0,0)^{\mathrm{T}},\\
f_6 &= (0,0,0,0,0,1,0,0,0)^{\mathrm{T}},\\
f_7 &= (0,0,0,0,0,0,1,0,0)^{\mathrm{T}},\\
f_8 &= (0,0,0,0,0,0,0,1,0)^{\mathrm{T}},\\
f_9 &= (0,0,0,0,0,0,0,0,1)^{\mathrm{T}},
\end{aligned}$$

and let F denote a linear space with the basis $\{f_i\}_{i=1}^9$, ie $F = \mathrm{span}\,\{f_i\}_{i=1}^9$. Denote the commuting relations among f_i, $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$, as follows

$$\begin{aligned}
&[f_1,f_2]=f_3,[f_1,f_3]=-f_2,[f_1,f_5]=f_6,[f_1,f_6]=-f_5,[f_1,f_8]=f_9,[f_1,f_9]=-f_8,[f_2,f_3]=f_1,\\
&[f_2,f_4]=-f_6,[f_2,f_6]=f_4,[f_2,f_7]=-f_9,[f_2,f_9]=f_7,[f_3,f_4]=f_5,[f_3,f_5]=-f_4,[f_3,f_7]=f_8,\\
&[f_3,f_8]=-f_7,[f_4,f_5]=f_9,[f_4,f_6]=f_8,[f_5,f_6]=f_7,[f_i,f_i]=0,1\le i\le 9,\\
&[f_1,f_4]=[f_1,f_7]=[f_2,f_5]=[f_2,f_8]=[f_3,f_6]=[f_3,f_9]=[f_4,f_7]=[f_4,f_8]=[f_4,f_9]=[f_5,f_7]\\
&=[f_5,f_8]=[f_5,f_9]=[f_6,f_7]=[f_6,f_8]=[f_6,f_9]=[f_7,f_8]=[f_7,f_9]=[f_8,f_9]=0, \qquad (1)
\end{aligned}$$

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where $[a,b]=ab-ba$. It is easy to verify that F is a Lie algebra, its resulting loop algebra is denoted by $\tilde{F}$ $=\mathrm{span}\{f_i(m)\}_{i=1}^{9}$, where $f_i(m)=f_i\lambda^m, 1\le i\le 9$. In terms of $\tilde{F}$, an isospectral problem is given by

$$\phi_x=[U,\phi],\quad U=(\lambda,u_1,u_2,0,u_3,u_4,0,u_5,u_6)^{\mathrm{T}}. \quad (2)$$

Set $V=\sum_{m\ge 0}(a_m,b_m,c_m,d_m,e_m,f_m,g_m,h_m,k_m)^{\mathrm{T}}\lambda^{-m}$, solving the stationary zero curvature equation

$$V_x=[U,V], \quad (3)$$

we get

$$\begin{cases} a_{mx}=u_1c_m-u_2b_m,\\ b_{mx}=u_2a_m-c_{m+1},\\ c_{mx}=b_{m+1}-u_1a_m,\\ d_{mx}=u_3c_m-u_2e_m+u_1f_m-u_4b_m,\\ e_{mx}=u_4a_m-u_2d_m-f_{m+1},\\ f_{mx}=e_{m+1}-u_3a_m-u_1d_m,\\ g_{mx}=u_5c_m-u_2h_m+u_1k_m-u_6b_m+u_3f_m-u_4e_m,\\ h_{mx}=u_6a_m+u_2g_m+u_4d_m-k_{m+1},\\ k_{mx}=h_{m+1}-u_5a_m-u_1g_m-u_3d_m,\\ a_0=\beta=\mathrm{const.},\qquad b_0=c_0=d_0=e_0=f_0=g_0=h_0=k_0=0,\\ a_1=d_1=g_1=0,\qquad b_1=\beta u_1,\qquad c_1=\beta u_2,\\ e_1=\beta u_3,\qquad f_1=\beta u_4,\qquad h_1=\beta u_5,\qquad k_1=\beta u_6,\quad \cdots. \end{cases} \quad (4)$$

Noting

$V_+^{(n)}=\sum_{m=0}^{n}(a_m,b_m,c_m,d_m,e_m,f_m,g_m,h_m,k_m)^{\mathrm{T}}\lambda^{n-m}$, $V_-^{(n)}=\lambda^nV-V_+^{(n)}$, we can write Eq.(3) as

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=V_{-x}^{(n)}-[U,V_-^{(n)}]. \quad (5)$$

It is easy to find that the terms of left-hand side are of degree ≥ 0, while the terms of the right-hand side are of degree ≤ 0. Hence,

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=\ -(0,-c_{n+1},b_{n+1},0,-f_{n+1},e_{n+1},0,-k_{n+1},h_{n+1})^{\mathrm{T}}.$$

Taking $V^{(n)}=V_+^{(n)}$, then from the zero curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0 \quad (6)$$

we get the integrable system

$$u_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{pmatrix}_t=\begin{pmatrix}-c_{n+1}\\b_{n+1}\\-f_{n+1}\\e_{n+1}\\-k_{n+1}\\h_{n+1}\end{pmatrix}=\begin{pmatrix}0&0&0&0&0&-1\\0&0&0&0&1&0\\0&0&0&-1&0&1\\0&0&1&0&-1&0\\0&-1&0&1&0&0\\1&0&-1&0&0&0\end{pmatrix}\begin{pmatrix}b_{n+1}+e_{n+1}+h_{n+1}\\c_{n+1}+f_{n+1}+k_{n+1}\\b_{n+1}+e_{n+1}\\c_{n+1}+f_{n+1}\\b_{n+1}\\c_{n+1}\end{pmatrix}=JP_{n+1}, \quad (7)$$

where J is a Hamiltonian operator. From Eq.(4), a recurrence operator L meets $P_{n+1}=LP_n$, where

$$L=\begin{pmatrix}A&B&E&F&K&L\\C&D&G&H&M&N\\0&0&A&B&E&F\\0&0&C&D&G&H\\0&0&0&0&A&B\\0&0&0&0&C&D\end{pmatrix}$$

with $A=-u_1\partial^{-1}u_2$, $B=\partial+u_1\partial^{-1}u_1$, $C=-\partial-u_2\partial^{-1}u_2$, $D=u_2\partial^{-1}u_1$, $E=-u_3\partial^{-1}u_2-u_1\partial^{-1}u_4$, $F=u_3\partial^{-1}u_1+u_1\partial^{-1}u_3$, $G=-u_4\partial^{-1}u_2-u_2\partial^{-1}u_4$, $H=u_4\partial^{-1}u_1+u_2\partial^{-1}u_3$, $K=-u_5\partial^{-1}u_2-u_1\partial^{-1}u_6-u_3\partial^{-1}u_4$, $L=u_5\partial^{-1}u_1+u_1\partial^{-1}u_5+u_3\partial^{-1}u_3$, $M=-u_6\partial^{-1}u_2-u_2\partial^{-1}u_6-u_4\partial^{-1}u_4$, $N=u_6\partial^{-1}u_1+u_2\partial^{-1}u_5+u_4\partial^{-1}u_3$.

Hence Eq.(7) can be written as

$$u_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{pmatrix}_t=JL^n\begin{pmatrix}\beta(u_1+u_3+u_5)\\\beta(u_2+u_4+u_6)\\\beta(u_1+u_3)\\\beta(u_2+u_4)\\\beta u_1\\\beta u_2\end{pmatrix}. \quad (8)$$

Reduction cases

Case 1 When $u_5=u_6=0$, we can take $g_m=h_m=k_m=0$, Eq.(8) reduces to Eq.(77) in Ref.[20].

Case 2 When $u_3=u_4=u_5=u_6=0$, we may take $d_m=e_m=f_m=g_m=h_m=k_m=0$, Eq.(8) is just the modified-AKNS hierarchy, which is denoted by m-AKNS hierarchy Eq.(79) in Ref.[20].

It is easy to find that $JL=L^*J$, hence we call Eq.(8) a Liouville integrable coupling system of the m-AKNS hierarchy.

As for $a,b\in\tilde{F}\left(a=\sum_{i=1}^{9}a_if_i=(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,)^{\mathrm{T}},b=\sum_{i=1}^{9}b_if_i=(b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8,b_9,)^{\mathrm{T}}\right)$, from Eq.(1), we have

$$[a,b]^{\mathrm{T}}=a^{\mathrm{T}}\begin{pmatrix}0&-b_3&b_2&0&-b_6&b_5&0&-b_9&b_8\\b_3&0&-b_1&b_6&0&-b_4&b_9&0&-b_7\\-b_2&b_1&0&-b_5&b_4&0&-b_8&b_7&0\\0&0&0&0&-b_3&b_2&0&-b_6&b_5\\0&0&0&b_3&0&-b_1&b_6&0&-b_4\\0&0&0&-b_2&b_1&0&-b_5&b_4&0\\0&0&0&0&0&0&0&-b_3&b_2\\0&0&0&0&0&0&b_3&0&-b_1\\0&0&0&0&0&0&-b_2&b_1&0\end{pmatrix}=a^{\mathrm{T}}R(b). \quad (9)$$

For the constant matrix $F=(f_{ij})_{9\times9}$, solving the matrix equation

$$F=F^{\mathrm{T}},R(b)F=-(R(b)F)^{\mathrm{T}}. \quad (10)$$

we have

$$F=\begin{pmatrix}1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0\\0&0&1&0&0&1&0&0&1\\1&0&0&1&0&0&0&0&0\\0&1&0&0&1&0&0&0&0\\0&0&1&0&0&1&0&0&0\\1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\end{pmatrix}. \quad (11)$$

Let

$$\begin{aligned}\{a,b\} = a^{\mathrm{T}}Fb = \; & (a_1+a_4+a_7)b_1+(a_2+a_5+a_8)b_2\\ & +(a_3+a_6+a_9)b_3+(a_1+a_4)b_4\\ & +(a_2+a_5)b_5+(a_3+a_6)b_6\\ & +a_1b_7+a_2b_8+a_3b_9. \end{aligned} \tag{12}$$

A direct computation gives

$$\begin{aligned}&\left\{V,\frac{\partial U}{\partial\lambda}\right\} = a+d+g,\\ &\left\{V,\frac{\partial U}{\partial u_1}\right\} = b+e+h,\\ &\left\{V,\frac{\partial U}{\partial u_2}\right\} = c+f+k,\\ &\left\{V,\frac{\partial U}{\partial u_3}\right\} = b+e,\quad \left\{V,\frac{\partial U}{\partial u_4}\right\} = c+f,\\ &\left\{V,\frac{\partial U}{\partial u_5}\right\} = b,\quad \left\{V,\frac{\partial U}{\partial u_6}\right\} = c,\end{aligned}$$

where $a=\sum_{m\geq 0}a_m\lambda^{-m}, b=\sum_{m\geq 0}b_m\lambda^{-m},\quad \cdots$.
Substituting the above formulae into the quadratic identity

$$\frac{\delta}{\delta u_i}\{V,U_\lambda\} = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{\gamma}\left\{V,\frac{\partial U}{\partial u_i}\right\}\right), \quad i=1,2,3,4,5,6, \tag{13}$$

we have

$$\frac{\delta}{\delta u}(a+d+g) = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{\gamma}\begin{pmatrix} b+e+h\\ c+f+k\\ b+e\\ c+f\\ b\\ c\end{pmatrix}\right). \tag{14}$$

Comparing the coefficients of λ^{-n-2} we get

$$\begin{aligned}&\frac{\delta}{\delta u}(a_{n+2}+d_{n+2}+g_{n+2})\\ =\;& (\gamma-n-1)\begin{pmatrix} b_{n+1}+e_{n+1}+h_{n+1}\\ c_{n+1}+f_{n+1}+k_{n+1}\\ b_{n+1}+e_{n+1}\\ c_{n+1}+f_{n+1}\\ b_{n+1}\\ c_{n+1}\end{pmatrix}.\end{aligned} \tag{15}$$

Taking $n=0$ then we have $\gamma=0$, and obtain the Hamiltonian function $H_{n+1}=-\frac{1}{n+1}(a_{n+2}+d_{n+2}+g_{n+2}), n\geq 0$, which satisfies $P_{n+1}=\frac{\delta H_{n+1}}{\delta u}$.

Hence Eq.(7) possesses the Hamiltonian form as follows:

$$u_t=\begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\end{pmatrix}_t = JP_{n+1}=J\frac{\delta H_{n+1}}{\delta u}, n\geq 0. \tag{16}$$

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