Vol 16 No 3, March 2007
1009-1963/2007/16(03)/0582-06
Chinese Physics
© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

The discrete variational principle and the first integrals of Birkhoff systems*

Zhang Hong-Bin(张宏彬)[a)b)†], Chen Li-Qun(陈立群)[b)c)], Gu Shu-Long(顾书龙)[a)], and Liu Chuan-Zhang(柳传长)[a)]

[a)] *Department of Physics, Chaohu College, Chaohu* 238000, *China*
[b)] *Shanghai Institute of Mathematics and Mechanics, Shanghai University, Shanghai* 200072, *China*
[c)] *Department of Mechanics, Shanghai University, Shanghai* 200436, *China*

(Received 30 March 2006; revised manuscript received 29 May 2006)

This paper shows that first integrals of discrete equation of motion for Birkhoff systems can be determined explicitly by investigating the invariance properties of the discrete Pfaffian. The result obtained is a discrete analogue of theorem of Noether in the calculus of variations. An example is given to illustrate the application of the results.

Keywords: discrete mechanics, Birkhoff system, discrete Pfaffian, Noether's theorem, first integral
PACC: 0270, 0320, 1130

1. Introduction

In 1927, the American mathematician Birkhoff presented a new form of the dynamical equation in Ref.[1], which is more general than the Hamilton equation. This equation is called the Birkhoff equation by Santilli.[2,3] It can be applied to hadron physics, statistical mechanics, space mechanics, engineering, biophysics and so on. All conservative and nonconservative, self-adjoint and non-self-adjoint ,unconstrained and nonholonomic constrained systems always admit a representation of Birkhoff's equations.[2,3] It is an important developing orientation in modern analytical mechanics to research the Birkhoff dynamics. Mei and co-workers[4−12] gave the Birkhoff equations of a nonholonomic constrained system, investigated the symmetry theory, global analysis, stability of motion and geometrical description for a Birkhoff system, and constructed the theoretical frame of Birkhoff dynamics. However, there are no any other algorithms for Birkhoff systems. In the Refs.[13,14], we investigate the properties of preserving Noether symmetries for discrete mechanics in Lagrangian formalism. This work expands the ideas of structure-preserving. The purpose of this paper is to study the properties of preserving Noether symmetries for discrete Birkhoff systems.

This paper is organized as follows. In Section 2, we review the differential equations of motion of Birkhoff systems. In Section 3, firstly, we give the variational principle of Birkhoff systems in continuous case, next we present the discrete variational principle and discrete version Noether theorem in extended configuration space. Finally, an example is given to illustrate the application of the results.

Throughout the paper, Einstein's summation convention is used. The index: $s = 1, 2, \ldots, n$; Greek indices: μ, ν, $\rho = 1, 2, \ldots, 2n$ and α, $\beta = 1, 2, \ldots, 2n+1$.

2. The differential equations of motion of Birkhoff systems

The Birkhoff equations can be written in the form

$$\Big(\frac{\partial R_\nu}{\partial a_\mu} - \frac{\partial R_\mu}{\partial a_\nu}\Big)\dot{a}_\nu - \frac{\partial B}{\partial a_\mu} - \frac{\partial R_\mu}{\partial t} = 0, \qquad (1)$$

where $B = B(t, \boldsymbol{a})$ is Birkhoffian, $R_\mu = R_\mu(t, \boldsymbol{a})$ are Birkhoff functions, here $\boldsymbol{a} = (a_1, a_2, \ldots, a_{2n})$, and a_μ are independent variables. The system described by Eq.(1) is called a Birkhoff system.

We introduce the symbol

$$\Omega_{\mu\nu} = \frac{\partial R_\mu}{\partial a_\nu} - \frac{\partial R_\nu}{\partial a_\mu}, \qquad (2)$$

*Project partially supported by the National Natural Science Foundation of China (Grant No 10172056) and the Science Research of the Education Bureau of Anhui Province, China (Grant No 2006KJ263B).
†Corresponding author. E-mail: hbzhang2002@eyou.com.

which is usually called the Birkhoff tensor. If Eq.(1) is nonsingular, i.e.

$$\det(\Omega_{\mu\nu}) \neq 0, \tag{3}$$

then all $\dot{a}_\mu$ can be solved from Eq.(1) as follows:

$$\dot{a}_\mu = \Omega^{\mu\nu}\Big(\frac{\partial B}{\partial a_\nu} - \frac{\partial R_\nu}{\partial t}\Big), \tag{4}$$

where

$$\Omega^{\mu\nu}\Omega_{\nu\rho} = \delta_{\mu\rho}, \tag{5}$$

and $\Omega^{\mu\nu}$ is the Birkhoff contravariant tensor.

It has been proven in the Ref.[4] that all holonomic constrained systems and nonholonomic constrained systems can be attributed to Birkhoff systems. Therefore, Birkhoff systems is a sort of more general constrained mechanical systems.

If we let

$$\begin{aligned} &a_\mu = q_\mu \ (\mu = 1, 2, \ldots, n), \\ &a_\mu = p_\mu \ (\mu = n+1, n+2, \ldots, 2n), \qquad (6) \\ &R_\nu = p_\nu \ (\nu = 1, 2, \ldots, n), \\ &R_\nu = 0 \ (\nu = n+1, n+2, \ldots, 2n), \qquad (7) \end{aligned}$$

$$B = H, \tag{8}$$

then Birkhoff equation (1) becomes the following Hamilton canonical equation:

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \tag{9}$$

$$\dot{p}_s = -\frac{\partial H}{\partial q_s}. \tag{10}$$

3. The variational principle and first integrals of Birkhoff systems in extended configuration space

3.1. The variational principle of Birkhoff systems in configuration space

We begin by recalling the ordinary variational principle of Birkhoff system. Suppose M denotes the configuration space with coordinates a_μ, $\mu = 1, 2, \ldots, 2n$. Consider a Birkhoffian $B : \boldsymbol{R} \times M \to \boldsymbol{R}$ and Birkhoff functions $R_\nu : \boldsymbol{R} \times M \to \boldsymbol{R}$, $\nu = 1, 2, \ldots, 2n$. The Pfaff action is defined by

$$J = \int_{t_1}^{t_2} [R_\mu(t, \boldsymbol{a})\mathrm{d}a_\mu - B(t, \boldsymbol{a})\mathrm{d}t], \tag{11}$$

where $a_\mu = a_\mu(t)$ is a C^2 curve in configuration space.

The variational principle of Pfaff–Birkhoff seeks the curves $a_\mu(t)$ for which the action functional J is stationary under variations of $a_\mu(t)$ with fixed end points. We first introduce the following infinitesimal transformation:

$$t^* = t, \tag{12}$$

$$a_\mu^* = a_\mu + \epsilon\xi_\mu(t, \boldsymbol{a}), \tag{13}$$

where ϵ is an infinitesimal parameter, $\xi_\mu(t, \boldsymbol{a})$ are the coefficient functions. For simplicity, we will use Einstein convention and omit the summation notation Σ in the following.

Next, we calculate the variation of J at $a_\mu(t)$ as follows:

$$\begin{aligned} \delta J &= \int_{t_1}^{t_2} [R_\mu(t, \boldsymbol{a}^*)\mathrm{d}a_\mu^* - B(t, \boldsymbol{a}^*)\mathrm{d}t] - \int_{t_1}^{t_2} [R_\mu(t, \boldsymbol{a})\mathrm{d}a_\mu - B(t, \boldsymbol{a})\mathrm{d}t] \\ &= \int_{t_1}^{t_2} \Big[R_\mu(t, \boldsymbol{a})\frac{\mathrm{d}}{\mathrm{d}t}\delta a_\mu + \frac{\partial R_\mu(t, \boldsymbol{a})}{\partial a_\nu}\dot{a}_\mu - \frac{\partial B(t, \boldsymbol{a})}{\partial a_\nu}\delta a_\nu\Big]\mathrm{d}t \\ &= \int_{t_1}^{t_2} \Big\{\frac{\mathrm{d}}{\mathrm{d}t}[R_\mu(t, \boldsymbol{a})\delta a_\mu] - \frac{\mathrm{d}}{\mathrm{d}t}[R_\mu(t, \boldsymbol{a})]\delta a_\mu + \frac{\partial R_\mu(t, \boldsymbol{a})}{\partial a_\nu}\delta a_\nu\dot{a}_\mu - \frac{\partial B(t, \boldsymbol{a})}{\partial a_\nu}\delta a_\nu\Big\}\mathrm{d}t \\ &= \int_{t_1}^{t_2} \Big\{\Big[\frac{\partial R_\nu(t, \boldsymbol{a})}{\partial a_\mu} - \frac{\partial R_\mu(t, \boldsymbol{a})}{\partial a_\nu}\Big]\dot{a}_\nu - \Big[\frac{\partial B(t, \boldsymbol{a})}{\partial a_\mu} + \frac{\partial R_\mu(t, \boldsymbol{a})}{\partial t}\Big]\Big\}\xi_\mu\mathrm{d}t + R_\mu(t, \boldsymbol{a})\xi_\mu|_{t_1}^{t_2}. \end{aligned} \tag{14}$$

If $\xi_\mu[\boldsymbol{a}(t_1)] = \xi_\mu[\boldsymbol{a}(t_2)] = 0$, then the requirement of $\delta J = 0$ yields the Birkhoff's equations for $[a_\mu(t)]$:

$$\Big[\frac{\partial R_\nu(t, \boldsymbol{a})}{\partial a_\mu} - \frac{\partial R_\mu(t, \boldsymbol{a})}{\partial a_\nu}\Big]\dot{a}_\nu - \Big[\frac{\partial B(t, \boldsymbol{a})}{\partial a_\mu} + \frac{\partial R_\mu(t, \boldsymbol{a})}{\partial t}\Big] = 0. \tag{15}$$

3.2. The variational principle of Birkhoff system in extended configuration space

Now we study a Birkhoff system in extended configuration space $\boldsymbol{R}^{2n+1}$ in which the coordinates of a point are a_μ ($\mu = 1, 2, \ldots, 2n$) and t. We introduce the notation

$$x_1 = t, \qquad x_{\mu+1} = a_\mu, \tag{16}$$

then all the variables x_α ($\alpha = 1, 2, \ldots, 2n+1$) may be given as a function of some parameter τ. A choice of it has no special meaning, e.g. see Synge (1960).[12] Let

$$x_\alpha = x_\alpha(\tau), \tag{17}$$

be some curves of class C^2 such that

$$x_\alpha{}' = \frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau}, \tag{18}$$

the $x_\alpha{}'$ are not all zero simultaneously. We set

$$\dot{x}_\alpha = \frac{\mathrm{d}x_\alpha}{\mathrm{d}t} = \frac{x_\alpha{}'}{x_1{}'}, \tag{19}$$

where $\dot{x}_\alpha$ denotes the derivative with respect to t and $x_\alpha{}'$ denotes the derivative with respect to τ.

It is straightforward to show that for any function $F = F(t, \boldsymbol{a}, \dot{\boldsymbol{a}})$[16,17]

$$\tilde{F}(x_\alpha, x_\alpha{}') = x_1{}' F\Big(x_\alpha, \frac{x_2{}'}{x_1{}'}, \ldots, \frac{x_{2n+1}{}'}{x_1{}'}\Big). \tag{20}$$

Obviously, the function $\tilde{F}$ is zero order homogeneous for $x_\alpha{}'$. For the Pfaffian $P(t, \boldsymbol{a}, \dot{\boldsymbol{a}})$ of a given system in configuration space, the Pfaffian $\Lambda(x_\alpha, x_\alpha{}')$ in extended configuration space is defined by the following equation:

$$\begin{aligned}\Lambda(x_\alpha, x_\alpha{}') &= x_1{}' P\Big(x_\alpha, \frac{x_2{}'}{x_1{}'}, \ldots, \frac{x_{2n+1}{}'}{x_1{}'}\Big) \\ &= \bar{R}_{\mu+1}(x_\alpha) x_{\mu+1}{}' - B(x_\alpha) x_1{}',\end{aligned} \tag{21}$$

where $\bar{R}_{\mu+1} = R_\mu$, and we have

$$\frac{\partial \Lambda(x_\alpha, x_\alpha{}')}{\partial x_\alpha{}'} x_\alpha{}' = \Lambda(x_\alpha, x_\alpha{}'). \tag{22}$$

Pfaff- Birkhoff's principle is

$$\begin{aligned}&\delta \tilde{J} = \int_{\tau_1}^{\tau_2} \delta \Lambda(x_\alpha, x_\alpha{}') \mathrm{d}\tau = 0, \\ &\delta x_\alpha|_{\tau=\tau_1} = \delta x_\alpha|_{\tau=\tau_2} = 0.\end{aligned} \tag{23}$$

Now we calculate the variation directly as follows:

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \delta \Lambda(x_\alpha, x_\alpha{}')\mathrm{d}\tau &= \int_{\tau_1}^{\tau_2} \Big\{ \bar{R}_{\mu+1}(x_\alpha + \delta x_\alpha)\Big[x_{\mu+1}{}' + \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x_{\mu+1})\Big] \\ &\quad - B(x_\alpha + \delta x_\alpha)\Big[x_1{}' + \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x_1)\Big]\Big\}\mathrm{d}\tau - \int_{\tau_1}^{\tau_2} [\bar{R}_{\mu+1}(x_\alpha) x_{\mu+1}{}' - B(x_\alpha) x_1{}']\mathrm{d}\tau \\ &= \int_{\tau_1}^{\tau_2} \Big\{ \bar{R}_{\mu+1}(x_\alpha) \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x_{\mu+1}) + \frac{\partial \bar{R}_{\mu+1}(x_\alpha)}{\partial x_\beta} \delta x_\beta x_{\mu+1}{}' - B(x_\alpha) \frac{\mathrm{d}}{\mathrm{d}\tau}(\delta x_1) - \frac{\partial B(x_\alpha)}{\partial x_\beta} \delta x_\beta x_1{}' \Big\}\mathrm{d}\tau \\ &= \int_{\tau_1}^{\tau_2} \Big\{ \Big[\Big(\frac{\partial \bar{R}_{\mu+1}}{\partial x_{\nu+1}} - \frac{\partial \bar{R}_{\nu+1}}{\partial x_{\mu+1}}\Big) x_{\nu+1}{}' - \Big(\frac{\partial B}{\partial x_{\nu+1}} + \frac{\partial \bar{R}_{\nu+1}}{\partial x_1}\Big) x_1{}'\Big] \delta x_{\mu+1} \\ &\quad + \Big(\frac{\partial B}{\partial x_{\mu+1}} + \frac{\partial \bar{R}_{\mu+1}}{\partial x_1}\Big) x_{\mu+1}{}' \delta x_1 \Big\}\mathrm{d}\tau + [\bar{R}_{\mu+1}(x_\alpha)\delta x_{\mu+1} - B(x_\alpha)\delta x_1]|_{\tau_1}^{\tau_2}.\end{aligned} \tag{24}$$

If $\delta x_\alpha(\tau_1) = \delta x_\alpha(\tau_2) = 0$, the requirement of $\delta \tilde{J} = 0$ for any choice of δx_α yields the parameter equations of motion for the Birkhoff system

$$\Big(\frac{\partial \bar{R}_{\mu+1}}{\partial x_{\nu+1}} - \frac{\partial \bar{R}_{\nu+1}}{\partial x_{\mu+1}}\Big) x_{\nu+1}{}' - \Big(\frac{\partial B}{\partial x_{\nu+1}} + \frac{\partial \bar{R}_{\nu+1}}{\partial x_1}\Big) x_1{}' = 0, \tag{25}$$

and

$$\Big(\frac{\partial B}{\partial x_{\mu+1}} + \frac{\partial \bar{R}_{\mu+1}}{\partial x_1}\Big) x_{\mu+1}{}' = 0. \tag{26}$$

3.3. The discrete variational principle and first integrals of Birkhoff system in extended configuration space

We now define the discrete action sum in extended configuration space to be

$$\tilde{J}(x_\alpha)=\sum_{k=1}^{N}\Lambda_{\rm d}[x_\alpha(k-1),x_\alpha(k)], \tag{27}$$

where $k\in\{0,1,2,\ldots,N\}$ and k is the discrete parameter, $\Lambda_{\rm d}[x_\alpha(k-1),x_\alpha(k)]$ is the discrete Pfaffian in extended configuration space, which can be obtained by substituting $x_\alpha(k)-x_\alpha(k-1)$ for $x_\alpha{}'$ and $x_\alpha(k-1)$ for x_α respectively in given Pfaffian. Then the discrete Pfaff–Birkhoff principle is given by

$$\begin{aligned}&\delta\tilde{J}(x_\alpha)=\sum_{k=1}^{N}\delta\Lambda_{\rm d}[x_\alpha(k-1),x_\alpha(k)]=0,\\&\delta x_\alpha(0)=\delta x_\alpha(N)=0.\end{aligned} \tag{28}$$

Equation (28) is the equivalent of Eq.(23).

Proposition In extended configuration space, for the Birkhoff system with the discrete Pfaffian $\Lambda_{\rm d}[x_\alpha(k-1),x_\alpha(k)]$, the discrete Birkhoff equations can be derived from principle (28) as follows:

$$\begin{aligned}&D_2\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]\\&\quad+D_1\Lambda_{{\rm d},\alpha}[x_\beta(k),x_\beta(k+1)]=0,\end{aligned} \tag{29}$$

where $D_i\Lambda_{{\rm d},\alpha}$ denotes the partial derivative of $\Lambda_{\rm d}$ with respect to the ith argument whose index is α.

Proof From Eq.(28), one has

$$\begin{aligned}\delta\tilde{J}(x_\alpha)&=\frac{\partial\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]}{\partial x_\alpha(k-1)}\delta x_\alpha(k-1)\\&\quad+\frac{\partial\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]}{\partial x_\alpha(k)}\delta x_\alpha(k)\\&=\Big\{\frac{\partial\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]}{\partial x_\alpha(k)}\\&\quad+\frac{\partial\Lambda_{{\rm d},\alpha}[x_\beta(k),x_\beta(k+1)]}{\partial x_\alpha(k)}\Big\}\\&\quad\times\delta x_\alpha(k)=0,\end{aligned} \tag{30}$$

where we have used a discrete integration by parts (rearranging the summation) and the fact that $\delta x_\alpha(0)=\delta x_\alpha(N)=0$. If we now require that Eq.(30) be zero for any choice of $x_\alpha(k)$, then we obtain the discrete Birkhoff equation (29) in extended configuration space.

As the continuous, a systematic procedure for the establishment of first integrals of the discrete Birkhoff equation can be developed from a direct study of the invariance properties the discrete Pfaffian $\Lambda_{\rm d}[x_\alpha(k-1),x_\alpha(k)]$.

Let us introduce the following infinitesimal transformation:

$$x_\alpha^*(k)=x_\alpha(k)+\epsilon u_\alpha(k), \tag{31}$$

where ϵ is a parameter and $u_\alpha(k)=u_\alpha[x_\beta(k),k]$ is a sequence depending upon k and $x_\beta(k)$, $k=1,2,\ldots,N$.

Definition The discrete Pfaffian $\Lambda_{\rm d}[x_\alpha(k-1),x_\alpha(k)]$ is difference invariant with respect to the infinitesimal transformation (31), if there exists a sequence $g[x_\alpha(k),k]$, $k=1,2,\ldots,N$, such that

$$\delta\Lambda_{\rm d}\{\epsilon[u_\beta(k),k]\}=\epsilon\{\Delta g[x_\alpha(k),k]\}, \tag{32}$$

for each k, where $\delta\Lambda_{\rm d}$ is given by following formula:[18]

$$\begin{aligned}&\delta\Lambda_{\rm d}[\delta x_\alpha(k)]=\{D_2\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]\\&\quad+D_1\Lambda_{{\rm d},\alpha}[x_\beta(k),x_\beta(k+1)]\}\delta x_\alpha(k)\\&\quad+\Delta\{-\delta x_\alpha(k-1)D_1\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]\}.\end{aligned} \tag{33}$$

where Δ is the difference operator, $\Delta x_\alpha(k)=x_\alpha(k+1)-x_\alpha(k)$.

Based on the discrete Pfaff–Birkhoff principle, the following theorem concerning first integrals can be proved.

Theorem If discrete Pfaffian $\Lambda_{\rm d}[x_\alpha(k-1),x_\alpha(k)]$ is difference invariant with respect to the infinitesimal transformation (31), and if Eq.(29) holds, then

$$\begin{aligned}&u_\alpha[x_\beta(k-1),k-1]D_1\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]\\&\quad+g[x_\beta(k),k]=\text{const.}\end{aligned} \tag{34}$$

Proof From Eqs.(32) and (33), one has

$$\begin{aligned}&\{D_2\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]\\&\quad+D_1\Lambda_{{\rm d},\alpha}[x_\beta(k),x_\beta(k+1)]\}\\&\quad\times\epsilon u_\alpha[x_\beta(k),k]-\Delta\{u_\alpha[x_\beta(k-1),k-1]\\&\quad\times D_1\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]\}=\epsilon\Delta g[x_\beta(k),k].\end{aligned} \tag{35}$$

Simplifying and using Eq.(29), one obtains

$$\begin{aligned}&\Delta\{u_\alpha[x_\beta(k-1),k-1]D_1\Lambda_{{\rm d},\alpha}[x_\beta(k-1),x_\beta(k)]\\&\quad+g[x_\beta(k),k]\}=0,\end{aligned} \tag{36}$$

whence Eq.(34) holds. This completes the proof.

4. An example

To illustrate various aspects of the theory developed in preceding sections we now give an example.

Example Considering a two-order Birkhoff system with Birkhoffian

$$B = \frac{1}{2}[(ta_1)^2 + (a_2)^2], \tag{37}$$

and Birkhoff functions

$$R_1 = 0\,, \quad R_2 = ta_1. \tag{38}$$

Let

$$x_1 = t, \quad x_2 = a_1, \quad x_3 = a_2. \tag{39}$$

Hence, in extended configuration space Pfaffian can be written as

$$\Lambda = x_1 x_2 {x_3}' - \frac{1}{2}[(x_1 x_2)^2 + (x_3)^2]{x_1}'. \tag{40}$$

Then the discrete Pfaffian is

$$\begin{aligned}\Lambda_{\rm d} = {} & x_1(k-1)x_2(k-1)[x_3(k) - x_3(k-1)] \\ & - \frac{1}{2}\{[x_1(k-1)x_2(k-1)]^2 \\ & + [x_3(k-1)]^2\}[x_1(k) - x_1(k-1)].\end{aligned} \tag{41}$$

From Eq.(29), the discrete Birkhoff equations can be obtained as follows:

$$\begin{aligned}& \frac{1}{2}\{[x_1(k)x_2(k)]^2 + [x_3(k)]^2\} \\ & \quad - \frac{1}{2}\{[x_1(k-1)x_2(k-1)]^2 \\ & \quad + [x_3(k-1)]^2\} + x_2(k)[x_3(k+1) - x_3(k)] \\ & \quad - x_1(k)x_2^2(k)[x_1(k+1) - x_1(k)] = 0,\end{aligned} \tag{42}$$

$$\begin{aligned}& x_1(k)[x_3(k+1) - x_3(k)] - x_1^2(k)x_2(k) \\ & \quad \times [x_1(k+1) - x_1(k)] = 0,\end{aligned} \tag{43}$$

$$\begin{aligned}& x_1(k-1)x_2(k-1) - x_1(k)x_2(k) - x_3(k) \\ & \quad \times [x_1(k+1) - x_1(k)] = 0.\end{aligned} \tag{44}$$

Direct verification shows that Eq.(41) is difference-invariant (with $g = 0$) with respect to the infinitesimal transformation

$$x_1^*(k) = x_1(k) - \epsilon, \tag{45}$$

$$x_2^*(k) = x_2(k) + \epsilon\frac{x_2(k)}{x_1(k)}, \tag{46}$$

$$x_3^*(k) = x_3(k). \tag{47}$$

Therefore, by theorem, we immediately obtain the first integral of Eqs.(42)–(44)

$$\begin{aligned}& u_1[x_\beta(k-1), k-1]D_1\Lambda_{{\rm d},1}[x_\beta(k-1), x_\beta(k)] \\ & \quad + u_2[x_\beta(k-1), k-1]D_1\Lambda_{{\rm d},2}[x_\beta(k-1), x_\beta(k)] \\ & \quad = \text{const},\end{aligned} \tag{48}$$

or

$$\frac{1}{2}\{[x_1(k-1)x_2(k-1)]^2 + [x_3(k-1)]^2\} = \text{const}. \tag{49}$$

If we choose $\Lambda_{\rm d}$ in Eq.(41) is

$$\begin{aligned}\Lambda_{\rm d} = {} & x_{1,k-1/2}x_{2,k-1/2}[x_3(k) - x_3(k-1)] \\ & - \frac{1}{2}[(x_{1,k-1/2}x_{2,k-1/2})^2 \\ & + (x_{3,k-1/2})^2][x_1(k) - x_1(k-1)],\end{aligned} \tag{50}$$

where

$$\begin{aligned}& x_{i,k-\frac{1}{2}} = \frac{x_i(k-1) + x_i(k)}{2}, \\ & x_{i,k+1/2} = \frac{x_i(k) + x_i(k+1)}{2}, \quad i = 1, 2, 3.\end{aligned} \tag{51}$$

In terms of Eq.(29), we can derive the corresponding discrete Birkhoff equation

$$\begin{aligned}& \frac{1}{2}x_{2,k-1/2}[x_3(k) - x_3(k-1)] - \frac{1}{2}[(x_{1,k-1/2}x_{2,k-1/2})^2 + (x_{3,k-1/2})^2] \\ & \quad - \frac{1}{2}x_{1,k-1/2}x_{2,k-1/2}^2[x_1(k) - x_1(k-1)] + \frac{1}{2}x_{2,k+1/2}[x_3(k+1) - x_3(k)] \\ & \quad + \frac{1}{2}[(x_{1,k+1/2}x_{2,k+1/2})^2 + (x_{3,k+1/2})^2] - \frac{1}{2}x_{1,k+1/2}x_{2,k+1/2}^2[x_1(k+1) - x_1(k)] = 0,\end{aligned} \tag{52}$$

$$\begin{aligned}& \frac{1}{2}x_{1,k-1/2}[x_3(k) - x_3(k-1)] - \frac{1}{2}x_{1,k-1/2}^2x_{2,k-1/2}[x_1(k) - x_1(k-1)] \\ & \quad + \frac{1}{2}x_{1,k+1/2}[x_3(k+1) - x_3(k)] - \frac{1}{2}x_{1,k+1/2}^2x_{2,k+1/2}[x_1(k+1) - x_1(k)] = 0,\end{aligned} \tag{53}$$

$$x_{1,k-1/2}x_{2,k-1/2} - \frac{1}{2}x_{3,k-1/2}[x_1(k) - x_1(k-1)] - x_{1,k+1/2}x_{2,k+1/2} - \frac{1}{2}x_{3,k+1/2}[x_1(k+1) - x_1(k)] = 0. \tag{54}$$

It is easy to verify that Eq.(50) is difference invariant (with $g = 0$) with respect to the following infinitesimal transformation:

$$x_1^*(k) = x_1(k) - \epsilon, \tag{55}$$

$$x_2^*(k) = x_2(k) + \epsilon \frac{x_{2,k-1/2}}{x_{1,k-1/2}}, \tag{56}$$

$$x_3^*(k) = x_3(k). \tag{57}$$

By theorem, one has

$$\frac{1}{2}[(x_{1,k-1/2}x_{2,k-1/2})^2 + (x_{3,k-1/2})^2] = \text{const.} \tag{58}$$

Obviously, there is no relation between the discrete first integral and the discrete form of Pfaffian.

Acknowledgement

We wish to thank the referees for their careful reading of the manuscript and their useful remarks which helped us to improve the quality of this paper.

References

[1] Birkhoff G D 1927 *Dynamical Systems* (Providence RI: AMS College Publications)
[2] Santilli R M 1978 *Foundations of Theoretical Mechanics* I (New York: Springer)
[3] Santilli R M 1983 *Foundations of Theoretical Mechanics* II (New York: Springer)
[4] Mei F X, Shi R C, Zhang Y F and Wu H B 1996 *Dynamics of Birkhoff System* (Beijing: Beijing Institute of Technology Press) (in Chinese)
[5] Chen X W and Mei F X 2000 *Mech. Res. Commun.* **27** 365
[6] Wu H B and Mei F X 1995 *Chin. Sin. Bull.* **40** 885
[7] Mei F X and Lèvesque B 1995 *Trans. CSME.* **19**(2) 59
[8] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[9] Guo Y X, Luo S K, Shang M and Mei F X 2001 *Rep. Math. Phys.* **47** 313
[10] Fu J L and Wang X M 2000 *Acta Phys. Sin.* **49** 1023 (in Chinese)
[11] Fu J L and Chen L Q 2003 *Acta Phys. Sin.* **52** 2664 (in Chinese)
[12] Fu J L and Chen L Q 2004 *Phys. Lett.* A **324** 95
[13] Zhang H B, Chen L Q and Liu R W 2005 *Chin. Phys.* **14** 238
[14] Zhang H B, Chen L Q and Liu R W 2005 *Chin. Phys.* **14** 888
[15] Synge J L 1960 *Classical Dynamics* (Berlin: Springer-Verlag)
[16] Mei F X 1990 *Acta Mech. Sin.* **6** 160
[17] Mei F X 2000 *Appl. Mech. Rev.* **53** 283
[18] Logan J D 1973 *Aequat. Math.* **9** 210