Vol 16 No 3, March 2007
1009-1963/2007/16(03)/0588-07

Chinese Physics

© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

Two types of loop algebras and their expanding Lax integrable models

Yue Chao(岳 超)[a)†], Zhang Yu-Feng(张玉峰)[b)‡], and Wei Yuan(魏 媛)[b)]

[a)] *School of Information Engineering, Taishan Medical University, Taian* 271016, *China*

[b)] *School of Information Science and Engineering, Shandong University of Science and Technology, Qingdao* 266510, *China*

(Received 10 June 2005; revised manuscript received 14 April 2006)

Though various integrable hierarchies of evolution equations were obtained by choosing proper U in zero-curvature equation $U_t - V_x + [U, V] = 0$, but in this paper, a new integrable hierarchy possessing bi-Hamiltonian structure is worked out by selecting V with spectral potentials. Then its expanding Lax integrable model of the hierarchy possessing a simple Hamiltonian operator $\widetilde{J}$ is presented by constructing a subalgebra $\widetilde{G}$ of the loop algebra $\widetilde{A}_2$. As linear expansions of the above-mentioned integrable hierarchy and its expanding Lax integrable model with respect to their dimensional numbers, their (2+1)-dimensional forms are derived from a (2+1)-dimensional zero-curvature equation.

Keywords: zero-curvature equation, integrable hierarchy, loop algebra
PACC: 0290, 0340K, 0365G

1. Introduction

Researching into new integrable hierarchies of soliton equations and their expanding integrable models is an important and interesting topic in soliton theory. In recent years, various integrable systems were obtained by choosing various U in zero-curvature equation $U_t - V_x + [U, V] = 0$.[1−5] In this paper, for given U, we choose an appropriate V with spectral potentials in the equation $U_t - V_x + [U, V] = 0$ to obtain a family of hierarchy of evolution equations, which possesses bi-Hamiltonian structure. As its reduction cases, a formalism of well-known Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy and the standard heart conduction equation are given. Then its expanding Lax integrable model of the hierarchy possessing a simple Hamiltonian operator $\widetilde{J}$ is presented by constructing a subalgebra $\widetilde{G}$ of the loop algebra $\widetilde{A}_2$.[6−12] In addition, as linear expansions of the above-mentioned integrable hierarchy and its expanding Lax integrable model with regard to their dimensional numbers, their (2+1)-dimensional forms are worked out from a (2+1)-dimensional zero-curvature equation.[13,14]

2. A new integrable hierarchy and its expanding Lax integrable model

Choose the loop algebra $\widetilde{A}_1$ with the basis

$$h(0) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e(0) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad f(0) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$[h, e] = f, \quad [h, f] = e, \quad [e, f] = -h, \quad h(n) = h(0)\lambda^n, \quad e(n) = e(0)\lambda^n, \quad f(n) = f(0)\lambda^n,$$

$$\deg(h(n)) = \deg(e(n)) = \deg(f(n)) = n\,. \tag{1}$$

Consider an isospectral problem

$$\phi_x = U\phi, \qquad \lambda_t = 0, \qquad U = \frac{1}{2}\begin{pmatrix} \lambda + s & q + r \\ q - r & -\lambda - s \end{pmatrix} = h(1) + sh(0) + qe(0) + rf(0). \tag{2}$$

†E-mail: yuechao_71@163.com
‡E-mail: zhang_yfshandong@163.com

Set

$$V = \frac{1}{2}\begin{pmatrix} \lambda a + sa & b + c + qa + ra \\ b - c + qa - ra & -\lambda a - sa \end{pmatrix} = \sum_{m\geq 0}(a_m h(1-m) + b_m e(-m) + c_m f(-m) + sa_m h(-m) + qa_m e(-m) + ra_m f(-m)).$$

Solving the adjoint equation

$$V_x = [U, V], \tag{3}$$

we have

$$\begin{aligned} &a_{m+1x} + (sa_m)_x = -qc_m + rb_m,\\ &b_{mx} + (qa_m)_x = c_{m+1} + sc_m,\\ &c_{mx} + (ra_m)_x = b_{m+1} + sb_m,\\ &a_0 = a_1 = 0, \quad b_0 = \alpha q, \quad c_0 = \alpha r,\\ &b_1 = \alpha(r_x - sq), \quad c_1 = \alpha(q_x - sr),\\ &a_2 = \frac{\alpha}{2}(r^2 - q^2),\\ &b_2 = \alpha(q_{xx} - (sr)_x - sr_x + s^2 q),\\ &c_2 = \alpha(r_{xx} - (sq)_x - sq_x + s^2 r). \end{aligned} \tag{4}$$

Denoting

$$\begin{aligned} V_+^{(n)} &= (\lambda^n V)_+ = \sum_{m=0}^{n}(a_m h(1+n-m)\\ &\quad + b_m e(n-m) + c_m f(n-m)\\ &\quad + sa_m h(n-m) + qa_m e(n-m)\\ &\quad + ra_m f(n-m)),\\ V_-^{(n)} &= \lambda^n V - V_+^{(n)}, \end{aligned}$$

we can write Eq.(3) as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]. \tag{5}$$

It is easy to verify that the terms on the left-hand side in Eq.(5) are of degree $\geqslant 0$, while the terms on the right-hand side in Eq.(5) are of degree $\leqslant 1$.
Thus, we have

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = a_{n+1x} h(0) - c_{n+1} e(0) - b_{n+1} f(0).$$

Taking $V_+^{(n)} = V^{(n)}$, we find that the zero-curvature equation $U_t - V_x^{(n)} + [U, V^{(n)}] = 0$ leads to

$$u_{t_n} = \begin{pmatrix} s \\ q \\ r \end{pmatrix}_{t_n} = \begin{pmatrix} -a_{n+1x} \\ c_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} -\partial & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} a_{n+1} \\ -b_{n+1} \\ c_{n+1} \end{pmatrix} = J_1 \begin{pmatrix} a_{n+1} \\ -b_{n+1} \\ c_{n+1} \end{pmatrix}, \tag{6}$$

where J_1 is a Hamiltonian operator.
From the recursive relation Eq.(4), we find that

$$\begin{pmatrix} a_{n+1} \\ -b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} -s & -\partial^{-1} r & -\partial^{-1} q \\ -\partial r & -s & -\partial \\ \partial q & -\partial & -s \end{pmatrix} \times \begin{pmatrix} a_n \\ -b_n \\ c_n \end{pmatrix} = L_1 \begin{pmatrix} a_n \\ -b_n \\ c_n \end{pmatrix}, \tag{7}$$

where L_1 is called the recursive operator.
Thus Eq.(6) can be written as

$$u_{t_n} = \begin{pmatrix} s \\ q \\ r \end{pmatrix}_{t_n} = J_1 L_1 \begin{pmatrix} a_n \\ -b_n \\ c_n \end{pmatrix} = J_1 L_1^n \begin{pmatrix} 0 \\ \alpha(-r_x + sq) \\ \alpha(q_x - sr) \end{pmatrix} \tag{8}$$

Specially, taking $s = 0$ in Eq.(8) we obtain a formalism of AKNS hierarchy as follows:

$$\tilde{u}_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \tilde{J}_1 \tilde{L}_1^n \begin{pmatrix} -\alpha r_x \\ \alpha q_x \end{pmatrix}, \tag{9}$$

where

$$\tilde{J}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{L}_1 = \begin{pmatrix} 0 & -\partial \\ -\partial & 0 \end{pmatrix}.$$

Remark Here $\tilde{L}_1$ is different from that of the well-known AKNS hierarchy. Hence, Eq.(9) is not an AKNS hierarchy.

As a reduction case of Eq.(9), taking $n = 1$, and $q = r = u$, we obtain the standard linear heat conduction equation $u_t = \alpha u_{xx}$. A direct calculation shows

$$\left\langle V, \frac{\partial U}{\partial s} \right\rangle = \frac{1}{2}(\lambda a + sa), \quad \left\langle V, \frac{\partial U}{\partial q} \right\rangle = \frac{1}{2}(b + qa),$$
$$\left\langle V, \frac{\partial U}{\partial r} \right\rangle = -\frac{1}{2}(c + ra), \quad \left\langle V, \frac{\partial U}{\partial \lambda} \right\rangle = \frac{1}{2}(\lambda a + sa). \quad (10)$$

Inserting Eq.(10) into the trace identity we get

$$\frac{\delta}{\delta u}(\lambda a + sa) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} \lambda a + sa \\ b + qa \\ -c - ra \end{pmatrix}.$$

Comparing the coefficients of λ^{-n-1} we get

$$\frac{\delta}{\delta u}(a_{n+2} + sa_{n+1}) = (\gamma - n)\begin{pmatrix} a_{n+1} + sa_n \\ b_n + qa_n \\ -c_n - ra_n \end{pmatrix}. \quad (11)$$

Substituting Eq.(4) into Eq.(11) we obtain $\gamma = -1$, thus

$$\frac{\delta H_n}{\delta u} = \begin{pmatrix} 0 & \partial^{-1}r & -\partial^{-1}q \\ q & -1 & 0 \\ -r & 0 & -1 \end{pmatrix} \times \begin{pmatrix} a_n \\ -b_n \\ c_n \end{pmatrix} = N\begin{pmatrix} a_n \\ -b_n \\ c_n \end{pmatrix},$$

where

$$H_n = -\frac{a_{n+2} + sa_{n+1}}{n+1}$$

are Hamiltonian functions, therefore, we have

$$\begin{pmatrix} a_n \\ -b_n \\ c_n \end{pmatrix} = N^{-1}\frac{\delta H_n}{\delta u}.$$

Hence the hierarchy of nonlinear equations (8) possesses the following bi-Hamiltonian structure

$$u_{t_n} = \begin{pmatrix} s \\ q \\ r \end{pmatrix}_{t_n} = J\frac{\delta H_n}{\delta u} = K\frac{\delta H_{n-1}}{\delta u}, \quad (12)$$

where

$$J = J_1 L_1 N^{-1}, \quad K = J_1 L_1 N^{-1} L_1,$$
$$N^{-1} = \begin{pmatrix} \frac{1}{2qr}\partial & \frac{1}{2q} & -\frac{1}{2r} \\ \frac{1}{2r}\partial & -\frac{1}{2} & -\frac{q}{2r} \\ -\frac{1}{2q}\partial & -\frac{r}{2q} & -\frac{1}{2} \end{pmatrix}.$$

It is easy to find that the loop algebra $\tilde{A}_1$ is equivalent to the following loop algebra:

$$e_1(0) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2(0) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3(0) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1, \quad e_i(n) = e_i(0)\lambda^n, \quad \deg(e_i(n)) = n, \quad i = 1, 2, 3. \quad (13)$$

It is easy to verify that using Eq.(13) and Tu-model may also engender Eq.(12). Furthermore, we may expand Eq.(13) into a subalgebra $\tilde{G}$ of the loop algebra $\tilde{A}_2$ as follows:

$$e_1(0) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2(0) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3(0) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4(0) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5(0) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = -e_1, \quad [e_1, e_4] = \frac{1}{2}e_5, \quad [e_1, e_5] = \frac{1}{2}e_4,$$

$$[e_2,e_4]=\frac{1}{2}e_4,\quad [e_2,e_5]=-\frac{1}{2}e_5,\quad [e_3,e_4]=\frac{1}{2}e_5,\quad [e_3,e_5]=-\frac{1}{2}e_4,\quad [e_4,e_5]=0,$$
$$e_i(n)=e_i(0)\lambda^n,\quad [e_i(m),e_j(n)]=[e_i(0),e_j(0)]\lambda^{m+n},\quad \deg(e_i(n))=n,\quad i,j=1,2,3,4,5.$$

An isospectral problem is established by

$$\phi_x=U\phi,\qquad \lambda_t=0,\quad U=e_1(1)+u_1e_1(0)+u_2e_2(0)+u_3e_3(0)+u_4e_4(0)+u_5e_5(0). \tag{14}$$

Let

$$\begin{aligned}V=\sum_{m\geq 0}&(a_me_1(1-m)+b_me_2(-m)+c_me_3(-m)+d_me_4(-m)+f_me_5(-m)\\&+u_1a_me_1(-m)+u_2a_me_2(-m)+u_3a_me_3(-m)+u_4a_me_4(-m)+u_5a_me_5(-m)).\end{aligned} \tag{15}$$

Solving the equation

$$V_x=[U,V], \tag{16}$$

we get

$$\begin{aligned}&a_{m+1x}+(u_1a_m)_x=-u_2c_m+u_3b_m,\\&b_{mx}+(u_2a_m)_x=c_{m+1}+u_1c_m,\\&c_{mx}+(u_3a_m)_x=b_{m+1}+u_1b_m,\\&2d_{mx}=f_{m+1}+u_1f_m+u_2d_m-u_3f_m-u_4b_m+u_5c_m,\\&2f_{mx}=d_{m+1}+u_1d_m-u_2f_m+u_3d_m-u_4c_m+u_5b_m,\\&a_0=a_1=d_0=f_0=0,\quad b_0=\alpha u_2,\quad c_0=\alpha u_3,\\&b_1=\alpha(u_{3x}-u_1u_2),\quad c_1=\alpha(u_{2x}-u_1u_3),\quad d_1=\alpha(u_3u_4-u_2u_5),\\&f_1=\alpha(u_2u_4-u_3u_5),\quad a_2=\frac{\alpha}{2}(u_3^2-u_2^3),\\&b_2=\alpha(u_{2xx}-(u_1u_3)_x-u_1u_{3x}+u_1^2u_2),\\&c_2=\alpha(u_{3xx}-(u_1u_2)_x-u_1u_{2x}+u_1^2u_3),\\&d_2=\alpha[2(u_2u_4-u_3u_5)_x+u_{2x}u_4-u_{3x}u_5+(u_2^2-u_3^3)u_4+2(u_1u_2u_5-u_1u_3u_4)],\\&f_2=\alpha[2(u_3u_4-u_2u_5)_x+u_{3x}u_4-u_{2x}u_5+(u_2^2-u_3^3)u_5+2(u_1u_3u_5-u_1u_2u_4)].\end{aligned} \tag{17}$$

Taking

$$\begin{aligned}V_+^{(n)}=\sum_{m=0}^{n}&(a_me_1(1+n-m)+b_me_2(n-m)+c_me_3(n-m)+d_me_4(n-m)+f_me_5(n-m)\\&+u_1a_me_1(n-m)+u_2a_me_2(n-m)+u_3a_me_3(n-m)+u_4a_me_4(n-m)+u_5a_me_5(n-m)),\\V_-^{(n)}=\lambda^nV-V_+^{(n)},&\end{aligned}$$

we can write Eq.(16) as

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=V_{-x}^{(n)}-[U,V_-^{(n)}], \tag{18}$$

similar to Eq.(5), the left-hand side of Eq.(18) reads

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=a_{n+1x}e_1(0)-c_{n+1}e_2(0)-b_{n+1}e_3(0)-\frac{1}{2}f_{n+1}e_4(0)-\frac{1}{2}d_{n+1}e_5(0).$$

Taking $V_+^{(n)} = V^{(n)}$, we find that thus the zero-curvature equation $U_t - V_x^{(n)} + [U, V^{(n)}] = 0$, determines the Lax integrable system as follows:

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}_t = \tilde{J} \begin{pmatrix} a_{n+1} \\ -b_{n+1} \\ c_{n+1} \\ -d_{n+1} \\ f_{n+1} \end{pmatrix} = \tilde{J}\tilde{L}^n \begin{pmatrix} 0 \\ \alpha(-u_{3x} + u_1u_2) \\ \alpha(u_{2x} - u_1u_3) \\ \alpha(-u_3u_4 + u_2u_5) \\ \alpha(u_2u_4 - u_3u_5) \end{pmatrix}, \tag{19}$$

where $\tilde{J}$ is a simple Hamiltonian operator.

$$\tilde{J} = \begin{pmatrix} -\partial & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \qquad \tilde{L} = \begin{pmatrix} -u_1 & -\partial^{-1}u_3 & -\partial^{-1}u_2 & 0 & 0 \\ -\partial u_3 & -u_1 & -\partial & 0 & 0 \\ \partial u_2 & -\partial & -u_1 & 0 & 0 \\ 0 & -u_5 & -u_4 & -u_1 - u_3 & -u_2 - 2\partial \\ 0 & -u_4 & -u_5 & -u_2 - 2\partial & u_3 - u_1 \end{pmatrix}.$$

Let $u_1 = 0$, taking $n = 1$ in Eq.(19), we have the following coupled equations:

$$\begin{aligned} &u_{2t} = \alpha u_{3xx}, \quad u_{3t} = \alpha u_{2xx}, \\ &u_{4t} = \frac{\alpha}{2}[2(u_3u_4 - u_2u_5)_x - u_{2x}u_5 \\ &\qquad + u_{3x}u_4 + (u_2^2 - u_3^2)u_5], \\ &u_{5t} = \frac{\alpha}{2}[2(u_2u_4 - u_3u_5)_x + u_{2x}u_4 \\ &\qquad - u_{3x}u_5 + (u_2^2 - u_3^2)u_4]. \end{aligned}$$

It is easy to find that when $u_1 = s$, $u_2 = q$, $u_3 = r$, $u_4 = u_5 = 0$, Eq.(19) reduces to Eq.(8). Comparing the constructions of J_1, L_1 in Eq.(8) with those of $\tilde{J}$, $\tilde{L}$ in Eq.(19), in terms of the definition of integrable coupling,[6−12] we conclude that Eq.(19) is the integrable coupling of Eq.(8), which is called a type of expanding Lax integrable model.

3. A direct expansion of loop algebra $\tilde{A}_1$—the (2+1)-dimensional integrable system and its integrable coupling

Denoting

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial w} = \frac{\partial}{\partial t} - \frac{\partial}{\partial y},$$

we find that the zero-curvature equation

$$U_w - V_z + [U, V] = 0 \tag{20}$$

may be written in a (2+1)-dimensional form

$$U_t + V_y + [U, V] - V_x - U_y = 0, \tag{21}$$

which is regarded as the compatibility of the following Lax pairs:

$$\phi_x - \phi_y = U\phi, \quad \phi_t - \phi_y = V\phi, \quad \lambda_t = 0. \tag{22}$$

We want to use Eq.(21) or Eq.(22) to produce a hierarchy of soliton equations systematically.
Solving the adjoint equation

$$V_x - V_y = [U, V], \tag{23}$$

we get

$$\begin{aligned} &a_{m+1x} + (sa_m)_x - a_{m+1y} - (sa_m)_y = -qc_m + rb_m, \\ &b_{mx} + (qa_m)_x - b_{my} - (qa_m)_y = c_{m+1} + sc_m, \\ &c_{mx} + (ra_m)_x - c_{my} - (ra_m)_y = b_{m+1} + sb_m, \\ &a_0 = a_1 = 0, \quad b_0 = \alpha q, \\ &c_0 = \alpha r, \quad b_1 = \alpha(r_x - r_y - sq), \\ &c_1 = \alpha(q_x - q_y - sr), \quad a_2 = \frac{\alpha}{2}(r^2 - q^2), \\ &b_2 = \alpha(q_{xx} + q_{yy} - 2q_{xy} - (sr)_x \\ &\qquad + (sr)_y - sr_x + sr_y + s^2q), \\ &c_2 = \alpha(r_{xx} + r_{yy} - 2r_{xy} - (sq)_x + (sq)_y \\ &\qquad - sq_x + sq_y + s^2r). \end{aligned} \tag{24}$$

Denoting

$$V_+^{(n)} = (\lambda^n V)_+ = \sum_{m=0}^{n} (a_m h(1 + n - m)$$

$$+ b_m e(n-m) + c_m f(n-m) + s a_m h(n-m)$$
$$+ q a_m e(n-m) + r a_m f(n-m)),$$
$$V_-^{(n)} = \lambda^n V - V_+^{(n)},$$

then by direct calculation we obtain

$$V_{+y}^{(n)} + [U, V_+^{(n)}] - V_{+x}^{(n)} = (a_{n+1x} - a_{n+1y})h(0) - c_{n+1}e(0) - b_{n+1}f(0).$$

Thus, the zero-curvature equation

$$U_t + V_y^{(n)} + [U, V^{(n)}] - V_x^{(n)} - U_y = 0$$

admits the (2+1)-dimensional integrable system

$$s_t - s_y = a_{n+1y} - a_{n+1x},$$
$$q_t - q_y = c_{n+1}, \quad r_t - r_y = b_{n+1}. \tag{25}$$

Taking $s = 0$, $n = 1$, $q = r = u$, we find that Eq.(25) reduces to the generalized (2+1)-dimensional linear heat conduction equation

$$u_t = \alpha(u_{xx} + u_{yy} - 2u_{xy}) + u_y\,.$$

Let $\tilde{U}$ and $\tilde{V}$ equal to the U in Eq.(14) and the V in Eq.(15) respectively, and solve the equation similar to Eq.(23), we obtain

$$\begin{aligned}
&a_{m+1x} + (u_1 a_m)_x - a_{m+1y} - (u_1 a_m)_y = -u_2 c_m + u_3 b_m,\\
&b_{mx} + (u_2 a_m)_x - b_{my} - (u_2 a_m)_y = c_{m+1} + u_1 c_m,\\
&c_{mx} + (u_3 a_m)_x - c_{my} - (u_3 a_m)_y = b_{m+1} + u_1 b_m,\\
&2(d_{mx} - d_{my}) = f_{m+1} + u_1 f_m + u_2 d_m - u_3 f_m - u_4 b_m + u_5 c_m,\\
&2(f_{mx} - f_{my}) = d_{m+1} + u_1 d_m - u_2 f_m + u_3 d_m - u_4 c_m + u_5 b_m,\\
&a_0 = a_1 = d_0 = f_0 = 0, \quad b_0 = \alpha u_2, \quad c_0 = \alpha u_3, \quad b_1 = \alpha(u_{3x} - u_{3y} - u_1 u_2),\\
&c_1 = \alpha(u_{2x} - u_{2y} - u_1 u_3), \quad d_1 = \alpha(u_3 u_4 - u_2 u_5), \quad f_1 = \alpha(u_2 u_4 - u_3 u_5), \quad a_2 = \frac{\alpha}{2}(u_3^2 - u_2^2),\\
&b_2 = \alpha(u_{2xx} + u_{2yy} - 2u_{2xy} - (u_1 u_3)_x + (u_1 u_3)_y - u_1 u_{3x} + u_1 u_{3y} + u_1^2 u_2),\\
&c_2 = \alpha(u_{3xx} + u_{3yy} - 2u_{3xy} - (u_1 u_2)_x + (u_1 u_2)_y - u_1 u_{2x} + u_1 u_{2y} + u_1^2 u_3),\\
&d_2 = \alpha[2(u_2 u_4 - u_3 u_5)_x - 2(u_2 u_4 - u_3 u_5)_y + u_{2x} u_4 - u_{3x} u_5 + (u_2^2 - u_3^2) u_4 + 2(u_1 u_2 u_5 - u_1 u_3 u_4)],\\
&f_2 = \alpha[2(u_3 u_4 - u_2 u_5)_x - 2(u_3 u_4 - u_2 u_5)_y + u_{3x} u_4 - u_{2x} u_5 + (u_2^2 - u_3^2) u_5 + 2(u_1 u_3 u_5 - u_1 u_2 u_4)].
\end{aligned} \tag{26}$$

Denoting

$$\tilde{V}_+^{(n)} = \sum_{m=0}^{n} (a_m e_1(1+n-m) + b_m e_2(n-m) + c_m e_3(n-m) + d_m e_4(n-m) + f_m e_5(n-m)$$
$$+ u_1 a_m e_1(n-m) + u_2 a_m e_2(n-m) + u_3 a_m e_3(n-m) + u_4 a_m e_4(n-m) + u_5 a_m e_5(n-m)),$$
$$\tilde{V}_-^{(n)} = \lambda^n \tilde{V} - \tilde{V}_+^{(n)},$$

then we have

$$\tilde{V}_{+y}^{(n)} + [\tilde{U}, \tilde{V}_+^{(n)}] - \tilde{V}_{+x}^{(n)} = (a_{n+1y} - a_{n+1x})e_1(0) - c_{n+1}e_2(0) - b_{n+1}e_3(0) - \frac{1}{2}f_{n+1}e_4(0) - \frac{1}{2}d_{n+1}e_5(0),$$

thus the zero-curvature equation

$$\tilde{U}_t + \tilde{V}_y^{(n)} + [\tilde{U}, \tilde{V}^{(n)}] - \tilde{V}_x^{(n)} - \tilde{U}_y = 0,$$

admits the (2+1)-dimensional integrable system

$$\begin{aligned}
&u_{1t} = a_{n+1y} - a_{n+1x} + u_{1y},\\
&u_{2t} = c_{n+1} + u_{2y},\\
&u_{3t} = b_{n+1} + u_{3y},\\
&u_{4t} = \frac{1}{2}f_{n+1} + u_{4y},\\
&u_{5t} = \frac{1}{2}d_{n+1} + u_{5y}.
\end{aligned} \tag{27}$$

Let

$$u_1 = 0, \quad n = 1,$$

in Eq.(27), and we have the following (2+1)-dimensional coupled equations:

$$u_{2t} = \alpha(u_{3xx} + u_{3yy} - 2u_{3xy}) + u_{2y},$$

$$u_{3t} = \alpha(u_{2xx} + u_{2yy} - 2u_{2xy}) + u_{3y},$$

$$u_{4t} = \frac{\alpha}{2}[2(u_3u_4 - u_2u_5)_x - 2(u_3u_4 - u_2u_5)_y - u_{2x}u_5 + u_{3x}u_4 + (u_2^2 - u_3^2)u_5] + u_{4y},$$

$$u_{5t} = \frac{\alpha}{2}[2(u_2u_4 - u_3u_5)_x - 2(u_2u_4 - u_3u_5)_y + u_{2x}u_4 - u_{3x}u_5 + (u_2^2 - u_3^2)u_4] + u_{5y}.$$

Obviously, when $u_1 = s$, $u_2 = q$, $u_3 = r$, $u_4 = u_5 = 0$, Eq.(27) reduces to Eq.(25), therefore, Eq.(27) is a type of expanding integrable model of Eq.(25). Furthermore, it is easy to find that Eqs.(25) and (27) are linear expansions of Eqs.(6) and (19) in respect of their dimensional numbers respectively.

Remark If we take the general Lie algebra $sl(n, R)$ to replace the Lie algebras presented in this paper, it is difficult to show such beautiful results because the commutators in the Lie algebras must be required during deducing the integrable hierarchies under the framework of the zero-curvature equation. However, as for the general Lie algebra, we cannot find the explicit commuting relations among the basis of the Lie algebra. But we could produce various integrable hierarchies of equations by using different Lie algebras. The advantages of the Lie algebras given in this paper are that it is explicit and efficient to deduce the integrable hierarchies, specially the integrable couplings of soliton equations. Concerning the problem of the applications for the bi-Hamiltonian systems and conserved flows, further study is required.

Acknowledgment

The authors are grateful to a referee's fine suggestions.

References

[1] Tu G Z 1989 *J. Math. Phys.* **30** 330
[2] Fan E G 2000 *J. Math. Phys.* **41** 7769
[3] Fan E G 2001 *Physica* A **301** 105
[4] Fan E G 2001 *J. Phys.* A: *Math. Gen.* **34** 513
[5] Hu X B 1997 *J. Phys.* A: *Math. Gen.* **30** 619
[6] Zhang Y F and Yan Q Y 2003 *Acta Phys. Sin.* **52** 2109 (in Chinese)
[7] Zhang Y F and Zhang H Q 2002 *J. Math. Phys.* **43** 466
[8] Zhang Y F, Zhang H Q and Yan Q Y 2002 *Phys. Lett.* A **299** 543
[9] Zhang Y F and Zhang H Q 2002 *J. Math. Res. Exposition* **22** 289 (in Chinese)
[10] Guo F K and Zhang Y F 2002 *Acta Phys. Sin.* **51** 13 (in Chinese)
[11] Zhang Y F 2004 *Ann. Diff. Eps.* **20** 423
[12] Zhang Y F 2004 *Chin. Phys.* **13** 132
[13] Zhou Z X 2002 *J. Phys. Jpn.* **71** 1857
[14] Zhang Y F and Guo X R 2006 *Chaos, Solitons and Fractals* **27** 555