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Ground-state entanglement in a three-spin transverse Ising model with energy current*

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The ground-state entanglement associated with a three-spin transverse Ising model is studied. By introducing an energy current into the system, a quantum phase transition to energy-current phase may be presented with the variation of external magnetic field; and the ground-state entanglement varies suddenly at the critical point of quantum phase transition. In our model, the introduction of energy current makes the entanglement between any two qubits become maximally robust.

Keywords: entanglement, energy current, quantum phase transition
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1. Introduction

It is well known that quantum entanglement is one of the significant ingredients for quantum-information processing and quantum computation (QC).[1,2] The security of quantum communication and the advantage of QC rely crucially on the quantum entanglement. On the other hand, it has been pointed out only recently that entanglement may play an essential role in the description and understanding of quantum phenomena presented in many-body condensed-matter systems, such as superconductivity[3] and quantum phase transition.[4] Therefore, much effort has been made in the past few years to investigate the quantum entanglement in various physical systems. Among such systems exactly solvable one-dimensional (1D) quantum spin systems (spin chains), e.g. the Ising model describing a chain of interacting spin-1/2 particles, offer an excellent theoretical framework for investigating the entanglement properties. A great number of spin chain models with various interactions and anisotropies have been extensively studied in the context of quantum information science.[5−7]

Exhibiting a variety of interesting phenomena signifying their quantum nature, 1D quantum spin systems have been the subject of intense theoretical and experimental studies. Especially, many 1D integrable quantum many-body systems, which are characterized by a macroscopic number of conservation laws, show anomalous dissipationless transport behaviours.[8] It has been shown, for example, that the transport of energy (energy current) is closely related to the first conservation law in these systems.[9] In Ref.[10], the authors suggested a method of studying the nonequilibrium steady state (i.e. the ground state) of a transverse Ising chain by imposing an energy current on the system, and investigated the effect of the current on some quantum properties, such as the long-range correlation, phase diagram and so on. Importantly, the energy current has been shown to have a measurable effect in a realistic experimental setup.[11] Motivated by these works, we will explore the entanglement properties of a transverse Ising chain, where the only conserved quantity is the energy, by imposing an energy current on the system. We aim to provide more information about entanglement properties in a nonequilibrium steady state of a dynamic system, and study the changes produced by the presence of a current in the quantum entanglement and in the quantum phase of the system. It is additionally hoped that the connection between quantum entanglement and quantum phase transition will be more clear with the investigation. Furthermore, it has been shown that the introduction of an energy current into the XX spin chain results in more entanglements in the system.[12] Thus

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it is natural to inquire whether a similar effect could be obtained in a transverse Ising spin chain.

The present paper further explores the well-known Ising model of a three-spin chain (ring) by imposing an energy current on the system. We construct an effective Hamiltonian of the Ising spin chain with energy current by using the Lagrange multiplier method (Section 2). We present the explicit spectrum of the effective Hamiltonian of the system, and investigate the properties of two-potential lowest-energy level (Section 3). We further analyse the pairwise entanglement and residual entanglement in the ground state, and discuss quantum phase transition at zero temperature (Section 4).

2. Ising model with energy current

The Hamiltonian of an Ising spin chain in a transverse magnetic field along z axis has the form[10]

$$H_I = -J_x \sum_i^N \sigma_i^x \otimes \sigma_{i+1}^x - \frac{h}{2} \sum_i^N \sigma_i^z, \tag{1}$$

where σ_i^α, $\alpha = x, y, z$ are the well-known Pauli operators on the ith spin. The first term in the Hamiltonian is the Ising interaction between two neighbouring spins with the coupling strength J_x, which is set to $J_x = 1$ throughout the paper. The transverse field h is measured in units of the coupling J_x. Here we will consider the system in a 1D periodic chain ($\sigma_{N+1}^\alpha = \sigma_1^\alpha$).

In the model, only the total energy is conserved. The conserved quantity of the system is represented by the energy current operator.[9] We construct the energy current operator as follows.[10] We can write down a continuity equation for the local energy ε_i at the ith site:

$$\dot{\varepsilon} = \mathrm{i}[H_I, \varepsilon] = j_i^{\mathrm{E}} - j_{i+1}^{\mathrm{E}}. \tag{2}$$

Here, j_i^{E} is defined as local energy current of the ith spin,

$$j_i^E = \frac{h}{2}(\sigma_{i-1}^x \sigma_i^y - \sigma_{i-1}^y \sigma_i^x), \tag{3}$$

and the macroscopic current J^{E} is expressed as the sum of local energy currents

$$J^{\mathrm{E}} = \frac{h}{2} \sum_i^N (\sigma_{i-1}^x \sigma_i^y - \sigma_{i-1}^y \sigma_i^x). \tag{4}$$

The next question is how to find current-carrying state in the system. We first construct an effective Hamiltonian whose eigenstates may carry an energy current. Easily, one can find that the macroscopic current commutes with the Hamiltonian (1), i.e. $[J^{\mathrm{E}}, H_I] = 0$. In other words, the system Hamiltonian and the macroscopic current operator have a series of common eigenstates. In order to impose a fixed energy current J^{E} on the states of the system, we add the current operator to the Hamiltonian H_I using the Lagrange multiplier method,

$$H = H_I - \lambda J^{\mathrm{E}}, \tag{5}$$

where, without loss of generality, $h \geq 0$ and $\lambda \geq 0$ are assumed. Now, an eigenstate of the effective Hamiltonian H, obviously, must still be a stationary state of H_I. However, these eigenstates may carry the energy current described by J^{E}, which is just desired in our investigation. In this paper, we mainly aim to investigate the lowest-energy states of H_I which carry a given energy current. The ground state of H gives a current-carrying steady state of H_I at zero temperature, and it can be analysed as a function of magnetic field h and energy current $J = \langle J^{\mathrm{E}}/N \rangle$. It is noteworthy that H is just another equilibrium Hamiltonian mathematically constructed to give nonequilibrium state of H_I, and the time evolution of the system is still governed by H_I.

3. Spectrum of the Hamiltonian

In order to investigate more easily the energy-carrying state in a transverse Ising chain, in this paper we only discuss a relatively simple system including only three spin-1/2 particles.

Due to the commutation of J^{E} and H_I, one can diagonalize H using the same method that diagonalizes H_I, and the current-carrying states can be obtained as stationary states of H_I.

The effective Hamiltonian can be directly diagonalized and its spectrum can be easily calculated,

$$\begin{aligned}
E_1 &= -\left(1 + \frac{h}{2} + \sqrt{4 - 2h + h^2}\right), \\
E_2 &= -\left(1 + \frac{h}{2} - \sqrt{4 - 2h + h^2}\right), \\
E_3 &= -\left(1 - \frac{h}{2} + \sqrt{4 + 2h + h^2}\right), \\
E_4 &= -\left(1 - \frac{h}{2} - \sqrt{4 + 2h + h^2}\right),
\end{aligned} \tag{6}$$

and

$$E_5 = 1 - \frac{h}{2} - \sqrt{3}h\lambda,$$
$$E_6 = 1 + \frac{h}{2} - \sqrt{3}h\lambda,$$
$$E_7 = 1 - \frac{h}{2} + \sqrt{3}h\lambda,$$
$$E_8 = 1 + \frac{h}{2} + \sqrt{3}h\lambda. \quad (7)$$

Obviously, the eigenvalues E_1, E_2, E_3, and E_4 are not related to λ. The energy levels from E_5 to E_8, on the contrary, are functions of λ, which implies that energy flows through the spin chain when the system occupies these levels. Additionally, the energy levels $E_5 = E_7$ and $E_6 = E_8$ are degenerate when $\lambda = 0$, and these levels will split with the introduction of energy current. Comparing all these levels, one can easily find that E_1 or E_5 may be the lowest energy level for different h and λ. In Fig.1, we plot the variations of levels E_1 and also level E_5 with h. We find a level crossing of E_1 and E_5, at which a quantum phase transition to energy-current phase occurs, for some h and λ. At the point of level crossing,

$$\lambda = \frac{2 + \sqrt{4 - 2h + h^2}}{\sqrt{3}h}. \quad (8)$$

With decreasing λ, the value of h corresponding to the level crossing increases. After a limit calculation, when $\lambda \to \lambda_c = \frac{1}{\sqrt{3}}$, $h \to \infty$. In other words, if the driving field λ exceeds the critical value λ_c the energy in the system at zero temperature starts to flow with the increase of the field h above its critical value h_c, and the quantum phase transition is introduced by the introduction of energy current.

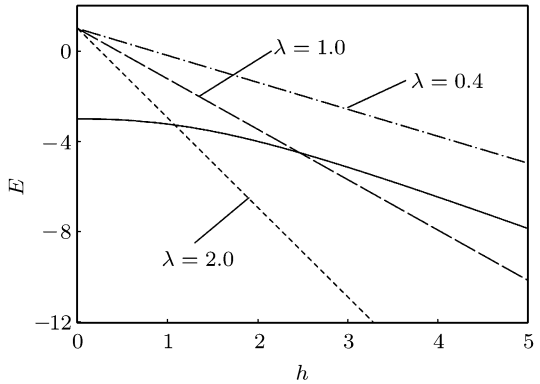


Fig.1. The variation of energy levels E_1 (solid line) and E_5 with magnetic field h for three values of λ (dotted and dashed lines).

4. Ground-state entanglement

In this section, we explore entanglement properties of the current-carrying ground state in a transverse Ising chain. Firstly, we give a kind of entanglement measure to quantify the entanglement. A well-known way to quantify the entanglement is by means of the concurrence.[13−15] To calculate this, we define the product matrix R corresponding to density matrix ρ of the two-qubit system as

$$R = \rho(\sigma_1^y \otimes \sigma_2^y)\rho^*(\sigma_1^y \otimes \sigma_2^y). \quad (9)$$

Now concurrence of entanglement between two particles in ρ is defined as

$$\mathcal{C} = \{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \quad (10)$$

where the quantities λ_i are the square roots of the eigenvalues of the operator R in descending order. In this method, the standard basis, $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, must be used. As usual for entanglement measures, the value of the concurrence ranges from 0 for an unentangled state, to 1, for a maximally entangled state. Here, we mainly focus on the pairwise entanglement associated with any two qubits of the system. In addition, we define the reduced density matrices as $\rho_{12} = \mathrm{Tr}_3[\rho_{123}]$, $\rho_{23} = \mathrm{Tr}_1[\rho_{123}]$, $\rho_{13} = \mathrm{Tr}_2[\rho_{123}]$, and we have $\rho_{12} = \rho_{23} = \rho_{13} = \rho$ under the symmetric and periodic condition assumed.

Moreover, Coffman *et al*[15] presented an entanglement measure, named the residual entanglement τ, for three-qubit pure states, $\tau = \mathcal{C}^2_{i(jk)} - \mathcal{C}^2_{ij} - \mathcal{C}^2_{ik}$, $(i, j, k \in \{1, 2, 3\})$. Here, $\mathcal{C}^2_{i(jk)}$ is bipartite entanglement of qubit i with the pair jk which is regarded as a single object. The residual entanglement is introduced to describe the global property of tripartite entanglement of a pure state. It has been shown that the residual entanglement in the measure of concurrence is 0, for a W-class state, and 1, for a GHZ-class state.

The potential ground state of the system for different h and λ can be directly calculated. Corresponding to E_1, the eigenstate of H is

$$\phi_1 = \frac{1}{\sqrt{a^2 + 3}}(a|000\rangle + |011\rangle + |101\rangle + |110\rangle), \quad (11)$$

and to E_5,

$$\phi_5 = \frac{1}{\sqrt{3}}(-\frac{1+\sqrt{3}\mathrm{i}}{2}|001\rangle - \frac{1-\sqrt{3}\mathrm{i}}{2}|010\rangle + |100\rangle). \tag{12}$$

Here, $a = h - 1 + \sqrt{4 - 2h + h^2}$. One can find that increasing a increases the probability of the product state $|000\rangle$ in ϕ_1 which only decreases the entanglement associated with ϕ_1, but ϕ_5 remains unchanged.

Consider now the properties of entanglement associated with the eigenvector ϕ_1. The reduced density matrix of any two qubits in the chain has the form

$$\rho = \frac{1}{a^2+3}\begin{pmatrix} a^2 & 0 & 0 & a \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ a & 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

Then, the concurrence of the pairwise entanglement can be obtained

$$\mathcal{C}_{12} = \frac{1}{a^2+3}\max\{2(a-1), 0\}. \tag{14}$$

The reduced density matrix ρ_i of the ith qubit is calculated as a trace over all remaining qubits in the chain. In our case, all the reduced density matrices of a single qubit have the same form due to the symmetric and periodic condition. Therefore, the density operator of the qubit 1 in the spin chain with the eigenstate ϕ_1 can be similarly calculated as

$$\rho_1 = \frac{1}{a^2+3}\begin{pmatrix} a^2+1 & 0 \\ 0 & 2 \end{pmatrix}. \tag{15}$$

The entanglement between a single qubit and the remaining two qubits, e.g. qubit 1 and 23, can be expressed as $\mathcal{C}_{1(23)} = 2\sqrt{|\rho_1|}$. Then the residual entanglement associated with the eigenstate ϕ_1 can be expressed as $\tau = \mathcal{C}^2_{1(23)} - 2\mathcal{C}^2_{12}$. In Fig.2 we present the three entanglements as functions of the magnetic field h. The figure is fully equivalent to Fig.2 given in Ref.[16]. For the pairwise entanglement, it has been shown that the maximum concurrence of entanglement between any pair of qubits of an N-qubit symmetric state is $2/N$. From Fig.2, we can see that the maximum pairwise entanglement is far less than the tightly bound.

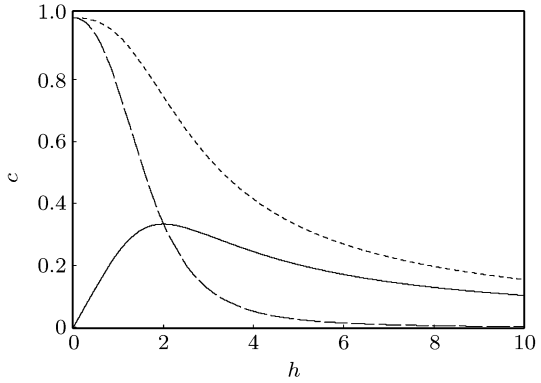


Fig.2. The entanglement in the ground state ϕ_1 of the transverse Ising chain as a function of h: the pairwise entanglement $\mathcal{C}_{12}$ between the qubits 1 and 2 (solid line), the bipartite entanglement $\mathcal{C}_{1(23)}$ between the first qubit and the remaining two qubits(dotted line), and the residual entanglement τ(dashed line).

For the eigenstate ϕ_5, it can be easily found that it is a W state.[17] Then, the concurrence of pairwise entanglement of any two qubits in ϕ_5 is equal to 2/3, and the residual entanglement is $\tau = 0$.

Now, we discuss the ground-state entanglement of a three-qubit system. As noted above, a level crossing in the ground state of the system may occur when $\lambda > 1/\sqrt{3}$, and no level crossing when $\lambda \leq 1/\sqrt{3}$. Here, we focus our attention on the former. For the case of $\lambda > 1/\sqrt{3}$, there exist two different phases in the ground state, i.e. the no-energy-current phase, $h < h_c$, and the energy-current phase, $h > h_c$. The critical point of quantum phase transition is located at the magnetic field point $h_c = 2(2\sqrt{3}\lambda - 1)/(3\lambda^2 - 1)$. In the no-energy-current phase, in which the eigenstate ϕ_1 is the ground state, its energy current $J = \langle J^E/N\rangle = 0$, and the pairwise entanglement and residual entanglement have been discussed above. When the magnetic field h exceeds the critical value h_c, the ground state jumps from ϕ_1 to ϕ_5 and turns into the energy-current phase with $\langle J^E/N\rangle = \sqrt{3}h/3$. In this phase, both the pairwise entanglement and the residual entanglement are constants, i.e. 2/3 and 0, respectively. In other words, at the critical point h_c the pairwise entanglement and the residual entanglement associated with the ground state undergo a sudden change with increasing h. The pairwise entanglement increases to 2/3 and the residual entanglement vanishes suddenly. This implies that the introduction of energy current enhances the entanglement between any two qubits and drives the ground state to a maximally robust entangled state under the disposal of any one of the three qubits. (see Fig.3).

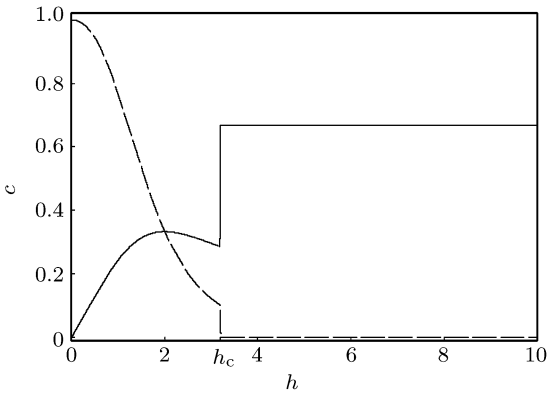


Fig.3. The variation of entanglement in the ground state of the transverse Ising chain for $\lambda = 3.2$ with h: the pairwise entanglement $\mathcal{C}_{12}$ between the qubits 1 and 2 (solid line), and the residual entanglement τ(dashed line).

5. conclusions

In conclusion, in this paper we have performed the investigation on the ground-state entanglement of a three-spin Ising model in an external magnetic field by introducing an energy current. We have shown that the introduction of an energy current can result in a quantum phase transition for $\lambda > 1/\sqrt{3}$ with the increase of external magnetic field. When the magnetic field exceeds its critical point, the system becomes the energy-current phase and the energy starts to flow. From no-energy-current phase to energy-current phase, the entangled ground state jumps to a W-class state, in which the entanglement between any two qubits becomes maximally robust and the residual entanglement vanishes suddenly.

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