Vol 16 No 2, February 2007
1009-1963/2007/16(02)/0340-06

Chinese Physics

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Evidence of parameter-induced aperiodic stochastic resonance with fixed noise*

Li Jian-Long(李建龙)†

Institute of Information and Communication Engineering, Zhejiang University, Hangzhou 310027, *China*

(Received 24 May 2006; revised manuscript received 31 August 2006)

Stochastic resonance (SR) is based on the cooperative effect between the stochastic dynamical system and the external forcing. As is well known, the cooperative effect is produced by adding noises. In this paper, we show the evidence that by changing the system parameters and the signal intensity, a nonlinear system in the presence of an input aperiodic signal can yield the cooperative effect, with the noise fixed. To quantify the nonlinear system output, we determine the theoretical bit error rate (BER). By numerical simulation, the validity of the theoretical derivation is checked. Besides, we show that parameter-induced SR is more realizable than SR via adding noises, especially when the noise intensity exceeds the resonance level, or when the characteristic of the noise is not known.

Keywords: aperiodic stochastic resonance, tuning parameters, bit error rate
PACC: 0540, 0250

1. Introduction

Since the early 1980s, there have appeared some extensions of stochastic resonance (SR), such as aperiodic SR,[1−4] suprathreshold SR,[5] non-Markovian SR[6] (see Refs.[7,8] for overviews), SR in a linear system,[9,10] and SR in the presence of different kinds of noise.[11−14] The above-mentioned phenomena of SR are realized by adding noises, which is referred to as conventional SR (CSR) in this paper. According to the concept of SR originally put forward by Benzi *et al*,[15] the SR is based on the cooperative effect between the stochastic dynamical system and the external periodic forcing. In communication systems, the external periodic forcing can be viewed as an input signal, and it was demonstrated that the CSR also could be realized with an input aperiodic signal.[1,2] Then, since the resonance occurs between the stochastic dynamical system and the input signal and changing the stochastic noise can realize the resonance effect, an idea naturally comes into being that changing the system or the input signal yields the phenomenon of the resonance. The system parameters or the signal intensity is then considered as an important tunable element in the study of SR.

To describe SR induced by tuning parameters, with noise fixed, consider a nonlinear dynamic system given by

$$\dot{x}(t) = ax(t) - bx^3(t) + cs(t) + \Gamma(t), \qquad (1)$$

where system parameters a and b are positive, $s(t)$ is the input aperiodic signal, parameter c can change the signal intensity, and $\Gamma(t)$ is an additive Gaussian white noise with zero-mean and correlation function

$$< \Gamma(0)\Gamma(t) >= 2D\delta(t), \qquad (2)$$

in which D is the noise intensity. Here, the realizing of parameter-induced aperiodic SR (PASR) means that the resonance phenomenon between the stochastic system and the input aperiodic signal is produced by tuning parameters a, b and c (or several of them), with noise intensity D fixed. To show the evidence of PASR, we carefully investigate the role of the noise. As is well known, when the noise plays a constructive role (the quantity that quantifies the output of the system achieves its extremum, i.e. the system attains to the 'resonance' state), it is considered that the system yields an SR. Here, we first make the system take those parameters that are adjusted for realizing PASR. Then, we vary D in a range that includes the initial value of the noise intensity to check whether the noise plays the constructive role at its initial value. Finally, if the answer is positive, then aperiodic SR is produced, with the parameters tuned and noise fixed.

*Project supported by the National Natural Science Foundation of China (Grant No 10332030) and the State Key Program for Basic Research of China (Grant No 5132103ZZT21B).
†E-mail: jlli@zju.edu.cn

2. The measurement of PASR

Assume the input signal $s(t)$ to be a pseudorandom baseband binary pulse amplitude modulated signal, which transmits a digit 0 by a waveform $s_0(t) = -A$ or digit 1 by a waveform $s_1(t) = +A$ in a time interval $[0\ T_\mathrm{b}]$. Here A is the signal level and T_b is the bit duration. Since the source digits are encoded according to the messages, digits 0 and 1 are often random in a sequence and $s(t)$ consists of the corresponding randomly arranged waveforms $s_0(t)$ and $s_1(t)$. $s(t)$ is then an aperiodic information-bearing binary signal, and SR discussed in the following is aperiodic.

For system (1), the statistically equivalent description for the probability density $\rho(x,t)$ is governed by the Fokker–Planck equation (FPE)

$$\begin{aligned}\frac{\partial\rho(x,t|\pm A)}{\partial t} &= \frac{\partial}{\partial x}[V'_{\pm A}(x)\rho(x,t|\pm A)] \\ &\quad + \frac{D}{2}\frac{\partial^2\rho(x,t|\pm A)}{\partial x^2},\end{aligned} \tag{3}$$

where $V'_{\pm A}(x)$ is the derivative of the potential $V_{\pm A}(x)$ with respect to x, and

$$V_{\pm A}(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4 \mp A. \tag{4}$$

In Eq.(3), the dynamical probability density function (PDF) $\rho(x,t|\pm A)$ can be approximately calculated from the FPE, which is discussed in Appendix A. It is also shown in Appendix A that $\rho(x,t|-A)$ and $\rho(x,t|A)$ are symmetric about the axis $x = 0$, so according to the maximum likelihood ratio decision rule, the optimal decision level for the output is $x_0 = 0$.

We employ the bit error rate (BER) as a measure to quantify the output of system (1), as is commonly used in communications. The BER is defined as the percentage of the wrong transmitted bits, and it represents the simplest way to evaluate the efficiency of a binary data channel. Given that the signal levels are statistically independent and equiprobable (i.e., $P_0 = P_1 = \frac{1}{2}$), the BER can be given by

$$\begin{aligned}\mathrm{BER} &= P_0P(A|-A) + P_1P(-A|A) \\ &= \frac{1}{2}[P(A|-A) + P(-A|A)],\end{aligned} \tag{5}$$

where $P(A|-A)$ is the error probability as $-A$ is transmitted and the decision is A; $P(-A|A)$ is the error probability as A is transmitted and the decision is $-A$. Assume that the current symbol takes the amplitude $+A$. If the number of the successive amplitudes with the same symbol is K ($K = 0, 1, \cdots$), and the amplitude of the $(K+1)$th symbol (after $(K+1)T_\mathrm{b}$ time interval) is $-A$, the probability of detection error for the ith (here, $0 < i \le (K+1)$) amplitude is

$$P_i(-A|A) = \int_{-\infty}^{0} \rho(x, iT_\mathrm{b}|A)\mathrm{d}x. \tag{6}$$

Because of the symmetry, we have $P_i(A|-A) = P_i(-A|A)$. Since the input amplitudes are statistically independent and equiprobable, the expression (5) can be expressed as

$$\begin{aligned}\mathrm{BER} &= \frac{1}{2}\int_{-\infty}^{0}\rho(x,T_\mathrm{b}|A)\mathrm{d}x \\ &\quad + \frac{1}{4}\int_{-\infty}^{0}\rho(x,2T_\mathrm{b}|A)\mathrm{d}x + \cdots \\ &= \sum_{i=1}^{+\infty}\frac{1}{2^i}\int_{-\infty}^{0}\rho(x,i\cdot T_\mathrm{b}|A)\mathrm{d}x,\end{aligned} \tag{7}$$

where $\frac{1}{2^i}$ is the probability of the occurrence of the corresponding successive amplitude with the same symbol. In the following calculation, we take $i = 1, 2, 3$ because the prior probability and the probability of detection error with $i > 3$ are small so that the corresponding terms could be neglected.

3. The method of realizing PASR

The preceding section determines the BER of the system output. In this section, we employ it to characterize an SR and introduce the method of realizing a PASR. Firstly, we rescale the variables to reduce the number of the parameters. Secondly, by tuning D, the theoretical CSR level of D can be obtained. Finally, with the help of several equations (the rescaled parameters before changing D are equal to those under which D is tuned for the CSR), the parameters of realizing the PASR can be evaluated, i.e. tuning D, with the system parameters for CSR fixed, can be viewed as tuning the system parameters, with D fixed. The detailed discussion is as follows.

Rescaling the variables as

$$\begin{aligned}&\bar{t} = at, \quad y = \sqrt{\frac{a}{D}}x, \\ &\bar{b} = \frac{bD}{a^2}, \quad \bar{c} = \frac{c}{\sqrt{aD}},\end{aligned} \tag{8}$$

then Eq.(1) will be rewritten as

$$\dot{y}(\bar{t}) = y(\bar{t}) - \bar{b}y^3(\bar{t}) + \bar{c}s(\bar{t}) + \xi(\bar{t}), \tag{9}$$

where

$$< \xi(\bar{t}) >= 0 \text{ and } < \xi(0)\xi(\bar{t}) >= 2\delta(\bar{t}). \tag{10}$$

In terms of $\bar{t} = at$, the bit duration for $s(\bar{t})$ is $\bar{T_{\rm b}} = aT_{\rm b}$. Obviously, the BER of system (9) with the new parameter scale is equivalent to that of system (1). Assume the initial parameter group (a, b, c, D) to be (a_0, b_0, c_0, D_0), then, according to the expressions (8), we will be able to transform the increase of initial noise intensity $D = D_0$ for realizing the CSR into the variation of a, b and c so that the SR is produced by tuning the parameters, with D_0 fixed. Although the BER does not change as the output x is multiplied by a constant (generally, the quantities for qualifying the system performance are not influenced by the output multiplied by a constant, for example, signal-to-noise ratio and normalized cross-correlation coefficient[2]) for the generality of the method, we adjust the parameters by

$$\begin{aligned} y &= \sqrt{\frac{a_0}{D_c}}x = \sqrt{\frac{a_1}{D_0}}x, \quad \bar{b} = \frac{b_0 D_c}{a_0^2} = \frac{b_1 D_0}{a_1^2}, \\ \bar{c} &= \frac{c_0}{\sqrt{a_0 D_c}} = \frac{c_1}{\sqrt{a_1 D_0}}, \end{aligned} \tag{11}$$

where the subscript c denotes the value of the noise intensity being the CSR level. The expressions (11) mean that tuning the noise intensity, with the parameters a_0, b_0 and c_0 fixed, can be viewed as tuning the parameters, with the noise intensity D_0 fixed. From these equations, one has

$$a_1 = \frac{a_0 D_0}{D_c}, \quad b_1 = \frac{b_0 D_0}{D_c}, \quad c_1 = \frac{c_0 D_0}{D_c}. \tag{12}$$

Hence, the output and the parameters of system (9) can be expressed as

$$\begin{aligned} y &= \sqrt{\frac{a_1}{D_0}}x, \quad \bar{b} = \frac{b_1 D_0}{a_1^2}, \\ \bar{c} &= \frac{c_1}{\sqrt{a_1 D_0}}, \quad \bar{T_b} = aT_b. \end{aligned} \tag{13}$$

Transform system (9), with their parameters taken as those in the expressions (13), into the form of the original parameter scale, i.e.

$$\begin{aligned} \frac{1}{\sqrt{a_0 D_0}}\dot{x}(t) = & \sqrt{\frac{a_0}{D_0}}x(t) - \frac{a_0\sqrt{a_0}b_1}{a_1^2\sqrt{D_0}}x^3(t) \\ & + \frac{c_1}{\sqrt{a_1 D_0}}s(t) + \frac{\Gamma(t)}{\sqrt{a_0 D_0}}, \end{aligned} \tag{14}$$

and multiply both sides of Eq.(14) by $\sqrt{a_0 D_0}$, and use the expression (12), then the system with SR produced will be given as

$$\dot{x}(t) = a_p x(t) - b_p x^3(t) + c_p s(t) + \Gamma(t), \tag{15}$$

where the subscript p denotes the values of the parameters obtained via the method of tuning parameters. In Eq.(15) the noise intensity is $D = D_0$ and the other parameters are

$$a_p = a_0, \quad b_p = \frac{b_0 D_c}{D_0}, \quad c_p = c_0\sqrt{\frac{D_0}{D_c}}. \tag{16}$$

From the above expressions, it is clear that the variation from D_0 to D_c is converted into that from b_0 and c_0 to b_p and c_p, and a_0 need not be changed. It is pointed out that the group of b_p and c_p may not be unique because a different method of tuning parameters may lead to a different group of b_p and c_p.

In the following, the role played by the noise, with D_0 fixed, is investigated theoretically and numerically in order to check whether there exists a resonance phenomenon in the system, with the parameters taken as those in the expression (16). Take $A = 1\,\text{mV}$, $T_{\rm b} = 1.0\,\text{s}$, $a_0 = 10$, $b_0 = 100\,\text{mV}^{-2}$ and $D_0 = 0.1\,\text{mV}^2$, where mV denotes the unit of voltage. From the expression (16), $a_p = 10$, $b_p = 150\,\text{mV}^{-2}$ and $c_p \approx 0.82$ can be evaluated. Figure 1 shows theoretical BER (lines) and numerical BER (symbols) as a function of noise intensity D. We note that theoretical and numerical results match well, so the theoretical BER introduced above is valid. As depicted in Fig.1, $D_c \approx 0.15\,\text{mV}^2$ can be obtained by varying the noise intensity. The block scheme of the numerical simulation is drawn from Ref.[16]. It can be implemented by using the simulation software MATLAB.

Figure 1 shows the minimal BER appearing at the initial noise intensity $D_0 = 0.1$, which indicates that the noise plays a constructive role when the system parameters are taken as those in Eq.(16), i.e. the system produces a resonance phenomenon at a fixed noise intensity $D = D_0$. Accordingly, the PASR is really realizable in the system (1) and by tuning the parameters it can be produced in the system (15). The minimal BERs in the two curves are equal, so the cooperative effects produced by the PASR and CSR are the same. In spite of the same results provided by the system (1) and system (15), they are completely different. The PASR could be interpreted as changing the intrinsic characteristic of the dynamical system and the signal intensity to yield the cooperative effect between the

stochastic nonlinear system and the input aperiodic signal. For the CSR, it is realized by changing the stochastic force to produce the cooperative effect.

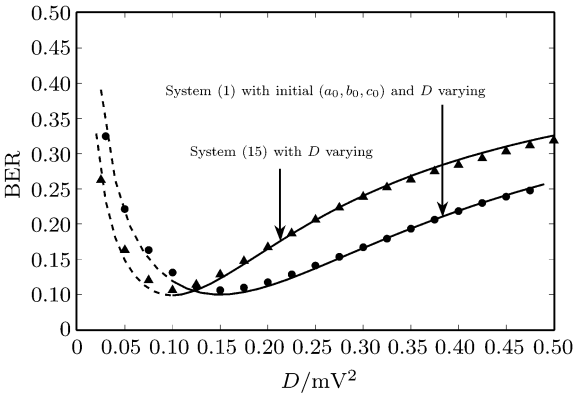


Fig.1. The BER as a function of noise intensity D with $A = 1\,\mathrm{mV}$, $T_\mathrm{b} = 1.0\,\mathrm{s}$, $a_0 = 10$, $b_0 = 100\,\mathrm{mV}^{-2}$, $c_0 = 1$ and $D_0 = 0.1\,\mathrm{mV}^2$. Lines denote theoretical results and symbols represent numerical results. Dashed lines denote the noise intensity that cannot be reduced. For $D_0 < D_c$, the CSR can be realized by adding the noise.

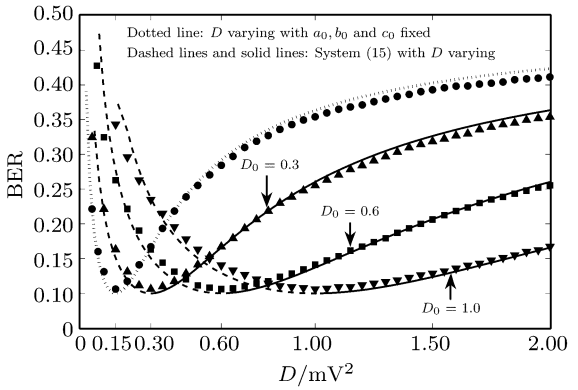


Fig.2. The BER as a function of noise intensity D with different values of D_0 (greater than resonance level $0.15\,\mathrm{mV}^2$) and other parameters being the same as those in Fig.1. Lines denote theoretical results, and symbols represent numerical results. Dashed lines denote the noise intensity that cannot be reduced. For $D_0 > D_c$, the CSR cannot be realized by increasing the noise intensity, while the PASR still can be realized at initial D_0 by taking the parameters as those in the expression (16).

It is well known that to realize a CSR, the parameter condition of signal amplitude $A \leq \sqrt{4a^3/(27b)}$ should be satisfied, but may not strictly.[17,18] However, for the PASR, because the parameters are adjustable, this condition is not necessary to be satisfied. Furthermore, as the noise cannot be reduced in practice, the CSR is realized by adding a noise only when the initial noise intensity is not greater than the resonance level, while for the PASR, since D_c and the parameters in Eq.(16) can always be obtained theoretically, at any given value of D_0, the PASR can be realized in practice as illustrated in Fig.2.

4. Conclusions

In conclusion, we have presented another method to realize an aperiodic SR, and theoretically and numerically demonstrated that the SR can be produced by tuning the system parameters and the signal intensity in a nonlinear system with an input aperiodic signal. And the results show that theoretical and numerical BERs have a good match. As the PASR is realized, the BER of the system shows a minimum, with the noise fixed at its initial intensity D_0. Hence, although the noise is not changed, by taking the parameters as those in the expression (16), i.e. by changing the intrinsic characteristic of the nonlinear system and the signal intensity, the stochastic system and the external forcing can yield the cooperative effect similar to that produced by adding the noise. Since the SR occurs between stochastic system and the external forcing, the resonance realized as well by changing the system and the aperiodic signal (external forcing) is reasonable as by tuning the stochastic noise. It seems the ordinary dynamical resonance in mechanics which could be realized by adjusting the inherent frequency of the system, or the frequency of an external periodic forcing, or both of them (it may not be necessary actually). Moreover, it has been shown that for a parameter-adjustable nonlinear system, the PASR is more realizable, especially when the input noise intensity exceeds the resonance level, or when we do not know the characteristic of the noise (generally, it is non-Gaussian).

Appendix A. Derivation of the dynamical probability density

Assume a solution of Eq.(3) to be

$$\rho(x,t|\pm A) = \psi(x,t)\exp\left[\frac{1}{2}V_{\pm A}(x)\right]. \qquad (17)$$

Substituting the expression (17) into Eq.(3) gives

$$\frac{\partial\psi(x,t)}{\partial t} = \frac{\partial^2\psi(x,t)}{\partial x^2} - F(x)\psi(x,t), \qquad (18)$$

where $F(x) = \frac{1}{2}V''_{\pm A}(x) + \frac{1}{4}V'^2_{\pm A}(x)$. Define $\psi(x,t)$ as $\psi(x,t) = u(x)\exp(-\lambda t)$. Substituting $\psi(x,t)$ into Eq.(18) yields

$$-\lambda u(x) = u''(x) - F(x)u(x). \tag{19}$$

This is an eigenvalue problem of the differential operator (18). Obviously, the right-hand side of Eq.(18) may be viewed as a self-conjugate differential operator acting on $\psi(x,t)$, so the eigenvalues of Eq.(19) are real and the eigenfunctions must be orthogonal to each other. Multiplying both sides of Eq.(19) by $u(x)$ and integrating it from minus infinity to infinity with respect to $\lim_{x\to\pm\infty}u(x) = \lim_{x\to\pm\infty}u'(x) = 0$, the equation can be transformed into a Rayleigh quotient form, i.e.

$$\lambda = \text{st.}_{u\neq 0}\frac{\int_{-\infty}^{+\infty}[F(x)u^2(x) + u'^2(x)]\mathrm{d}x}{\int_{-\infty}^{+\infty}u^2(x)\mathrm{d}x}, \tag{20}$$

where st. denotes the stationary value in variational problems. The evaluation of λ is introduced in Ref.[19], which introduces a method of optimizing system parameters. The minimum eigenvalue $\lambda_0 = 0$ and the corresponding eigenfunction $u_0(x) = \exp\left[\frac{1}{2}V_{\pm A}(x)\right]$, so $\rho_0(x,t|\pm A) = \exp[V_{\pm A}(x)]$ is the steady state solution. The minimum nonzero eigenvalue λ_1 is the system response speed. The solution corresponding to eigenvalue λ_j for $j = 1, 2, \cdots$ is

$$\begin{aligned}\rho_j(x,t|\pm A) &= u_j(x)\exp\left[\frac{1}{2}V_{\pm A}(x)\right]\exp(-\lambda_j t)\\ &= \varphi_j^{\pm A}(x)\exp(-\lambda_j t).\end{aligned} \tag{21}$$

Therefore, the complete solution of Eq.(3) can be expressed as

$$\rho(x,t|\pm A) = \sum_{j=0}^{\infty}c_j\varphi_j^{\pm A}(x)\exp(-\lambda_j t), \tag{22}$$

where $\varphi_0^{\pm A}(x) = \exp[V_{\pm A}(x)]$, c_0 can be obtained from the normalization condition of the steady state probability density, and constant coefficients c_j's ($j = 1, 2, \cdots$) can be determined by the initial condition and the orthogonality of the eigenfunctions.

Assume that a bit symbol takes the level $-A$ and after time $t = iT_b$, the level $+A$ is transmitted, and use the expression (22), then we will have

$$\begin{aligned}\rho(x,t|+A) \simeq &\sum_{j=0}^{n}c_j\varphi_j^{+A}(x)\exp(-\lambda_j t)\\ &+Q(x)\exp(-\lambda_{n+1}t)\\ &+O(\mathrm{e}^{-\lambda_{n+2}t}),\end{aligned} \tag{23}$$

where $Q(x)\exp(-\lambda_{n+1}t)$ is a truncated term. Here, we assume $\lambda_1 < \lambda_2 < \cdots < \lambda_{n+2}$. The above equation fulfils the initial condition

$$\rho(x,0|+A) = \rho_0(x|-A), \tag{24}$$

where $\rho_0(x|-A)$ is the steady state solution with the input level $-A$. Hence, omitting the trace quantity $O(\mathrm{e}^{-\lambda_{n+2}t})$, from the expressions (23) and (24), $Q(x)$ can be obtained by

$$Q(x) = \rho_0(x|-A) - \sum_{j=0}^{n}c_j\varphi_j^{+A}(x). \tag{25}$$

Substituting the expression (25) into the expression (23) gives

$$\begin{aligned}\rho(x,t|+A) \simeq &\sum_{j=0}^{n}c_j\varphi_j^{+A}(x)\exp(-\lambda_j t)\\ &+\left[\rho_0(x|-A) - \sum_{j=0}^{n}c_j\varphi_j^{+A}(x)\right]\\ &\times\exp(-\lambda_{n+1}t).\end{aligned} \tag{26}$$

Obviously, with the system close to the steady state (i.e. $t\to+\infty$), the above equation is still satisfied. Similarly, $\rho(x,t|-A)$ can be evaluated.

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