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Entanglement-enhanced classical communication through generalized amplitude damping channel*

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The problem of sending a single classical bit through a generalized amplitude damping channel is considered. When two transmissions through the channel are available as a resource, we find that two entangled transmissions can enhance the capability of receiver's judging information correctly under certain conditions compared with two product-state transmissions. In addition, we find a special case in which the two entangled transmissions can always make a classical bit more effectively disable the noise influence.

Keywords: two transmissions, entanglement, generalized amplitude damping channel, the average probability of error

PACC: 0365

1. Introduction

A quantum channel is a completely positive and trace-preserving mapping into quantum states. We often use quantum channels as communication channels,[1−7] and therefore how to reliably transmit information through them is one of the most basic problems in quantum information theory. In order to solve this problem, many researchers have provided different methods. For example, Bowen[8] showed that quantum superdense coding using previously shared entanglement could enhance classical communication; Sohma and Hirota [9] indicated that two-mode squeezed states could enhance classical communication over quantum Gaussian channel, and Yeo and Skeen [10] proved that the use of entangled input signals might enhance the information transmission compared with the use of product input signals through time-correlated quantum amplitude damping channel. In Ref.[11], Bennett *et al* demonstrated that two entangled transmissions could enhance classical communication through two-pauli channel and predicted that this effect could be a property of most noisy quantum channels. However, to the best of our knowledge, further demonstration of this forecast of using other quantum channels has not been reported yet so far.

The generalized amplitude damping channel (GADC) is one of very important noisy quantum channels and it can describe the effect of dissipation to an environment at a finite temperature.[12] In this paper, we use a GADC as effectively as possible to perform the communication of a single bit 0 or 1, with only two transmissions through this channel permitted. Alice, the sender, may take any way, as long as it is permitted by the laws of quantum mechanics, for encoding the bit in the two transmissions; Bob, the receiver, may also use any means permitted by the laws of quantum mechanics for measuring the output of the channel in an attempt to infer the actual bit. Classically, any attempt to protect the bit from noise by redundancy always fails in using only two transmissions, because Bob cannot perform any sort of 'majority vote' error correction to two transmissions. However, in a quantum field, these things will become very interesting. Making use of the numerical calculation, we obtain the average probability of error in the receiver's identifying the bit that the sender has sent him. Also we prove that the entangled transmissions through the GADC are more effective on disabling the noise influence than the product-state transmissions under certain conditions.

The rest of this paper is organized as follows. In Section 2, we develop the general formula about the average probability of error in the receiver's inferring a

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classical bit that sender has sent him through a noisy quantum channel, and introduce the GADC. In Section 3, we study the encoding for two transmissions through the GADC and find that a particular entangled encoding for two transmissions does enhance classical communication compared with an encoding for two product-state transmissions. Finally, we review our results and draw some conclusions in Section 4.

2. Noisy quantum channels

The concept of noisy quantum channel arose from quantum communication. Information is encoded in quantum states and transmitted across some channel where the receiver decodes the information to retrieve the original message. The ability to reliably send messages can be affected by the noise present in the channel. The influence of the noise consists in capturing an initial quantum state and then transforming it into another quantum state. The noisy quantum channel is determined by a mapping

$$\rho \longrightarrow \phi(\rho), \tag{1}$$

which maps a quantum state described by a density operator ρ into a quantum state described by a density operator $\phi(\rho)$. There are certain restrictions on the class of mappings that generate legitimate density operators. In the most general case, we restrict our attention to completely positive, trace-preserving mappings,[13,14] which are called stochastic mappings.

Noise in a channel may be considered as a reservoir with which the quantum state of interest is coupled. The state and the reservoir interact unitarily for some time and they become correlated. If we are now interested only in the system, we trace over the environment degrees of freedom. One may think of the reservoir that can extract the information from the system as it has mapped pure states into mixed states. This noise process can be described by a quantum operator involving only operations on the system of interest. This is called a Kraus decomposition and has the form

$$\phi(\rho) = \sum_i A_i \rho A_i^+, \tag{2}$$

with the condition

$$\sum_i A_i^+ A_i = I \tag{3}$$

ensuring that the unit trace is preserved at all times.[15] If an operation has a Kraus decomposition, then it is completely positive, and *vice versa*. Therefore, all stochastic mappings have a Kraus decomposition.

In terms of the Kraus theorem,[15] the process of two transmissions through a channel is described by the mapping

$$R \longrightarrow \Phi(R) = \sum_{i,j} (A_i \otimes A_j) R (A_i \otimes A_j)^+, \tag{4}$$

where R denotes a density operator in the d^2-dimensional Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_d$.

In our particular problem, Alice encodes the classical bit that she wishes to transmit to Bob by preparing a quantum system in either of two states, R_0 and R_1, in $\mathcal{H}_d \otimes \mathcal{H}_d$. The action of the channel *via* Eq.(4) leads to one of two possible output density operators ($\tilde{R}_0$ or $\tilde{R}_1$) for Bob at the receiving end.

In order to distinguish between the two density operators, $\tilde{R}_0$ and $\tilde{R}_1$, we imagine Bob performing a general quantum mechanical measurement, or a positive operator-valued measure (POVM) $\{E_\mathrm{b}\}$, and then using the acquired data to venture a guess at the identity of the density operator. Depending upon which bit s Alice has sent, the probability of Bob's measurement outcomes will be given by $\mathrm{tr}(\tilde{R}_s E_\mathrm{b})$. Clearly, the best strategy for Bob to identify the bit, upon finding an outcome b, is to guess at such a value of s that $\mathrm{tr}(\tilde{R}_s E_\mathrm{b})$ is the largest. We assume that each input is an equally likely event, therefore an average probability of error is given by[11]

$$P_\mathrm{e}(\{E_\mathrm{b}\}) = \frac{1}{2} \sum_b \min\{\mathrm{tr}(\tilde{R}_0 E_b), \mathrm{tr}(\tilde{R}_1 E_b)\}. \tag{5}$$

Thus, it is clear that the best measurement on Bob's part is to choose $\{E_\mathrm{b}\}$ such that this expression is minimized. It turns out that this measurement can be described by a standard von Neumann measurement of the Hermitian operator $\Gamma = \tilde{R}_1 - \tilde{R}_0$.[16] Moreover, with this measurement, the expression (5) reduces to[17]

$$\begin{aligned} P_\mathrm{e} = & \frac{1}{2} - \frac{1}{4}\mathrm{tr}|\tilde{R}_0 - \tilde{R}_1| \\ = & \frac{1}{2} - \frac{1}{4}\mathrm{tr}|\Phi(R_0) - \Phi(R_1)|. \end{aligned} \tag{6}$$

Here $\mathrm{tr}|A|$, for any Hermitian operator A, should be interpreted as the sum of the absolute value of A's eigenvalues.

The GADC can be determined by following Krausian A_i operators,

$$A_1 = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\eta} \end{pmatrix},$$

$$A_2 = \sqrt{p}\begin{pmatrix} 0 & \sqrt{1-\eta} \\ 0 & 0 \end{pmatrix}, \tag{7}$$

$$A_3 = \sqrt{1-p}\begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & 1 \end{pmatrix},$$

$$A_4 = \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{1-\eta} & 0 \end{pmatrix}, \tag{8}$$

where $p \in [0,1]$ is a parameter that represents the temperature of the environment, and $\eta \in [0,1]$ is a parameter that describes dissipation, that is, $\eta = \mathrm{e}^{-\frac{t}{T}}$ where t is time and T is a constant characterizing the speed of the process. The stationary state of GADC is

$$\rho_\infty = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}, \tag{9}$$

which satisfies $\phi_{\mathrm{GADC}}(\rho_\infty) = \rho_\infty$. When $p = 0$ or 1, the channel reduces to an amplitude damping channel.[12]

3. Two transmissions

3.1. Two product-state transmissions

Firstly, we consider two product-state transmissions through a GADC. We assume two inputs to be in the following forms:

$$R_0 = \rho_+ \otimes \rho_+ \quad \text{and} \quad R_1 = \rho_- \otimes \rho_-, \tag{10}$$

where $\rho_\pm = \frac{1}{2}(I \pm \boldsymbol{a}\cdot\boldsymbol{\sigma})$ are in the Bloch sphere representation. Here $\boldsymbol{a} = (a_1, a_2, a_3)$ is an arbitrary real vector with a length of 1 or less, and $\boldsymbol{\sigma}$ is the vector of Pauli matrices.

By performing the mapping as indicated in the expression (4), we obtain

$$\Phi(R_0) - \Phi(R_1) = \begin{pmatrix} a_{11} & A & A & 0 \\ A^* & a_{22} & 0 & B \\ A^* & 0 & a_{33} & B \\ 0 & B^* & B^* & a_{44} \end{pmatrix}, \tag{11}$$

where

$$a_{11} = \eta[2p + (1-2p)\eta]a_3, \tag{12}$$

$$a_{22} = a_{33} = \eta(1-\eta)(1-2p)a_3, \tag{13}$$

$$a_{44} = \eta[-2(1-p) + (1-2p)\eta]a_3, \tag{14}$$

$$A = \frac{1}{2}\sqrt{\eta}[2p + (1-2p)\eta](a_1 - \mathrm{i}a_2), \tag{15}$$

$$B = \frac{1}{2}\sqrt{\eta}[2(1-p) - (1-2p)\eta](a_1 - \mathrm{i}a_2). \tag{16}$$

If we now specifically consider the channel for communication purpose, we should make the final states as distinguishable as possible. This means that we should pick the vector $\boldsymbol{a}$ such that R_0 and R_1 are optimal inputs. The Appendix of Ref.[11] has proved that the optimal input states for leading to the maximal distinguishability of the associated outputs will be orthogonal pure states. So, in this case, the optimal inputs will be orthogonal pure states but with $a_1 = a_2 = 0$ and $a_3^2 = 1$.

Therefore, Eq.(11) becomes

$$\Phi(R_0) - \Phi(R_1) = \begin{pmatrix} a'_{11} & 0 & 0 & 0 \\ 0 & a'_{22} & 0 & 0 \\ 0 & 0 & a'_{33} & 0 \\ 0 & 0 & 0 & a'_{44} \end{pmatrix} \tag{17}$$

with

$$a'_{11} = \pm\eta[2p + (1-2p)\eta], \tag{18}$$

$$a'_{22} = a'_{33} = \pm\eta(1-\eta)(1-2p), \tag{19}$$

$$a'_{44} = \pm\eta[-2(1-p) + (1-2p)\eta]. \tag{20}$$

Thus, the probability of error in guessing the identity of the states can be worked out easily, and it is

$$P_{\mathrm{e}} = \frac{1}{2} - \frac{1}{4}\{\eta[2p(1-\eta)+\eta] + 2\eta(1-\eta)|1-2p| + \eta|2(1-p) - (1-2p)\eta|\}. \tag{21}$$

From the expression (21), we know that P_{e} is determined by the channel parameters η and p when we choose the optimal input state, i.e. $P_{\mathrm{e}} = P_{\mathrm{e}}(p,\eta)$. Through numerical results, we have the following conclusions. (1) The value of P_{e} is symmetric about the channel parameter $p = 0.5$ as shown in Fig.1. This can also be proved by the equality $P_{\mathrm{e}}(0.5+x,\eta) = P_{\mathrm{e}}(0.5-x,\eta)$ for any $x \in [-0.5, 0.5]$. We also know that, for a fixed η ($\eta \neq 0, 1$), the value of P_{e} is maximal when $p = 0.5$. This shows that, when $p = 0.5$, the corresponding temperature of the environment is the worst in the sense of improving the capability of receiver's judging information correctly for the fixed η, because p is a parameter that represents the temperature of the environment. In particular, when $p = 0.5$, the expression (21) will reduce to

$$P_{\mathrm{e}} = \frac{1}{2}(1-\eta). \tag{22}$$

(2) For a fixed p, we find that the bigger the η is, the less the $P_{\rm e}$ is. This is obviously shown in Fig.2. Furthermore, for any p, when $\eta = 0$ we have $P_{\rm e} = 0.5$, and when $\eta = 1$ we have $P_{\rm e} = 0$. This is well identical with the fact that η is a parameter describing dissipation. The maximal value of $P_{\rm e}$ is 0.5.

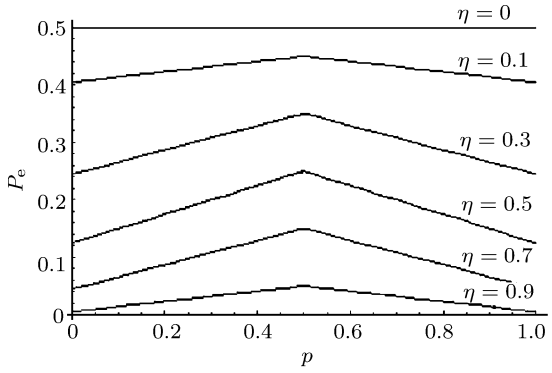


Fig.1. The average probabilities of error $P_{\rm e}$ of two product-state transmissions versus p for different values of η.

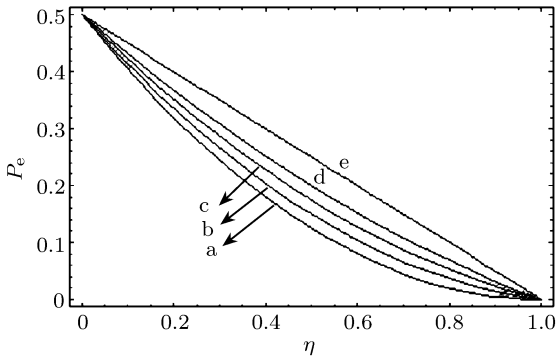


Fig.2. The average probabilities of error $P_{\rm e}$ of two product-state transmissions versus η for (a) $p = 0$, (b) $p = 0.1$, (c) $p = 0.2$, (d) $p = 0.3$, and (e) $p = 0.5$ respectively.

3.2. Two entangled transmissions

Now, we turn to the problem of sending entangled states through a GADC. A convenient basis, with which the input states for two transmissions are written, is the Bell operator basis, where the four orthogonal basis vectors are

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle \pm |\downarrow\rangle|\downarrow\rangle) \quad \text{and} \quad |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle \pm |\downarrow\rangle|\uparrow\rangle). \tag{23}$$

We use the usual notation as follows:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{24}$$

for the eigenvectors of σ_3. With respect to this basis, an arbitrary set of two input states can be written as

$$|0\rangle = a_0|\Phi^+\rangle + b_0|\Phi^-\rangle + c_0|\Psi^+\rangle + d_0|\Psi^-\rangle, \tag{25}$$

and

$$|1\rangle = a_1|\Phi^+\rangle + b_1|\Phi^-\rangle + c_1|\Psi^+\rangle + d_1|\Psi^-\rangle. \tag{26}$$

In order to make the entangled input states the orthogonal pure states, we must choose appropriate a_i, b_i, c_i and d_i $(i = 0, 1)$. The analysis shows that we can choose many equivalent optimal entangled signals. However, two input states can always be taken as being of the form

$$|0\rangle = \cos\alpha|\Phi^+\rangle + \sin\alpha|\Psi^+\rangle, \tag{27}$$

and

$$|1\rangle = -\sin\alpha|\Phi^+\rangle + \cos\alpha|\Psi^+\rangle \tag{28}$$

without any loss of performance.

Our goal is to calculate the average probability of error in Bob's judgement of the signal, i.e. $P_{\rm e}$. From the expression (6), we must know the eigenvalues of $\Phi(|0\rangle\langle 0|) - \Phi(|1\rangle\langle 1|)$. After tedious calculation, we obtain

$$\Phi(|0\rangle\langle 0|) - \Phi(|1\rangle\langle 1|) = \begin{pmatrix} A & B & B & C \\ B & -A & -C & D \\ B & -C & -A & D \\ C & D & D & A \end{pmatrix}, \tag{29}$$

where

$$A = \frac{1}{2}\eta^2\cos 2\alpha, \tag{30}$$

$$B = \frac{1}{2}[2p(1-\eta)+\eta]\sqrt{\eta}\sin 2\alpha, \tag{31}$$

$$C = \frac{1}{2}\eta\cos 2\alpha, \tag{32}$$

$$D = \frac{1}{2}[2(1-p)(1-\eta)+\eta]\sqrt{\eta}\sin 2\alpha. \tag{33}$$

Let $y = \cos 2\alpha$, then $-1 \le y \le 1$. Thus $P_{\rm e}$ should be determined by the input state parameter y and channel parameters η and p, i.e. $P_{\rm e} = P_{\rm e}(y, \eta, p)$. Because

$P_{\rm e}(y,\eta,p)=P_{\rm e}(-y,\eta,p)$, we consider only $0\le y\le 1$ in this paper. Through the numerical computation, we obtain the following conclusions. (1) For the fixed η and y, the values of $P_{\rm e}$ are symmetric about the channel parameter $p=0.5$, i.e. $P_{\rm e}(y,\eta,0.5+x)=P_{\rm e}(y,\eta,0.5-x)$, where $x\in[-0.5,0.5]$. For example, when we choose $y=0.4$, the values of $P_{\rm e}$ as a function of p for a fixed η are shown in Fig.3. (2) When p and y take certain values, we find that the bigger the η is, the less the $P_{\rm e}$ is. This conclusion is consistent with the fact that η is a parameter describing dissipation. In particular, when $\eta=0$, we have $P_{\rm e}(y,0,p)=0.5$; and when $\eta=1$, we have $P_{\rm e}(y,1,p)=0$. This shows that the maximal value of $P_{\rm e}$ is 0.5. These can be seen from Fig.4 where we still choose the input state parameter $y=0.4$. (3) When p and η ($\eta\neq 0,1$) are given, we can find that there exists a certain value $y=y_0$ that can make the value of $P_{\rm e}$ minimal. For example, if we choose $p=0.3$ and $\eta=0.4$, then $P_{\rm e}=0.1692$ is minimal when $y=0.2045$. It shows that the most optimal entangled inputs always exist for the fixed channel parameters p and η.

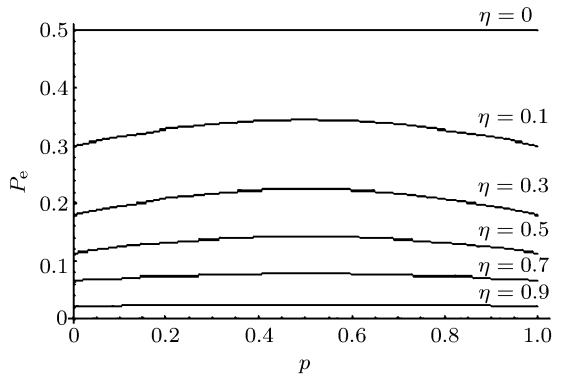


Fig.3. The average probabilities of error $P_{\rm e}$ of two entangled transmissions versus p for different values of η when input state parameter $y=0.4$.

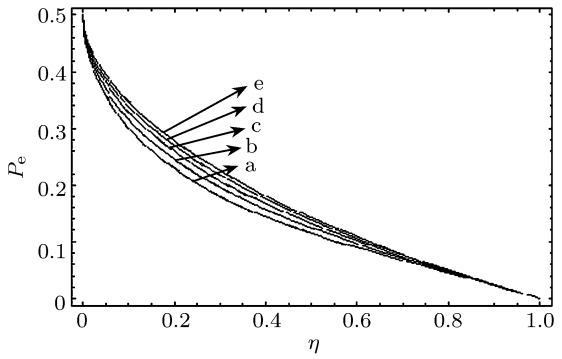


Fig.4. The average probabilities of error $P_{\rm e}$ of two entangled transmissions versus η for different values of p when input state parameter $y=0.4$. (a) $p=0$; (b) $p=0.1$; (c) $p=0.2$; (d) $p=0.3$; (e) $p=0.5$.

3.3. Comparisons

From the above analyses, we know that the variations of the average error probability $P_{\rm e}$ with channel parameters p and η are similar no matter what transmissions are used: two entangled transmissions or two product-state transmissions, when input states are given. However, numerical results show that two entangled transmissions are more reliable for carrying the bit from a sender to a receiver than two product-state transmissions under certain conditions. These conditions depend on the channel parameters p and η. For example, when $p=0.2$ and $\eta=0.3$, numerical results demonstrate that for $y<0.8576$, the optimal inputs are entangled states; however for $y>0.8576$, product-state inputs will give the minimal error probabilities.

Now we examine a particular example. When the channel parameter $p=0.5$, the entangled transmissions through the GADC always can enhance the capability of receiver's judging the information correctly compared with the product-state transmissions for any input parameter y and channel parameter η ($\eta\neq 0,1$).

When $p=0.5$, Eq.(29) reduces to

$$\Phi(|0\rangle\langle 0|)-\Phi(|1\rangle\langle 1|)=\begin{pmatrix} A' & B' & B' & C' \\ B' & -A' & -C' & D' \\ B' & -C' & -A' & D' \\ C' & D' & D' & A' \end{pmatrix}, \quad (34)$$

where

$$A'=\frac{1}{2}\eta^2 y, \quad (35)$$

$$B'=\pm\frac{1}{2}\sqrt{\eta}\sqrt{1-y^2}, \quad (36)$$

$$C'=\frac{1}{2}\eta y, \quad (37)$$

$$D'=\pm\frac{1}{2}\sqrt{\eta}\sqrt{1-y^2}. \quad (38)$$

The corresponding probability of error is

$$P_{\rm e}=\frac{1}{4}\{2-y(1-y)\eta -\sqrt{\eta[4+y^2(\eta^3+2\eta^2+\eta-4)]}\}. \quad (39)$$

To check the effectiveness of entangled transmissions, we plot $P_{\rm e}$ vs η in Fig.5 according to the expressions (22) and (39). From Fig.5, we know that, when the channel parameter $p=0.5$, the average error probability of entangled transmissions ($P_{\rm ee}$) is less than or equal to the average error probability of optimal product states transmissions ($P_{\rm ep}$), i.e. $P_{\rm ee}\le P_{\rm ep}$.

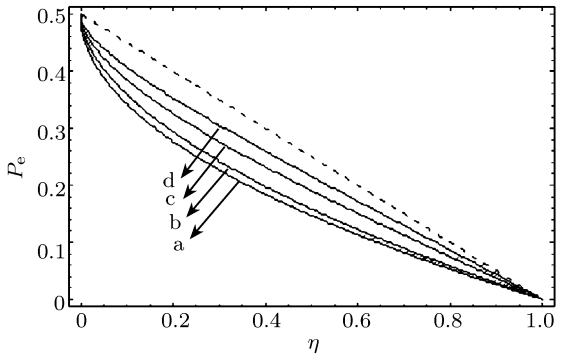


Fig.5. The comparison between the values of P_e for optimal product-state transmissions(dashed line) and the entangled transmissions(solid line)with different values of input state parameter y when $p = 0.5$ for (a) $y = 0$; (b) $y = 0.6$; (c) $y = 0.8$; (d) $y = 0.9$.

In other words, the entangled transmissions through the GADC can always enhance the capability of receiver's judging information correctly in this situation. This is very useful for the information communication in quantum information. Through choosing appropriate channel parameters and encoding the bits in appropriate input states, we can improve the capability of receiver's judging information correctly.

4. Conclusions

In conclusion, we have considered the problem of sending a single classical bit through a GADC. By increasing our resources from one to two transmissions and allowing those transmissions to be entangled, we can make a classical bit sent through the GADC more effectively disable the noise influence under certain conditions. These conditions depend on the channel parameters. In particular, we find a particular case in which two entangled transmissions always enhance classical communication compared with two product-state transmissions. This has no analogue with classical information theory. More work is underway to make further considerations about this effect of two entangled transmissions through other noisy quantum channels. In addition, we will consider in detail the question whether the classical information capacity of a noisy quantum channel can be increased by two entangled transmissions.

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