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# Multifractal analysis of the Yellow River flows*

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This paper deals with time series of the Yellow River daily flows at Tongguan hydrological station, from the year 2000 to 2005. Power spectrum analysis and statistical moment scaling function on a range of scales revealed scaling qualities of the data. The partition function, which displayed a convex curvature, and the generalized dimension function showed that multifractality is presented. The singularity spectrum, which is single-humped, has shown strong multifractality degree.

**Keywords:** multifractal, scale, time series, river flows
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## 1. Introduction

Fractal analysis has been given considerable attention since it was introduced by Mandelbrot. Monofractals are homogeneous in the sense that they have the same scaling properties, characterized by a single singularity exponent, generally a fractal dimension.[1] But scaling properties of complex processes, such as turbulence and rainfall, need more than one scaling exponent to describe. This leads to multifractals. The concepts of multifractals were originally introduced by Mandelbrot in the discussion of turbulence[2] and expanded to many other contexts. Schertzer and Lovejoy[3] showed that a multifractal field can be regarded as the result of multiplicative cascade process. Telesca *et al*[4] used a set of multifractal parameters, defined from the shape of the multifractal spectrum to obtain the degree of multifractality in short-term time dynamics of the ultra low frequency (ULF) geomagnetic field.

River flows have been known to be scaling for over 50 years since Hurst showed that the stream flows of various rivers exhibit long-range statistical dependencies in 1951. It took over 30 years for its significance to be fully appreciated. During the early 1980s multifractal notions have developed rapidly and many geophysical fields have shown to be multifractal over various ranges of scales. Using the framework of universal multifractals, Tessier *et al*[5] provided the first multifractal analysis of the time series of river runoff in French. It has been recognized that multifractal futures need to be accounted for towards a better understanding of river flows.

In view of this the purpose of the present study is to perform a multifractal characterization of the time series of the Yellow River flows so as to investigate the possibility of a multifractal approach. The river flows observed on a hydrological station, Tongguan, in Shaanxi, China, are analysed and the study deals with the multifractal analysis only in time. The existence of multifractality is investigated by employing three methods: (1) power spectrum; (2) statistical moment scaling function; and (3) multifractal spectrum function which can quantify the degree of multifractality.

The organization of this paper is as follows. First, a brief review of the methods employed in the present study to identify the presence of multifractals in a time series is given in Section 2. Details of the data used, analysis carried out and results achieved are presented in Section 3. Finally, important conclusions drawn from this study are provided in Section 4.

## 2. Methodology

Research over the past two decades has led to the development of a wide variety of methods to identify fractals in a time series.[6−9] However, before applying any fractal identification method, general information about the fractal behaviour may be obtained

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by using standard statistical descriptions, such as the power spectrum. This method is, therefore, employed in the present study. In addition, the statistical moment scaling function can provide important information regarding the existence of a fractal and its type, whether monofractal or multifractal. Finally, the multifractal spectrum method is employed to display the degree of multifractality in the time series.

## 2.1. Power spectrum

Power spectrum is a standard tool in fractal investigations in hydrological process.[5,8,10] When all or part of the spectrum obeys the following power law

$$E(\omega) \propto \omega^{-\beta},$$

where $\omega$ is the frequency and $\beta$ is an exponent, called the spectral exponent, the data are scaling in that range, that is, the scaling regime. The absence of characteristic timescales indicates that fractal behaviour may be assumed to hold.

## 2.2. Statistical moments

The scaling of time series can also be investigated by statistical moments.

First, the time series is divided into pairwise disjointed intervals (sometimes we call them cells) and $\lambda$ is the ratio of the maximum interval length to the ensemble interval. For different ratios, $\lambda$, we compute the average river flow intensity in each interval, $i$, and denote it by $\epsilon(\lambda, i)$:

$$\epsilon(\lambda, i) = N(\lambda, i)/M,$$

where $N(\lambda, i)$ is the river flow measurements in time interval $i$, $M$ is the sum of all the values in the time series. Raising $\epsilon(\lambda, i)$ to a power $q$, we obtain the statistical moments $M(\lambda, q)$:

$$M(\lambda, q) = \sum_i \epsilon(\lambda, i)^q.$$

Here we assume $0^0 = 0$. For a power $q$ chosen, the statistical moment as a function of $\lambda$, has the following form:

$$M(\lambda, q) \propto \lambda^{\tau(q)},$$

where $\tau(q)$ is regarded as a characteristic function of the fractal behaviour and is called the partition function. If $\tau(q)$ versus $q$ is linear, the data set is monofractal. If $\tau(q)$ versus $q$ is convex, the data set has multifractal property.

The generalized dimension $D(q)$ introduced by Grassberger,[11] Hentschel and Procaccia[12] is defined by

$$D(q) = \begin{cases} \dfrac{\tau(q)}{q-1} & q \neq 1, \\ \lim\limits_{\lambda \to 0} \dfrac{\sum\limits_i \epsilon(\lambda, i) \cdot \log \epsilon(\lambda, i)}{\log \lambda} & q = 1, \end{cases} \tag{1}$$

where $D(0)$ is the capacity dimension, $D(1)$ is the information dimension, and $D(2)$ is the correlation dimension.

## 2.3. Multifractal spectrum

Maybe the most adequate method of dealing with multifractals is to study their singularity spectra.[13]

If $\epsilon(\lambda, j)$ has the power-law dependence, $\epsilon(\lambda, j) \propto \lambda^{\alpha(j)}$, the exponent $\alpha$, which depends on the cell $j$, is called the Hölder exponent. If all the cells have the same exponent $\alpha$, the sample is monofractal, otherwise we ask how many cells have a given exponent $\alpha$ and denote this number as $N_\alpha(\lambda)$. Multifractal spectrum function $f(\alpha)$ is defined as

$$f(\alpha) = \lim_{\lambda \to 0} \frac{\log N_\alpha(\lambda)}{-\log \lambda}.$$

The quantities $\alpha$ and $f(\alpha)$ can be related with $q$ and $\tau(q)$ by means of the Legendre transform

$$\begin{cases} \alpha(q) = \dfrac{\mathrm{d}\tau(q)}{\mathrm{d}q}, \\ f(\alpha) = q\alpha(q) - \tau(q). \end{cases} \tag{2}$$

The curve $f(\alpha)$ is single-humped for a multifractal, it reduces to a point for a monofractal. $\alpha$ is defined in $[\alpha_{\min}, \alpha_{\max}]$, where $f(\alpha) \geq 0$ and $\alpha_{\min} = \dfrac{\mathrm{d}\tau(q)}{\mathrm{d}q}|_{q \to +\infty}$, $\alpha_{\max} = \dfrac{\mathrm{d}\tau(q)}{\mathrm{d}q}|_{q \to -\infty}$.[14]

We use three parameters to describe the complexity of the time series. One is $f_{\max}$, the maximum value of $f(\alpha)$, another is the width of the range $[\alpha_{\min}, \alpha_{\max}]$. To obtain the third parameter, we fit the spectrum function to a quadratic function around the position of its maximum at $\alpha_0$ by a least square method:

$$\tilde{f}(\alpha) = A(\alpha - \alpha_0)^2 + B(\alpha - \alpha_0) + C.$$

Parameter $B$ measures the asymmetry of the spectrum curve, which is zero for symmetric shapes, positive or negative for left-skewed or right-skewed shapes respectively. High maximum, large width and low asymmetry denote a time series with 'strong' multifractality degree.[4]

# 3. Results and discussion

## 3.1. Data description

The Yellow River, originating from the Yueguzonglie basin on the northern part of the Bayankela Mountain in Tibet highlands at el. 4500 m, flows through nine provinces of China, with a total length of 5464 km, basin area 795000 $km^2$, and finally empties into the Bohai sea. It is the second largest river in China. The total population within river basin is 107 million. It cultivated the North China Plain and we all consider the Yellow River as the cradle of Chinese civilization; the Yellow River is well known as our 'mother river'.

A large part of the Yellow River basin belongs to arid or semi-arid area, where the rainfall is deficient and soil erosion is very serious. In history the broad masses of people waged long years of struggle against the flood disasters of the Yellow River. Since new China was founded, the state has devoted much importance to the harnessing of the Yellow River. Many hydrological stations have been set up. By 1990 there was in place a large network including 451 hydrological stations. There are 4.28 hydrological stations per 10000 square kilometres, which is 8.23 times that in 1949. Along with the improvement of hydrological stations, the collating and editing of hydrological data and the analysis of hydrology were also progressed. In this paper data were collected at 8:00 a.m. every day from one of the hydrological stations, Tongguan, which is in Shaanxi Province and located in the middle reaches of the Yellow River. The database comprises river flow measurements of 1755 days (26th June 2000–15th April 2005) and the daily stream flow records are shown in Fig.1. The mean flow rate is 620.2882 $m^3/s$.

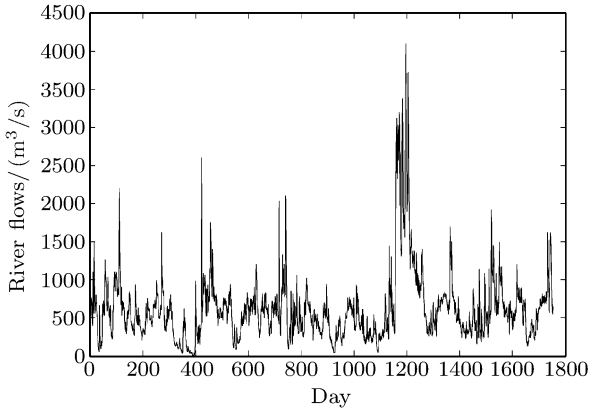



**Fig.1.** The Yellow River daily flows from 26th June 2000 up to 15th April 2005.

## 3.2. Power spectrum

Before performing the multifractal properties of the river flows, we firstly calculate the power spectrum $E(\omega)$ and give the log–log plot of $E(\omega)$ against the frequency $\omega$. The result is shown in Fig.2 (solid). The spectrum appears to consist of a power law form from 1 day to 100 days, the estimated value of $\beta$ for this low-frequency river flow regime is $\beta \approx 1.6328$. Thus the river flows may be expected to exhibit a scaling behaviour during this time interval. This result is close to that obtained by Tessier *et al*[5] for them the estimated value of $\beta$ for a high-frequency river flow regime was $\beta = 1.6 \pm 0.5$.

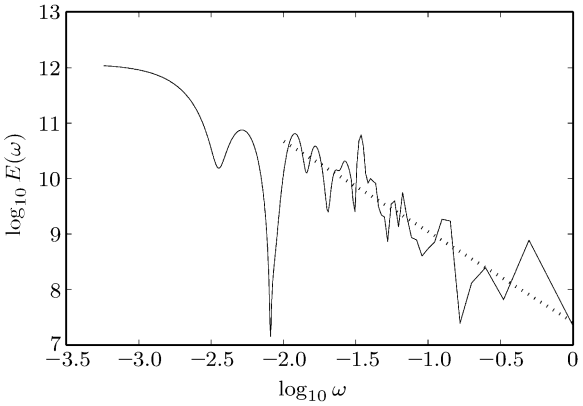



**Fig.2.** Power spectrum for the Yellow River flows.

## 3.3. Statistical moments

Figure 3 shows the $q$th moments, $M(\lambda, q)$, of the river flows on the time scales between 1 day ($\lambda = 0.0005634$) and 25 days ($\lambda = 0.0140845$), against the scale ratio $\lambda$. The $q$-moments plotted are indicated in the legend. The curves in Fig.3(a), for moments of $q = 0.0, 0.5, 1.0, \ldots, 4.5$, exhibit a well-defined straight line behaviour over the whole scaling regime. However this occurs only for a range of $q$-moments. Moments of high orders do not follow straight lines for the time series. An example can be seen in Fig.3(b), for moments of $q = 60, 70, 80, \ldots, 150$. We have already known that large moments highlight the high river flows and small moments highlight small flows when we calculate the statistical moments. Kumar *et al*[15] conveyed that high-order empirical moments are dominated by large observations and are therefore highly variable. The river flow measuring device and data processing procedure will lead to inaccuracies in representing very high flows. This may affect the behaviour of the moments.

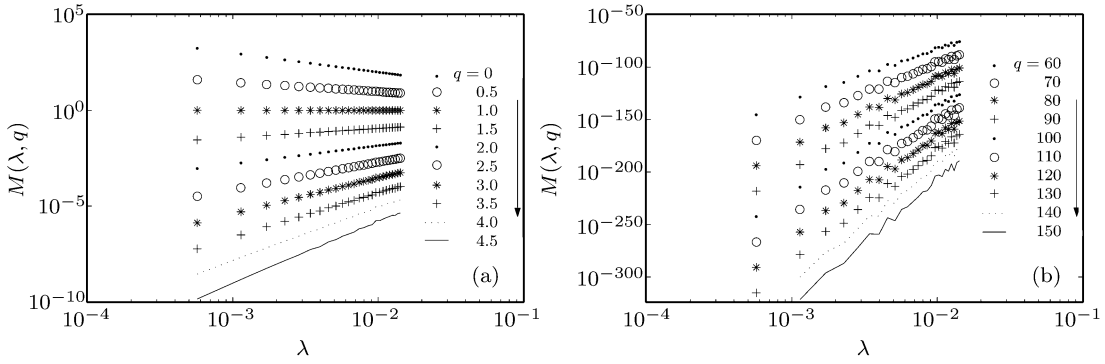



**Fig.3.** The $q$th statistical moments, on time scales between 1 day and 25 days for moments of: (a) $q = 0.0, 0.5, \ldots,$ 4.5 and (b) $q = 60, 70, \ldots,$ 150.

The partition function $\tau(q)$ can be obtained by using the equation $\tau(q) = \lim_{\lambda \to 0} \dfrac{\log M(\lambda, q)}{\log \lambda}$, so we estimate the slopes of the fitted lines in the log–log plot of $M(\lambda, q)$ against $\lambda$. The $\tau(q)$ function is shown for orders from $-5$ to 5 (see Fig.4). It is convex, so the river flow time series is multifractal.

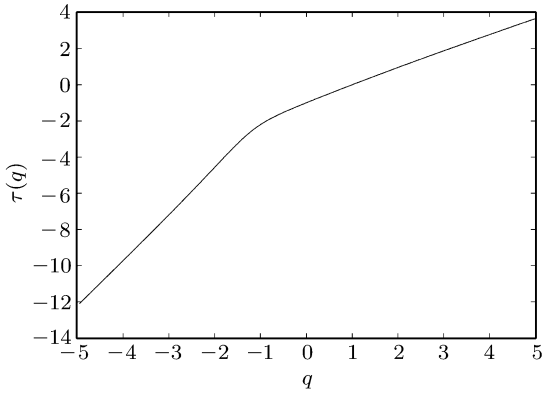



**Fig.4.** The partition function $\tau(q)$.

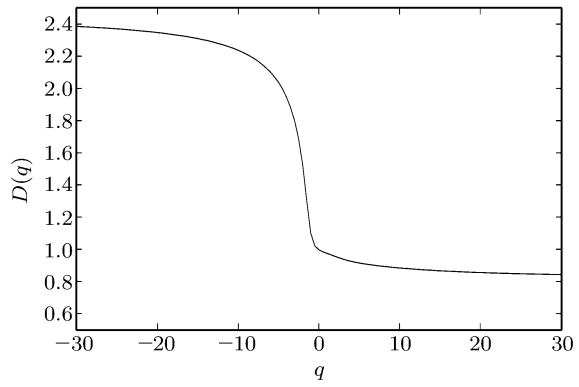



**Fig.5.** The generalized function $D(q)$.

Having known $\tau(q)$, we calculate $D(q)$ from Eq.(1). The result is shown in Fig.5. For $q = 0$, the capacity dimension $D(0)$ equals to 0.9972, which is the dimension of the 'support' of the time series. And this suggests that the block of the flow for these six years is very little. That is consistent with the fact that the break of the flow did not occur since the river was managed by the Yellow River Conservancy Committee in the year 1999. The information dimension $D(1)$ is 0.9747, and the correlation dimension $D(2)$ is 0.9559.

## 3.4. Multifractal spectrum

We can plot the multifractal spectrum by using Legendre transform of $\tau(q)$. For discrete time series, the spectrum is calculated by $f(\alpha) = \min_{-\infty < q < \infty} (\alpha \cdot q - \tau(q))$.[13] We recover the $f(\alpha)$ curve (solid) shown in Fig.6. The maximum value of the $f(\alpha)$ curve occurs at $\alpha_0 = 1.028$ and $f_{\max} = 0.9972$, which equals to $D(0)$, the fractal dimension of the 'support' of the river flows. $\alpha_{\min} = 0.8146$ and $\alpha_{\max} = 2.1396$, so the width of $[\alpha_{\min}, \alpha_{\max}]$ is 1.3250. We fit $f(\alpha)$ to a quadratic function (dashed curve shown in Fig.6) around $\alpha_0$ by a least square method:

$$\begin{aligned} \tilde{f}(\alpha) = \; & -0.9734(\alpha - 1.028)^2 \\ & +0.2174(\alpha - 1.028) + 0.8553. \end{aligned}$$

So the fit parameter $B$ is 0.2174. The high maximum, large width and low asymmetry indicated that the river flow time series has shown 'strong' multifractality degree.

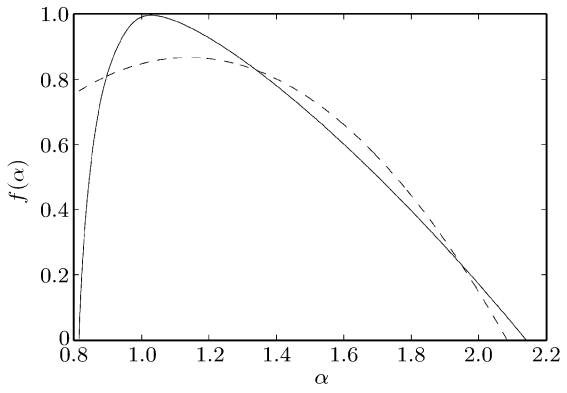



**Fig.6.** The multifractal spectrum $f(\alpha)$ (solid) and the fitted curve (dashed).

## 4. Conclusions

In this study we dealt with time series of the Yellow River daily flows at Tongguan hydrological station from the year 2000 to 2005. We have shown that the time dynamics can be characterized as multifractal signals.

Power spectrum analysis of the time series indicated that the scaling regime was approximately 1–100 days. The data displayed linear behaviour of the $M(\lambda, q)$ curve for moments of orders up to 4.5, thus revealed scaling qualities of the data. The partition function $\tau(q)$, displaying a convex curvature, suggested multifractality of the time series. The generalized dimension $D(q)$ is characterized by some important generalized features of the time series, such as information entropy and the correlation dimension. The singularity spectrum, $f(\alpha)$, which is single-humped, has shown strong multifractality degree.

Many studies have viewed scaling in the river flow process essentially from a stochastic perspective, because it has traditionally been assumed to be a result of a very large number of variables. But our study shows that the Yellow River flows can be represented by a nonlinear model, multifractal model. It must be emphasized that the scale range of the data was limited and that the outcome of the analyses might have been different for a large range. In practical hydrology, detailed data over larger-scale ranges are seldom available. Thus investigations such as the present study may be important steps in direction towards development of practically useful tools for the Yellow River flow data processing based on a multifractal approach. In the future detailed understanding and short-term prediction of the Yellow River flows can be obtained by using multifractal models.

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