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Asymptotic solution of a sea–air oscillator for ENSO mechanism*

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The EI Niño/La Niña–Southern Oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific ocean–atmosphere interactions. In this paper, a class of coupled system of the ENSO mechanism is considered. Based on a class of oscillator of ENSO model, the asymptotic solution of a corresponding problem is studied by employing the approximate method. It is proved from the results that the perturbation method can be used for analysing the sea surface temperature anomaly in the equatorial eastern Pacific and the thermocline depth anomaly of the atmosphere–ocean oscillation for the ENSO model.

Keywords: nonlinear system, asymptotic method, El Niño–Southern Oscillator model
PACC: 0230, 0200, 0545

1. Introduction

The El Niño–Southern Oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific ocean–atmosphere interactions. It is a very attractive object of study in the international academic circles.[1−6] Mo and Lin and co-workers also considered some problems in atmospheric physics, ocean weather, dynamic system and so on.[7−19] Superimposing the interdecadal fluctuation on this natural mode of the coupled ocean–atmosphere system we find that it contributes to the irregularity of the southern oscillation. In the eastern equatorial tropical Pacific a shoaling of subsurface isotherms signals the end of El Niño and the return of colder surface waters. The oscillatory nature of ENSO requires both positive and negative ocean–atmosphere feedbacks. This anomaly reduces the zonal sea surface temperature (SST) gradient and the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn drive the ocean circulation and changes and then reinforce the SST anomaly. This positive ocean–atmosphere feedback or coupled ocean–atmosphere instability leads the equatorial Pacific to an occurring of never-ending warm state. Recently, many scholars have investigated the relevant nonlinear perturbation problem.[20−28] Approximate methods have been developed and improved, including the method of averaging, boundary layer method, matched asymptotic expansion method and multiple scales method. In this paper we consider a sea–air oscillator model by using a simple and special method for perturbation theory. The conceptual oscillator model should consider the variations of both the eastern and western Pacific anomaly patterns. As will be shown below, the oscillator model can be reduced to a simple oscillator model by further simplification and assumptions.

2. A model of sea–air oscillator

The western Pacific oscillator[4,5] emphasizes the role of the western Pacific anomaly patterns in ENSO. In particular, off-equatorial SST and sea level pressure (SLP) variations initiate equatorial wind anomalies in the western Pacific. These wind anomalies force ocean to proceed eastward to affect anomalies in the equa-

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torial eastern Pacific. If all delay parameters are set to be zero,[6] we can understand why the slow ocean dynamical adjustment does not necessarily need the explicit role of wave propagation. Now we consider the sea–air oscillator of an ENSO model, which may be described by the following equations:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = a\tau_1 + b\tau_2 - \varepsilon T^3, \tag{1}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -R_\mathrm{h}h - c\tau_1, \tag{2}$$

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}t} = dT - R_1\tau_1, \tag{3}$$

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}t} = eh - R_2\tau_2, \tag{4}$$

where T is the SST anomaly in the equatorial eastern Pacific, h is the thermocline depth anomaly, τ_i, $i = 1, 2$, are the equatorial zonal wind stress anomalies in the central Pacific and the western Pacific respectively, the model parameters a, b, c, d and e are positive constants and the ε, R_h, R_1 and R_2 are positive damping coefficients for the T, h, τ_1 and τ_2 respectively.

3. Perturbation method

We also assume that parameter ε is small. Then we can obtain the asymptotic solution for the coupled system Eqs.(1)–(4) of the ENSO model.

It is easy to obtain that the solutions ($T(t,\varepsilon)$, $h(t,\varepsilon)$, $\tau_1(t,\varepsilon)$, $\tau_2(t,\varepsilon)$) of the nonlinear coupled system Eqs.(1)–(4) are

$$T(t,\varepsilon) = \sum_{i=0}^{\infty} T_i(t)\varepsilon^i, \quad h(t,\varepsilon) = \sum_{i=0}^{\infty} h_i(t)\varepsilon^i,$$
$$\tau_i(t,\varepsilon) = \sum_{i=0}^{\infty} \tau_{ii}(t)\varepsilon^i, \quad i = 1,2\,. \tag{5}$$

Substituting Eq.(5) into Eqs.(1)–(4), let $\varepsilon = 0$, we have the reduced system of original coupled system Eqs.(1)–(4)

$$\frac{\mathrm{d}T_0}{\mathrm{d}t} = a\tau_{10} + b\tau_{20}, \tag{6}$$

$$\frac{\mathrm{d}h_0}{\mathrm{d}t} = -R_\mathrm{h}h_0 - c\tau_{10}, \tag{7}$$

$$\frac{\mathrm{d}\tau_{10}}{\mathrm{d}t} = dT_0 - R_1\tau_{10}, \tag{8}$$

$$\frac{\mathrm{d}\tau_{20}}{\mathrm{d}t} = eh_0 - R_2\tau_{20}. \tag{9}$$

From the linear coupled system Eqs.(6)–(9) we can obtain a series of solutions

$$T_0(t) = \sum_{i=1}^{4} C_{i0}\exp(\lambda_i t), \tag{10}$$

$$h_0(t) = \sum_{i=1}^{4} \frac{(\lambda_i + R_2)(\lambda_i(\lambda_i + R_1) - ad)C_{i0}}{be(\lambda_i + R_1)} \times \exp(\lambda_i t), \tag{11}$$

$$\tau_{10}(t) = \sum_{i=1}^{4} \frac{dC_{i0}}{(\lambda_i + R_1)}\exp(\lambda_i t), \tag{12}$$

$$\tau_{20}(t) = \sum_{i=1}^{4} \frac{(\lambda_i(\lambda_i + R_1) - ad)C_{i0}}{b(\lambda_i + R_1)}\exp(\lambda_i t), \tag{13}$$

where $C_{i0}\,(i = 1, 2, 3, 4)$, are arbitrary constants and λ_i, $(i = 1, 2, 3, 4)$, are characteristic roots of the characteristic equation

$$\begin{vmatrix} -\lambda & 0 & a & b \\ 0 & -(R_\mathrm{h} + \lambda) & -c & 0 \\ d & 0 & -(R_1 + \lambda) & 0 \\ 0 & e & 0 & -(R_2 + \lambda) \end{vmatrix} = 0,$$

i.e.

$$\lambda^4 + (R_\mathrm{h} + R_1 + R_2)\lambda^3 + (R_\mathrm{h}R_1 + R_\mathrm{h}R_2 + R_1R_2 - ad)\lambda^2 + (R_\mathrm{h}R_1R_2 - ad(R_\mathrm{h} + R_2))\lambda + d(bce - aR_\mathrm{h}R_2) = 0.$$

Substituting Eqs.(5) and (10)–(13) into Eqs.(1)–(4), for the coefficient of ε, we have

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = a\tau_{11} + b\tau_{21} - \sum_{i,j,k=1}^{4} C_{i0}C_{j0}C_{k0} \times \exp(\lambda_i + \lambda_j + \lambda_k)t, \tag{14}$$

$$\frac{\mathrm{d}h_1}{\mathrm{d}t} = -R_\mathrm{h}h_1 - c\tau_{11}, \tag{15}$$

$$\frac{\mathrm{d}\tau_{11}}{\mathrm{d}t} = dT_1 - R_1\tau_{11}, \tag{16}$$

$$\frac{\mathrm{d}\tau_{21}}{\mathrm{d}t} = eh_1 - R_2\tau_{21}. \tag{17}$$

A series of particular solutions ($\bar{T}_1(t)$, $\bar{h}_1(t)$, $\bar{\tau}_{11}(t)$, $\bar{\tau}_{21}(t)$) of the Eqs.(14)–(17) are

$$\bar{T}_1(t) = \sum_{i,j,k=1}^{4} \frac{(\lambda_i + \lambda_j + \lambda_k) + R_1}{d}K \times \exp(\lambda_i + \lambda_j + \lambda_k)t, \tag{18}$$

$$\bar{h}_1(t) = \sum_{i,j,k=1}^{4} \frac{-c}{\lambda_i + \lambda_j + \lambda_k + R_\mathrm{h}}K \times \exp(\lambda_i + \lambda_j + \lambda_k)t, \tag{19}$$

$$\bar{\tau}_{11}(t) = \sum_{i,j,k=1}^{4} K\exp(\lambda_i + \lambda_j + \lambda_k)t, \tag{20}$$

$$\bar{\tau}_{21}(t)=\sum_{i,j,k=1}^{4}\frac{-ce}{(\lambda_i+\lambda_j+\lambda_k+R_{\rm h})(\lambda_i+\lambda_j+\lambda_k+R_2)}\times K\exp(\lambda_i+\lambda_j+\lambda_k)t, \tag{21}$$

where

$$K=\frac{C_{i0}C_{j0}C_{k0}}{a-bce(\lambda_i+\lambda_j+\lambda_k)(\lambda_i+\lambda_j+\lambda_k+R_1)}.$$

Thus the general solutions ($T_1(t)$, $h_1(t)$, $\tau_{11}(t)$, $\tau_{21}(t)$) of linear coupled system Eqs.(14)–(17) is

$$T_1(t)=\sum_{i=1}^{4}C_{i1}\exp(\lambda_i t)+\varepsilon\bar{T}_1(t), \tag{22}$$

$$h_1(t)=\sum_{i=1}^{4}\frac{(\lambda_i+R_2)(\lambda_i(\lambda_i+R_1)-ad)C_{i1}}{be(\lambda_i+R_1)}\times\exp(\lambda_i t)+\varepsilon\,\bar{h}_1(t), \tag{23}$$

$$\tau_{11}(t)=\sum_{i=1}^{4}\frac{dC_{i1}}{(\lambda_i+R_1)}\exp(\lambda_i t)+\varepsilon\bar{\tau}_{11}(t), \tag{24}$$

$$\tau_{21}(t)=\sum_{i=1}^{4}\frac{(\lambda_i(\lambda_i+R_1)-ad)C_{i1}}{b(\lambda_i+R_1)}\times\exp(\lambda_i t)+\varepsilon\bar{\tau}_{21}(t), \tag{25}$$

where C_{i1}, $(i=1,2,3,4)$ are arbitrary constants.

Therefore, from Eqs.(5), (10)–(13) and (22)–(25), we obtain first asymptotic solution for the ENSO model Eqs.(1)–(4):

$$T(t)=\sum_{i=1}^{4}C_i\exp(\lambda_i t)+\bar{T}_1(t)+O(\varepsilon^2), \tag{26}$$

$$h(t)=\sum_{i=1}^{4}\frac{(\lambda_i+R_2)(\lambda_i(\lambda_i+R_1)(1+\varepsilon)-ad)C_i}{be(\lambda_i+R_1)}\times\exp(\lambda_i t)+\bar{h}_1(t)+O(\varepsilon^2), \tag{27}$$

$$\tau_1(t)=\sum_{i=1}^{4}\frac{dC_i}{(\lambda_i+R_1)}\exp(\lambda_i t)+\bar{\tau}_{11}(t)+O(\varepsilon^2), \tag{28}$$

$$\tau_2(t)=\sum_{i=1}^{4}\frac{(\lambda_i(\lambda_i+R_1)(1+\varepsilon)-ad)C_i}{b(\lambda_i+R_1)}\times\exp(\lambda_i t)+\bar{\tau}_{21}(t)+O(\varepsilon^2), \tag{29}$$

where $C_i=C_{i0}+C_{i1}\varepsilon$ $(i=1,2,3,4)$ are arbitrary constants, $\bar{T}_1$, $\bar{h}_1$, $\bar{\tau}_1$ and $\bar{\tau}_2$ are expressed by Eqs.(18)–(21).

4. Precision of perturbed solution

In order to evaluate the precision of the first asymptotic solutions Eqs.(25)–(29), we give a set of parameters as follows:[5] $a=1.5\times10^{-2}\,℃\cdot m^2\cdot N^{-1}\cdot yr^{-1}$, $b=7.5\times10^{2}\,℃\cdot m^2\cdot N^{-1}\cdot yr^{-1}$, $c=1.5\times\,m^3\cdot N^{-1}\cdot yr^{-1}$, $d=3.6\times10^{-2}\,℃^{-1}\cdot N\cdot m^{-2}\cdot yr^{-1}$, $e=3.0\times10^{-3}N\cdot m^{-3}\cdot yr^{-1}$, $R_{\rm h}=5.0\,yr^{-1}$, $R_1=R_2=2.0\,yr^{-1}$ and let $\varepsilon=0.1℃^{-2}\cdot yr^{-1}$ and $\varepsilon=0.01\,℃^{-2}\cdot yr^{-1}$, $t\in[0,4]$, $[T(0),h(0),\tau_1(0),\,\tau_2(0)]=[0,0,0,0]$. Then we have the values of the SST anomaly T as shown in Table 1 to make a comparison between the simulation solution and perturbation asymptotic solution.

Table 1. A comparison of the values of between $T_{\rm exa}$ and $T_{\rm asy}$.

ε	T	$t=0.00$	$t=0.25$	$t=0.50$	$t=0.75$	$t=1.00$	$t=1.25$	$t=1.50$	$t=1.75$	$t=2.00$
0.1	$T_{\rm exa}$	1.000	1.000	0.999	1.026	1.171	1.448	1.283	0.281	−1.368
	$T_{\rm asy}$	1.000	1.000	0.997	1.026	1.142	1.340	1.094	0.035	−1.422
0.01	$T_{\rm exa}$	1.000	1.000	1.001	1.038	1.210	1.587	1.587	0.658	−1.073
	$T_{\rm asy}$	1.000	1.000	1.001	1.037	1.207	1.577	1.579	0.654	−1.077

From the above table, we can see that the first asymptotic solution is a good precision by means of the perturbation method. And as ε is smaller then the precision of perturbation asymptotic solution is better.

5. Conclusions

Using the perturbation method, we can also obtain higher asymptotic solution for the coupled system (1)–(4). Using the theory of differential inequality,[29] we can prove that for small parameter ε, Eqs.(26)–(29) are uniformly valid asymptotic expansions of solution for the nonlinear coupled system Eqs.(1)–(4). Thus using Eqs.(26)–(29) we can make the climate prediction for the sea–air oscillator at Pacific area and other analyses.

The perturbation method is an approximate an-

alytic method, which is different from the general numerical method. The expansions of a solution by means of the perturbation method can be made in the analytic operation. Thus, from Eqs.(26)–(29), we can further study the quality and quantitative behaviours of the temperature anomaly and the zonal wind stress anomaly in the equatorial eastern Pacific. But in this paper we do not discuss this any further.

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