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Synchronization of chaotic systems with different orders*

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A controller is designed to realize the synchronization between chaotic systems with different orders. The structure of the controller, the error equations and the Lyapunov functions are determined based on stability theory. Hyperchaotic Chen system and Rossler system are taken for example to demonstrate the method to be effective and feasible. Simulation results show that all the state variables of Rossler system can be synchronized with those of hyperchaotic Chen system by using only one controller, and the error signals approach zero smoothly and quickly.

Keywords: chaos synchronization, hyperchaotic Chen system, Rossler system,chaotic systems with different orders

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1. Introduction

Chaos synchronization has attracted a great deal of attention from the nonlinear science field. Since Pecora and Carroll[1] introduced a method to synchronize two identical chaotic systems with different initial conditions, many methods and techniques for synchronization have been developed, such as the Pecora–Carroll (PC) method, variable coupling method, adaptive control method, variable feedback method and so on.[2−17] Most of the methods are used to synchronize two identical chaotic systems. However, it is difficult to find identical chaotic systems in reality, such as in physics, autocontrol, information and secure communications. Therefore, it is much more important to synchronize two different chaotic systems, especially chaotic systems with different orders, such as the hyperchaotic Chen system that is a fourth-order system,[4] and the Rossler system that is a third-order system.[18] Since they have different nonlinear functions, they are different chaotic systems with different orders. A controller is designed in this paper to realize the synchronization between chaotic systems with different orders. The structure of the controller, the error equations and the Lyapunov functions are designed based on stability theory. Hyperchaotic Chen system and Rossler system are taken for example to demonstrate the method. Simulation results show that all the state variables of Rossler system can be synchronized with those of hyperchaotic Chen system by using only one controller, and the error signals approach zero smoothly and quickly.

2. Principle of synchronization

Assume the hyperchaotic Chen system to be a drive system

$$\begin{cases} \dot{x}_1 = a_1(y_1 - x_1) + v_1, \\ \dot{y}_1 = d_1 x_1 - x_1 z_1 + c_1 y_1, \\ \dot{z}_1 = x_1 y_1 - b_1 z_1, \\ \dot{v}_1 = y_1 z_1 + g_1 v_1, \end{cases} \tag{1}$$

and Rossler system to be a response system

$$\begin{cases} \dot{x}_2 = -y_2 - z_2, \\ \dot{y}_2 = x_2 + a_2 y_2, \\ \dot{z}_2 = b_2 + x_2 z_2 - c_2 z_2 + u(t), \end{cases} \tag{2}$$

where $a_1 = 35$, $b_1 = 3$, $c_1 = 12$, $d_1 = 7$, $g_1 = 0.5$, $a_2 = b_2 = 0.2$, $c_2 = 5.7$, and $u(t)$ is the controller designed. Before $u(t)$ is added, systems (1)and (2) have different nonlinear functions and different orders, the phase maps are also different in shape and type, the

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orbit of hyperchaotic Chen system is a chaotic attractor of double helixes, while that of Rossler system is a single helix (shown in Fig.1 and Fig.2). They are different chaotic systems with different orders.

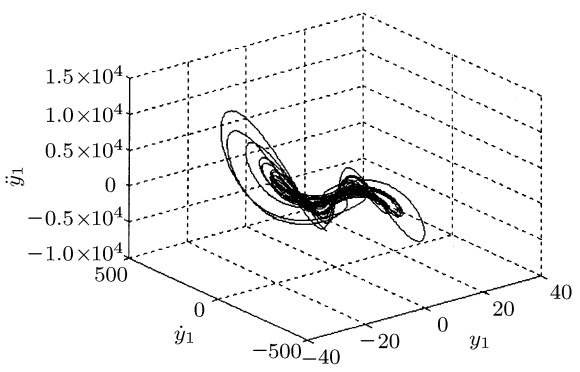


Fig.1. Phase map of $(y_1, \dot{y}_1, \ddot{y}_1)$ in system (1).

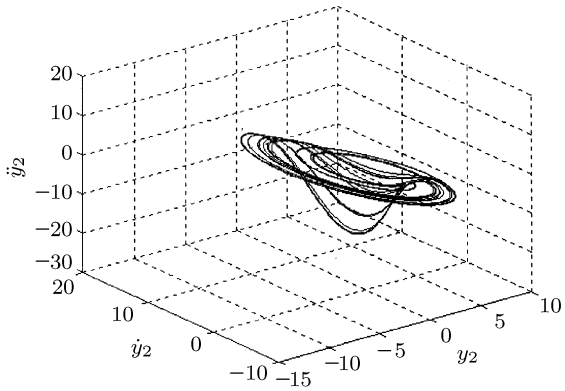


Fig.2. Phase map of $(y_2, \dot{y}_2, \ddot{y}_2)$ in system (2).

Lemma:[19] Consider the systems

$$\dot{x} = F(x, y), \tag{3}$$

$$\dot{y} = G(x, y), \quad (x, y) \in R^{n\times n}. \tag{4}$$

Assume $H_1(x)$, $H_2(x)$, ..., $H_k(x)$ and $S_1(y)$, $S_2(y)$, ..., $S_k(y)$ to be observable variables of systems (3) and (4), respectively. Systems (3) and (4) are synchronized with respect to (H_1, H_2, ..., H_k) and (S_1, S_2, ..., S_k), if

$$\lim_{t\to\infty}[S_i(y(t)) - H_i(x(t))] = 0, \quad (i = 1, 2, \ldots, k). \tag{5}$$

Theorem Suppose that observable variables of systems (1) and (2) are

$$\begin{aligned} S_1(x_2, y_2, z_2) &= y_2, \\ S_2(x_2, y_2, z_2) &= x_2 + (\alpha + a_2)y_2, \\ S_3(x_2, y_2, z_2) &= z_2 - (\alpha + \beta + a_2)x_2 - (\beta + a_2)(\alpha + a_2)y_2; \end{aligned} \tag{6}$$

$$\begin{aligned} H_1(x_1, y_1, z_1) &= y_1, \\ H_2(x_1, y_1, z_1) &= \alpha y_1 + d_1 x_1 - x_1 z_1 + c_1 y_1, \\ H_3(x_1, y_1, z_1) &= \beta(z_1 - d_1)x_1 - (1 + \alpha\beta + c_1\beta)y_1 + (z_1 - d_1)v_1 + x_1^2 y_1 \\ &\quad +[-\alpha d_1 + d_1 a_1 - c_1 d_1 + (\alpha - b_1 - a_1 + c_1)z_1]x_1 + (-\alpha c_1 \\ &\quad -d_1 a_1 + a_1 z_1 - c_1^2)y_1. \end{aligned} \tag{7}$$

The systems (1) and (2) can be synchronized if the controller is of the following form:

$$u(t) = -\gamma e_3 + e_2 - W(t), \tag{8}$$

where e_i are error variables, and $W(t)$ are combinations of the state variables of the two systems.

Proof Introduce e_1 as an error variable of the systems and let $e_1 = y_2 - y_1$. According to the systems (1) and (2), we obtain

$$\dot{e}_1 = x_2 + a_2 y_2 - d_1 x_1 + x_1 z_1 - c_1 y_1. \tag{9}$$

The first partial Lyapunov function is constructed as

$$V_1 = \frac{1}{2}e_1^2. \tag{10}$$

The derivative of V_1 is obtained as follows:

$$\dot{V}_1 = e_1\dot{e}_1 = -\alpha e_1^2 + e_1(\alpha e_1 + x_2 + a_2 y_2 - d_1 x_1 + x_1 z_1 - c_1 y_1), \tag{11}$$

where $\alpha > 0$.

Let e_2 be the another error variable of the systems and let $e_2 = x_2 - k_1$, where

$$k_1 = -\alpha e_1 - a_2 y_2 + d_1 x_1 - x_1 z_1 + c_1 y_1, \tag{12}$$

and we can obtain

$$\begin{aligned} \dot{e}_2 = \; & (\alpha + a_2)x_2 + (\alpha a_2 + a_2^2 - 1)y_2 - z_2 + (z_1 - d_1)v_1 + x_1^2 y_1 + [-\alpha d_1 + d_1 a_1 - c_1 d_1 \\ & + (\alpha - b_1 - a_1 + c_1)z_1]x_1 + (-\alpha c_1 - d_1 a_1 + a_1 z_1 - c_1^2)y_1. \end{aligned} \tag{13}$$

The second partial Lyapunov function is constructed as

$$V_2 = V_1 + \frac{1}{2}e_2^2. \tag{14}$$

And we have

$$\begin{aligned} \dot{V}_2 = \dot{V}_1 + e_2\dot{e}_2 = \; & -\alpha e_1^2 - \beta e_2^2 + e_2\{-z_2 + e_1 + \beta e_2 + (\alpha + a_2)x_2 + (\alpha a_2 + a_2^2 - 1)y_2 \\ & + (z_1 - d_1)v_1 + x_1^2 y_1 + [-\alpha d_1 + d_1 a_1 - c_1 d_1 + (\alpha - b_1 - a_1 + c_1)z_1]x_1 \\ & + (-\alpha c_1 - d_1 a_1 + a_1 z_1 - c_1^2)y_1\}, \end{aligned} \tag{15}$$

where $\beta > 0$.

Introduce e_3 as the third error variable of the system, and let $e_3 = z_2 - k_2$, where

$$\begin{aligned} k_2 = \; & e_1 + \beta e_2 + (\alpha + a_2)x_2 + (\alpha a_2 + a_2^2 - 1)y_2 + (z_1 - d_1)v_1 + x_1^2 y_1 \\ & + [-\alpha d_1 + d_1 a_1 - c_1 d_1 + (\alpha - b_1 - a_1 + c_1)z_1]x_1 + (-\alpha c_1 - d_1 a_1 + a_1 z_1 - c_1^2)y_1, \end{aligned} \tag{16}$$

and we will have

$$\dot{e}_3 = \dot{z}_2 - \dot{k}_2 = b_2 + x_2 z_2 - c_2 z_2 + u(t) - \dot{k}_2 = u(t) + W(t), \tag{17}$$

where

$$\begin{aligned} W(t) = \; & b_2 + x_2 z_2 - c_2 z_2 - \dot{k}_2 \\ = \; & b_2 + x_2 z_2 - c_2 z_2 - \dot{e}_1 - \beta\dot{e}_2 - (\alpha + a_2)\dot{x}_2 - (\alpha a_2 + a_2^2 - 1)\dot{y}_2 - (z_1 - d_1)\dot{v}_1 - \dot{z}_1 v_1 \\ & - x_1^2\dot{y}_1 - 2x_1 y_1 \dot{x}_1 - [-\alpha d_1 + d_1 a_1 - c_1 d_1 + (\alpha - b_1 - a_1 + c_1)z_1]\dot{x}_1 - (\alpha - b_1 - a_1 + c_1)x_1\dot{z}_1 \\ & - (-\alpha c_1 - d_1 a_1 + a_1 z_1 - c_1^2)\dot{y}_1 - a y_1 \dot{z}_1. \end{aligned}$$

The Lyapunov function is constructed as

$$V_3 = V_2 + \frac{1}{2}e_3^2. \tag{18}$$

The derivative of V_3 is

$$\begin{aligned} \dot{V}_3 = \dot{V}_2 + e_3\dot{e}_3 = \; & -\alpha e_1^2 - \beta e_2^2 - \gamma e_3^2 + e_3(\gamma e_3 - e_2 + \dot{e}_3) \\ = \; & -\alpha e_1^2 - \beta e_2^2 - \gamma e_3^2 + e_3[\gamma e_3 - e_2 + u(t) + W(t)], \end{aligned} \tag{19}$$

where $\gamma > 0$.

According to system (2), if the controller satisfies we have

$$u(t) = -\gamma e_3 + e_2 - W(t), \tag{20}$$

$$\dot{V} = -\alpha e_1^2 - \beta e_2^2 - \gamma e_3^2 \le 0. \tag{21}$$

According to Lyapunov stability theory,[20] the synchronization of the two systems is then achieved and the error variables of the system $e_i(t)$ approach zero with the lapses of times, i.e.

$$\lim_{t\to\infty} e_i(t) = 0, \quad (i = 1, 2, 3). \tag{22}$$

Suppose that

$$\begin{aligned}
S_1(x_2, y_2, z_2) = &\ y_2,\\
S_2(x_2, y_2, z_2) = &\ x_2 + (\alpha + a_2)y_2,\\
S_3(x_2, y_2, z_2) = &\ z_2 - (\alpha + \beta + a_2)x_2 - (\beta + a_2)(\alpha + a_2)y_2;\\
H_1(x_1, y_1, z_1) = &\ y_1,\\
H_2(x_1, y_1, z_1) = &\ \alpha y_1 + d_1 x_1 - x_1 z_1 + c_1 y_1,\\
H_3(x_1, y_1, z_1) = &\ \beta(z_1 - d_1)x_1 - (1 + \alpha\beta + c_1\beta)y_1 + (z_1 - d_1)v_1 + x_1^2 y_1\\
&+[-\alpha d_1 + d_1 a_1 - c_1 d_1 + (\alpha - b_1 - a_1 + c_1)z_1]x_1 + (-\alpha c_1\\
&-d_1 a_1 + a_1 z_1 - c_1^2)y_1,
\end{aligned}$$

and we will have

$$e_1 = y_2 - y_1 = S_1(x_2, y_2, z_2) - H_1(x_1, y_1, z_1), \tag{23}$$

$$e_2 = x_2 - k_1 = S_2(x_2, y_2, z_2) - H_2(x_1, y_1, z_1), \tag{24}$$

$$e_3 = z_2 - k_2 = S_3(x_2, y_2, z_2) - H_3(x_1, y_1, z_1), \tag{25}$$

Then

$$\lim_{t\to\infty} e_i(t) = \lim_{t\to\infty} S_i(x_2, y_2, z_2) - H_i(x_1, y_1, z_1) = 0 \quad (i = 1, 2, 3) \tag{26}$$

holds. This completes the proof.

3. Simulation

The simulation of synchronization between hyperchaotic Chen system and Rossler system is made to verify the conclusion above. System parameters are kept constant during the simulation; initial values of the states are taken as $x_1(0) = 3$, $y_1(0) = -4$, $z_1(0) = 2$, $v_1(0) = 2$, $x_2(0) = 0$, $y_2(0) = 2.9$ and $z_2(0) = -1$. Under the controlling of the controller $u(t)$, the three variables of Rossler system are synchronized with those of hyperchaotic Chen system, and the error signals approach zero smoothly and rapidly as shown in Fig.3. The variation of state variables with time is shown in Fig.4. The orbit of Rossler system in the phase map changes from single helix into double helixes, which is exactly the same as that of hyperchaotic Chen system as shown in Figs.5 and 6.

4. Conclusion

A controller is designed to realize the synchronization between chaotic systems with different orders. The structure of the controller, the error equations and the Lyapunov functions are designed based on the stability theory. Simulation results show that the method is effective and feasible. All the state variables of the Rossler system can be synchronized with those of the hyperchaotic Chen system by using only one controller, and the controller is simple, efficient, and cost-effective.

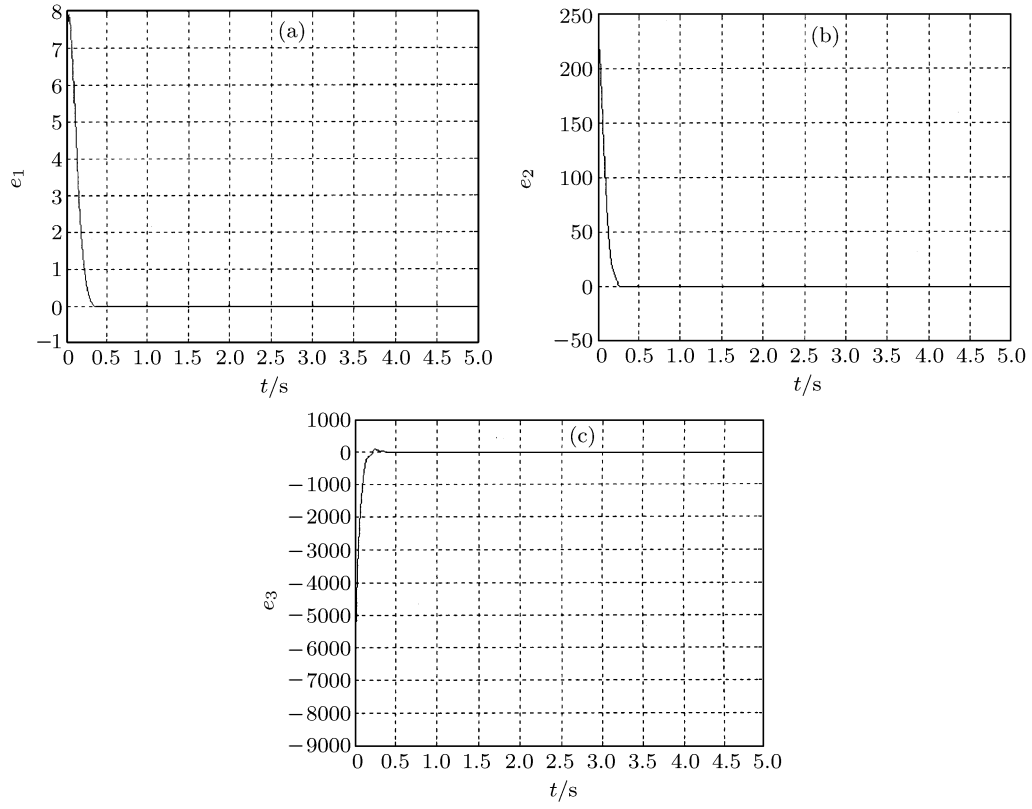


Fig.3. Error variable e_1, e_2 and e_3 vs time t ($\alpha = 6$, $\beta = 8$ and $\gamma = 8$).

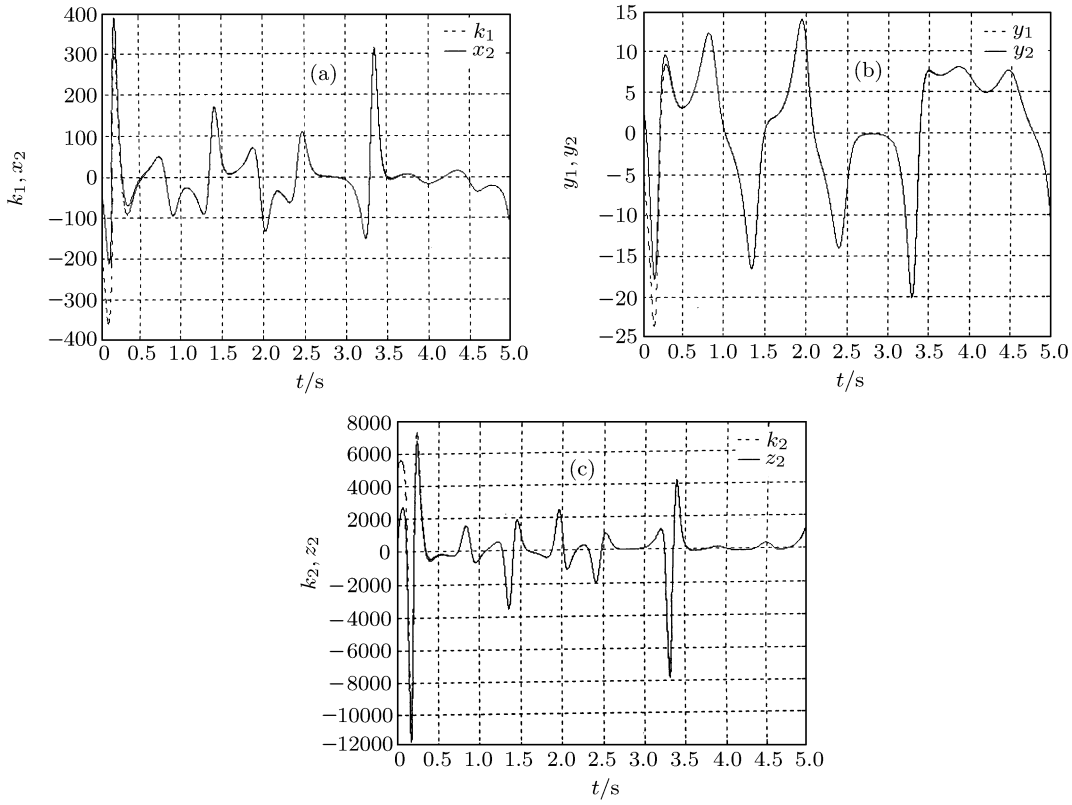


Fig.4. State variable vs time t ($\alpha = 6$, $\beta = 8$ and $\gamma = 8$).

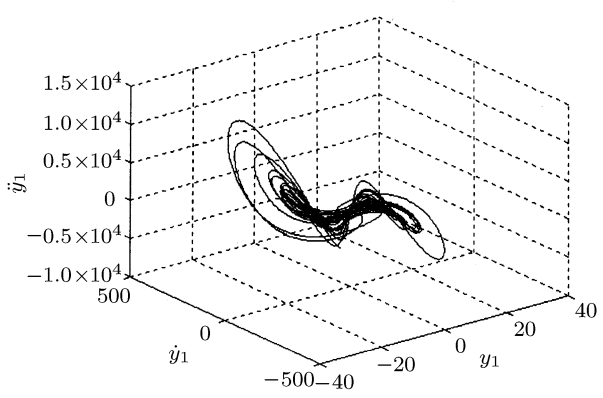


Fig.5. Synchronization phase map of $(y_1, \dot{y}_1, \ddot{y}_1)$ ($\alpha = 6$, $\beta = 8$ and $\gamma = 8$).

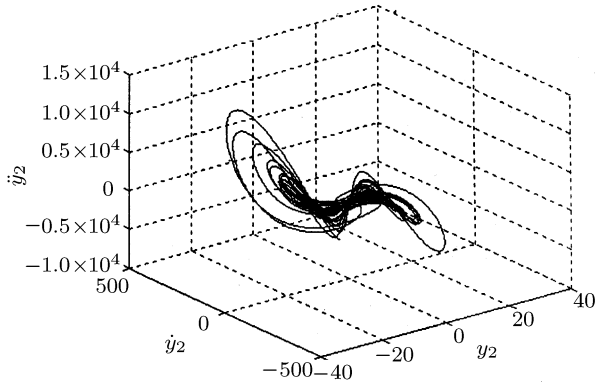


Fig.6. Synchronization phase map of $(y_2, \dot{y}_2, \ddot{y}_2)$ ($\alpha = 6$, $\beta = 8$ and $\gamma = 8$).

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