Vol 16 No 2, February 2007
1009-1963/2007/16(02)/0310-08

Chinese Physics

© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

New travelling wave solutions for combined KdV−mKdV equation and (2+1)-dimensional Broer−Kaup−Kupershmidt system*

Yang Xian-Lin(杨先林)† and Tang Jia-Shi(唐驾时)

Department of Engineering Mechanics, Hunan University, Changsha 410082, *China*

(Received 4 June 2006; revised manuscript received 3 July 2006)

Some new exact solutions of an auxiliary ordinary differential equation are obtained, which were neglected by Sirendaoreji *et al* in their auxiliary equation method. By using this method and these new solutions the combined Korteweg–de Vries (KdV) and modified KdV (mKdV) equation and (2+1)-dimensional Broer–Kaup–Kupershmidt system are investigated and abundant exact travelling wave solutions are obtained that include new solitary wave solutions and triangular periodic wave solutions.

Keywords: auxiliary equation, Korteweg–de Vries (KdV)–modified KdV (mKdV) equation, Broer–Kaup–Kupershmidt system

PACC: 0340K, 0290

1. Introduction

It is well known that the nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of sciences, such as physics, biology, chemistry, etc. Therefore, seeking exact solutions of NPDEs are very important and significant in the nonlinear sciences. In the past decades, a great effort has been made for this task and many powerful methods have been presented, such as homogeneous balance method,[1−5] the sine-cosine method,[6] Jacobi elliptic function method,[7−12] tanh-function method[13−16] and so on.

Among these methods, the tanh-function method is one of most effectively straightforward algebraic methods to construct exact solutions of NPDEs. One can write the solitary wave solutions of a nonlinear equation as the polynomials of hyperbolic functions and change it into a nonlinear system of algebraic equations. In recent years, much research work has concentrated on the various extensions and applications of the tanh-function method. In Refs.[17,18], Sirendaoreji and Sun presented a useful auxiliary equation method of solving NPDEs; they introduced the following auxiliary ordinary differential equation:

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2 = c_2\varphi^2(\xi) + c_3\varphi^3(\xi) + c_4\varphi^4(\xi), \qquad (1)$$

where c_2, c_3 and c_4 are real parameters. The exact solutions of Eq.(1) are as follows:

$$\varphi_1(\xi) = \frac{-c_2c_3\mathrm{sech}^2\left(\pm\frac{\sqrt{c_2}}{2}\xi\right)}{c_3^2 - c_2c_4\left(1-\tanh\left(\pm\frac{\sqrt{c_2}}{2}\xi\right)\right)^2}, \quad \text{when} \quad c_2 > 0, \qquad (2)$$

$$\varphi_2(\xi) = \frac{2c_2\mathrm{sech}\sqrt{c_2}\xi}{\sqrt{c_3^2 - 4c_2c_4} - c_3\mathrm{sech}\sqrt{c_2}\xi}, \quad \text{when} \quad c_3^2 - 4c_2c_4 > 0, \quad c_2 > 0. \qquad (3)$$

*Project supported by the National Natural Science Foundation of China (Grant No 10472029).

†E-mail: yxlhnu@sohu.com

They used the solutions of Eq.(1) to replace the tanh function in the tanh-function method. Zhao *et al*[19] applied this method to solving the combined Korteweg–de Vries (KdV) and modified KdV(mKdV) equation. Liu and Liu[20] picked up some solutions of the auxiliary ordinary differential equation (1), which were neglected by Sirendaoreji and Sun.[17] They classified the solutions of the auxiliary ordinary differential equation (1) according to three parameters. Besides the solution (3), the remaining exact solutions of Eq.(1) are as follows:

$$\varphi_3(\xi)=\frac{-2c_2\mathrm{sech}\sqrt{c_2}\xi}{\sqrt{c_3^2-4c_2c_4}+c_3\mathrm{sech}\sqrt{c_2}\xi},\quad \text{when}\quad c_3^2-4c_2c_4>0,\quad c_2>0, \tag{4}$$

$$\varphi_4(\xi)=\frac{2c_2\mathrm{csch}\sqrt{c_2}\xi}{\sqrt{4c_2c_4-c_3^2}-c_3\mathrm{csch}\sqrt{c_2}\xi},\quad \text{when}\quad c_3^2-4c_2c_4<0,\quad c_2>0, \tag{5}$$

$$\varphi_5(\xi)=\frac{-2c_2\mathrm{csch}\sqrt{c_2}\xi}{\sqrt{4c_2c_4-c_3^2}+c_3\mathrm{csch}\sqrt{c_2}\xi},\quad \text{when}\quad c_3^2-4c_2c_4<0,\quad c_2>0, \tag{6}$$

$$\varphi_6(\xi)=-\frac{c_2}{c_3}\left(1\pm\tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)\right),\quad \text{when}\quad c_3^2-4c_2c_4=0,\quad c_2>0, \tag{7}$$

$$\varphi_7(\xi)=-\frac{c_2}{c_3}\left(1\pm\coth\left(\frac{\sqrt{c_2}}{2}\xi\right)\right),\quad \text{when}\quad c_3^2-4c_2c_4=0,\quad c_2>0. \tag{8}$$

In this paper, we extend the solutions of the auxiliary ordinary differential equation (1), which were neglected by Sirendaoreji *et al* in their auxiliary equation method. Then, by using the auxiliary equation method the combined KdV–mKdV equation and the Broer–Kaup–Kupershmidt (BKK) system are investigated, and many exact travelling wave solutions are obtained that include new solitary wave solutions and triangular periodic wave solutions.

The rest of this paper is organized as follows. In Section 2, we revisit the general process of the auxiliary equation method, and some new solutions of the auxiliary ordinary differential equation are obtained. In Section 3, the KdV–mKdV equation and the BKK system are investigated and abundant exact solutions are obtained. In Section 4, some conclusions are provided.

2. The auxiliary equation method and new solutions of the auxiliary equation

For the auxiliary equation method, its main steps are outlined as follows: [17,21]

Step1. Reduce a given NPDE with physical fields $u_i(x)$ in variables $x=(x_0=t,x_1,x_2,\ \ldots,\ x_l)$

$$N_i(u_i,u_{i,x_j},u_{i,x_j,x_k},\ \ldots\ )=0,\quad (i=1,2,\ \ldots;\ j,\ k=0,1,2,\ \ldots,\ l), \tag{9}$$

by using the wave transformation

$$u_i(x)=U_i(\xi),\quad \xi=\sum_{j=0}^{l}k_jx_j, \tag{10}$$

where $k_j(j=0,1,2,\ \ldots,\ l)$ are constants to be determined later, to the following nonlinear ordinary differential equations (NODE):

$$H_i(U_i,U_i',U_i'',\ \ldots\ )=0. \tag{11}$$

Step2. Expand the solution of Eq.(11) in the form

$$u_i(x,t)=U_i(\xi)=\sum_{k=0}^{n}a_{ik}\varphi^k, \tag{12}$$

where the variable $\varphi=\varphi(\xi)$ satisfies Eq.(1). The derivative with respect to the variable ξ becomes the derivative with respect to the variable φ as shown below:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\rightarrow\ \pm\sqrt{\sum_{j=2}^{4}c_j\varphi^j}\frac{\mathrm{d}}{\mathrm{d}\varphi}, \tag{13}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \to \pm\left[\frac{1}{2}\sum_{j=2}^{4} jc_j\varphi^{j-1}\frac{\mathrm{d}}{\mathrm{d}\varphi} + \sum_{j=2}^{4} c_j\varphi^j\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\right], \quad \ldots \tag{14}$$

Step3. Substitute the expression (12) into Eq.(11) and compare the highest derivative terms in the resulting equation with the nonlinear terms in Eq.(11) by making use of the expressions (13) and (14), to obtain a value of the positive integer n.

Step4. Substitute the expression (12) and Eq.(1) into Eq.(11) and set all coefficients of $\varphi^k\left(\sqrt{\sum_{j=2}^{4} c_j\varphi^j}\right)^v$ $(v = 0, 1; k = 0, 1, 2, \ \ldots\ )$ to be zero to have a system of nonlinear algebraic equations of a_{ik} and $c_j (k = 0, 1, \ \ldots, \ n; j = 2, 3, 4)$ and then solve the system of algebraic equations with the aid of the Mathematica to obtain a_{ik} and c_j.

Step5. Substitute the constants $c_j (j = 2, 3, 4)$ into Eq.(1) to obtain all the possible solutions. The solutions of Eq.(9) depend on the explicit solvability of Eq.(1).

Motivated by Refs.[22,23], in what follows we give some new particular solutions of Eq.(1) which are related to the parameters c_2, c_3 and c_4.

Case1. When $c_3^2 - 4c_2c_4 > 0$, and $c_2 > 0$, Eq.(1) has the following solutions:

$$\varphi_8(\xi) = \frac{2c_2\mathrm{sech}^2\frac{\sqrt{c_2}}{2}\xi}{2\sqrt{c_3^2-4c_2c_4} - (\sqrt{c_3^2-4c_2c_4}+c_3)\mathrm{sech}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{15}$$

$$\varphi_9(\xi) = \frac{2c_2\mathrm{sech}^2\frac{\sqrt{c_2}}{2}\xi}{-2\sqrt{c_3^2-4c_2c_4} + (\sqrt{c_3^2-4c_2c_4}-c_3)\mathrm{sech}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{16}$$

$$\varphi_{10}(\xi) = \frac{2c_2\mathrm{csch}^2\frac{\sqrt{c_2}}{2}\xi}{2\sqrt{c_3^2-4c_2c_4} + (\sqrt{c_3^2-4c_2c_4}-c_3)\mathrm{csch}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{17}$$

$$\varphi_{11}(\xi) = \frac{2c_2\mathrm{csch}^2\frac{\sqrt{c_2}}{2}\xi}{-2\sqrt{c_3^2-4c_2c_4} - (\sqrt{c_3^2-4c_2c_4}+c_3)\mathrm{csch}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{18}$$

Case 2. When $c_3^2 - 4c_2c_4 > 0$, and $c_2 < 0$, Eq.(1) obtains the following solutions:

$$\varphi_{12}(\xi) = \frac{2c_2\sec^2\frac{\sqrt{-c_2}}{2}\xi}{2\sqrt{c_3^2-4c_2c_4} - (\sqrt{c_3^2-4c_2c_4}+c_3)\sec^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{19}$$

$$\varphi_{13}(\xi) = \frac{2c_2\sec^2\frac{\sqrt{-c_2}}{2}\xi}{-2\sqrt{c_3^2-4c_2c_4} + (\sqrt{c_3^2-4c_2c_4}-c_3)\sec^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{20}$$

$$\varphi_{14}(\xi) = \frac{2c_2\csc^2\frac{\sqrt{-c_2}}{2}\xi}{2\sqrt{c_3^2-4c_2c_4} - (\sqrt{c_3^2-4c_2c_4}+c_3)\csc^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{21}$$

$$\varphi_{15}(\xi) = \frac{2c_2\csc^2\frac{\sqrt{-c_2}}{2}\xi}{-2\sqrt{c_3^2-4c_2c_4} + (\sqrt{c_3^2-4c_2c_4}-c_3)\csc^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{22}$$

Case 3. When $c_0 = \frac{c_2^2}{4c_4}, c_2 < 0, c_3 = 0$, and $c_4 > 0$, i.e. $\varphi'^2 = \frac{c_2^2}{4c_4} + c_2\varphi^2 + c_4\varphi^4$, Eq.(1) has the following solutions:

$$\varphi_{16}(\xi) = \sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{\frac{-c_2}{2}}\xi\right), \tag{23}$$

$$\varphi_{17}(\xi) = \sqrt{-\frac{c_2}{2c_4}}\coth\left(\sqrt{\frac{-c_2}{2}}\xi\right). \tag{24}$$

3. The travelling wave solutions for the KdV−mKdV equation and BKK system

In this section, we will apply our proposed method in the second section to two NPDEs: the KdV–mKdV equation and the BKK system.

3.1. The Combined KdV and mKdV equation

The KdV–mKdV equation

$$u_t - 6uu_x + 6u^2u_x + u_{xxx} = 0 \tag{25}$$

arises from a variety of fields, such as solid-state physics, plasma physics, fluid dynamics, and quantum field theory.[19,24,25] We perform the following transformation:

$$u(x,t) = U(\xi), \quad \text{and} \quad \xi = kx - \omega t, \tag{26}$$

where k, and ω are constants. Substituting the expression (26) into Eq.(25) and integrating it with respect to ξ and then taking the integration constant as zero yield an NODE

$$-\omega U - 3kU^2 + 2kU^3 + k^3U'' = 0. \tag{27}$$

We take the solution of Eq.(27) as

$$U(\xi) = \sum_{i=0}^{n} a_i\varphi^i, \tag{28}$$

where the variable φ satisfies Eq.(1). Balancing U'' with U^3 in Eq.(27) gives $n = 1$. Thus the solution of Eq.(27) can be taken as

$$U = a_0 + a_1\varphi. \tag{29}$$

Substituting the expression (29) into Eq.(27) along with Eq.(1) leads to the following system of algebraic equations:

$$\begin{aligned}
&-\omega a_0 - 3ka_0^2 + 2ka_0^3 = 0,\\
&-\omega a_1 + k^3c_2a_1 - 6ka_0a_1 + 6ka_0^2a_1 = 0,\\
&\frac{3}{2}k^3c_3a_1 - 3ka_1^2 + 6ka_0a_1^2 = 0,\\
&2k^3c_4a_1 + 2ka_1^3 = 0.
\end{aligned} \tag{30}$$

By the use of Mathematica, solving the over-determined algebraic equations, we have the following results:

$$\begin{aligned}
&a_0 = 0,\ k \neq 0,\ a_1 = \pm k\sqrt{-c_4},\\
&\omega = k^3c_2, \quad \text{and} \quad k = \pm\frac{2}{c_3}\sqrt{-c_4}
\end{aligned} \tag{31}$$

where $c_4 < 0$.

Substituting the expression (31) into the expression (29) along with the expressions (2)–(8) and (15)–(24) yields the following solitary wave solutions and triangular periodic wave solutions to Eq.(25):

$$u_1(x,t) = \pm k\sqrt{-c_4}\frac{-c_2c_3\operatorname{sech}^2\left(\pm\frac{\sqrt{c_2}}{2}\xi\right)}{c_3^2 - c_2c_4\left(1 - \tanh\left(\pm\frac{\sqrt{c_2}}{2}\xi\right)\right)^2}, \tag{32}$$

where $\xi = kx - \omega t, c_2 > 0$, and $c_4 < 0$;

$$u_2(x,t)=\pm k\sqrt{-c_4}\frac{2c_2\mathrm{sech}\sqrt{c_2}\xi}{\sqrt{c_3^2-4c_2c_4}-c_3\mathrm{sech}\sqrt{c_2}\xi}, \tag{33}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2>0$, and $c_4<0$;

$$u_3(x,t)=\pm k\sqrt{-c_4}\frac{-2c_2\mathrm{sech}\sqrt{c_2}\xi}{\sqrt{c_3^2-4c_2c_4}+c_3\mathrm{sech}\sqrt{c_2}\xi}, \tag{34}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2>0$, and $c_4<0$;

$$u_4(x,t)=\pm k\sqrt{-c_4}\frac{2c_2\mathrm{sech}^2\frac{\sqrt{c_2}}{2}\xi}{2\sqrt{c_3^2-4c_2c_4}-(\sqrt{c_3^2-4c_2c_4}+c_3)\mathrm{sech}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{35}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2>0$, and $c_4<0$;

$$u_5(x,t)=\pm k\sqrt{-c_4}\frac{2c_2\mathrm{sech}^2\frac{\sqrt{c_2}}{2}\xi}{-2\sqrt{c_3^2-4c_2c_4}+(\sqrt{c_3^2-4c_2c_4}-c_3)\mathrm{sech}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{36}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2>0$, and $c_4<0$;

$$u_6(x,t)=\pm k\sqrt{-c_4}\frac{2c_2\mathrm{csch}^2\frac{\sqrt{c_2}}{2}\xi}{2\sqrt{c_3^2-4c_2c_4}+(\sqrt{c_3^2-4c_2c_4}-c_3)\mathrm{csch}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{37}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2>0$, and $c_4<0$;

$$u_7(x,t)=\pm k\sqrt{-c_4}\frac{2c_2\mathrm{csch}^2\frac{\sqrt{c_2}}{2}\xi}{-2\sqrt{c_3^2-4c_2c_4}-(\sqrt{c_3^2-4c_2c_4}+c_3)\mathrm{csch}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{38}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2>0$, and $c_4<0$;

$$u_8(x,t)=\pm k\sqrt{-c_4}\frac{2c_2\mathrm{sec}^2\frac{\sqrt{-c_2}}{2}\xi}{2\sqrt{c_3^2-4c_2c_4}-(\sqrt{c_3^2-4c_2c_4}+c_3)\mathrm{sec}^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{39}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2<0$, and $c_4<0$;

$$u_9(x,t)=\pm k\sqrt{-c_4}\frac{2c_2\mathrm{sec}^2\frac{\sqrt{-c_2}}{2}\xi}{-2\sqrt{c_3^2-4c_2c_4}+(\sqrt{c_3^2-4c_2c_4}-c_3)\mathrm{sec}^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{40}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2<0$, and $c_4<0$;

$$u_{10}(x,t)=\pm k\sqrt{-c_4}\frac{2c_2\mathrm{csc}^2\frac{\sqrt{-c_2}}{2}\xi}{2\sqrt{c_3^2-4c_2c_4}-(\sqrt{c_3^2-4c_2c_4}+c_3)\mathrm{csc}^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{41}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2<0$, and $c_4<0$;

$$u_{11}(x,t)=\pm k\sqrt{-c_4}\frac{2c_2\mathrm{csc}^2\frac{\sqrt{-c_2}}{2}\xi}{-2\sqrt{c_3^2-4c_2c_4}+(\sqrt{c_3^2-4c_2c_4}-c_3)\mathrm{csc}^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{42}$$

where $\xi=kx-\omega t, c_3^2-4c_2c_4>0, c_2<0$, and $c_4<0$.

Remark: u_1–u_7 and u_8–u_{11} are respectively solitary wave solutions and triangular periodic wave solutions of Eq.(25), and u_4–u_{11} have not been given yet in the literature to our knowledge.

3.2. (2+1)-dimensional BKK system

We consider the celebrated (2+1)-dimensional BKK system[26−29]

$$\begin{aligned} &u_{ty} - u_{xxy} + 2(uu_x)_y + 2v_{xx} = 0, \\ &v_t + v_{xx} + 2(uv)_x = 0. \end{aligned} \tag{43}$$

This system has been widely applied in many branches of physics like plasma physics, fluid dynamics, nonlinear optics, etc. and it can be derived from the celebrated Kadomtsev–Petviashvili equation by using the symmetrical constraint.[30] Using some suitable dependent and independent variable transformations, Chen and Li[31] have proved that the (2+1)-dimensional BKK system can be transformed into the (2+1)-dimensional dispersive long wave equation and the (2+1)-dimensional Ablowitz–Kaup–Newell–Segur system. Using the following transformation:

$$\begin{aligned} &u(x,y,t) = U(\xi),\ v(x,y,t) = V(\xi), \\ &\xi = kx + \beta y + \lambda t, \end{aligned} \tag{44}$$

Eq.(43) reduces to

$$\beta\lambda U'' - k^2\beta U''' + 2k\beta(UU')' + 2k^2V'' = 0, \tag{45a}$$

$$\lambda U' + k^2 V'' + 2k(UV)' = 0. \tag{45b}$$

Integrating (45a) with respect to ξ twice and taking the integration constants as zero yield the NODE

$$V(\xi) = \frac{\beta}{2}U' - \frac{\beta}{2k}U^2 - \frac{\beta\lambda}{2k^2}U. \tag{46}$$

Substituting Eq.(46) into Eq.(45b) and integrating it once lead to

$$\frac{\beta k^2}{2}U'' - \beta U^3 - \frac{3\beta\lambda}{2k}U^2 - \frac{\beta\lambda}{2k^2}U = 0. \tag{47}$$

Solving Eq.(47) in the same way as solving Eqs.(28)–(30), we have

$$U = a_0 + a_1\varphi, \tag{48}$$

Substituting Eq.(48) into Eq.(47) along with Eq.(1) leads to the following system of algebraic equations

$$\begin{aligned} &-\frac{\beta\lambda}{2k^2}a_0 - \frac{3\beta\lambda a_0^2}{2k} - \beta a_0^3 = 0, \\ &-\frac{\beta\lambda}{2k^2}a_1 - \frac{3\beta\lambda a_0 a_1}{k} - 3\beta a_0^2 a_1 + \frac{1}{2}k^2\beta a_1 c_2 = 0, \\ &-\frac{3\beta\lambda a_1^2}{2k} - 3\beta a_0 a_1^2 + \frac{3}{4}k^2\beta a_1 c_3 = 0, \\ &-\beta a_1^3 + k^2\beta a_1 c_4 = 0. \end{aligned} \tag{49}$$

Solving the over determined algebraic equations with the aid of Mathematica, we obtain the following results:

$$a_0 = 0,\ a_1 = \frac{c_3}{2kc_2},\ \lambda = \frac{c_3^2}{4c_2c_4},\ \text{and } k^4 = \frac{\lambda}{c_2}, \tag{50}$$

where $c_2 \neq 0, c_4 \neq 0$, and $k \neq 0$.

Substituting the expressions (50) into the expression (48) along with the expressions (2)–(8) and (15)–(24) yields the following solitary wave solutions and triangular periodic wave solutions to Eq.(43):

$$u_1(x,y,t) = -\frac{1}{2k}\frac{c_3^2\mathrm{sech}^2\left(\pm\frac{\sqrt{c_2}}{2}\xi\right)}{c_3^2 - c_2c_4\left(1 - \tanh\left(\pm\frac{\sqrt{c_2}}{2}\xi\right)\right)^2}, \tag{51}$$

where $\xi = kx + \beta y + \lambda t$, and $c_2 > 0$;

$$u_2(x,y,t) = \frac{c_3}{k}\frac{\mathrm{sech}\sqrt{c_2}\xi}{\sqrt{c_3^2 - 4c_2c_4} - c_3\mathrm{sech}\sqrt{c_2}\xi}, \tag{52}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 > 0$;

$$u_3(x,y,t) = -\frac{c_3}{k}\frac{\text{sech}\sqrt{c_2}\xi}{\sqrt{c_3^2 - 4c_2c_4} + c_3\text{sech}\sqrt{c_2}\xi}, \tag{53}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 > 0$;

$$u_4(x,y,t) = \frac{c_3}{k}\frac{\text{csch}\sqrt{c_2}\xi}{\sqrt{4c_2c_4 - c_3^2} - c_3\text{csch}\sqrt{c_2}\xi}, \tag{54}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 < 0$, and $c_2 > 0$;

$$u_5(x,y,t) = -\frac{c_3}{k}\frac{\text{csch}\sqrt{c_2}\xi}{\sqrt{4c_2c_4 - c_3^2} + c_3\text{csch}\sqrt{c_2}\xi}, \tag{55}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 < 0$, and $c_2 > 0$;

$$u_6(x,y,t) = -\frac{1}{2k}\left(1 \pm \tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)\right), \tag{56}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 = 0$, and $c_2 > 0$;

$$u_7(x,y,t) = -\frac{1}{2k}\left(1 \pm \coth\left(\frac{\sqrt{c_2}}{2}\xi\right)\right), \tag{57}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 = 0$, and $c_2 > 0$;

$$u_8(x,y,t) = \frac{c_3}{k}\frac{\text{sech}^2\frac{\sqrt{c_2}}{2}\xi}{2\sqrt{c_3^2 - 4c_2c_4} - (\sqrt{c_3^2 - 4c_2c_4} + c_3)\text{sech}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{58}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 > 0$;

$$u_9(x,y,t) = \frac{c_3}{k}\frac{\text{sech}^2\frac{\sqrt{c_2}}{2}\xi}{-2\sqrt{c_3^2 - 4c_2c_4} + (\sqrt{c_3^2 - 4c_2c_4} - c_3)\text{sech}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{59}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 > 0$;

$$u_{10}(x,y,t) = \frac{c_3}{k}\frac{\text{csch}^2\frac{\sqrt{c_2}}{2}\xi}{2\sqrt{c_3^2 - 4c_2c_4} + (\sqrt{c_3^2 - 4c_2c_4} - c_3)\text{csch}^2\frac{\sqrt{c_2}}{2}\xi} \tag{60}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 > 0$;

$$u_{11}(x,y,t) = \frac{c_3}{k}\frac{\text{csch}^2\frac{\sqrt{c_2}}{2}\xi}{-2\sqrt{c_3^2 - 4c_2c_4} - (\sqrt{c_3^2 - 4c_2c_4} + c_3)\text{csch}^2\frac{\sqrt{c_2}}{2}\xi}, \tag{61}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 > 0$;

$$u_{12}(x,y,t) = \frac{c_3}{k}\frac{\text{sec}^2\frac{\sqrt{-c_2}}{2}\xi}{2\sqrt{c_3^2 - 4c_2c_4} - (\sqrt{c_3^2 - 4c_2c_4} + c_3)\text{sec}^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{62}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 < 0$;

$$u_{13}(x,y,t) = \frac{c_3}{k}\frac{\text{sec}^2\frac{\sqrt{-c_2}}{2}\xi}{-2\sqrt{c_3^2 - 4c_2c_4} + (\sqrt{c_3^2 - 4c_2c_4} - c_3)\text{sec}^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{63}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 < 0$;

$$u_{14}(x,y,t) = \frac{c_3}{k}\frac{\csc^2\frac{\sqrt{-c_2}}{2}\xi}{2\sqrt{c_3^2-4c_2c_4}-(\sqrt{c_3^2-4c_2c_4}+c_3)\csc^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{64}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 < 0$;

$$u_{15}(x,y,t) = \frac{c_3}{k}\frac{\csc^2\frac{\sqrt{-c_2}}{2}\xi}{-2\sqrt{c_3^2-4c_2c_4}+(\sqrt{c_3^2-4c_2c_4}-c_3)\csc^2\frac{\sqrt{-c_2}}{2}\xi}, \tag{65}$$

where $\xi = kx + \beta y + \lambda t, c_3^2 - 4c_2c_4 > 0$, and $c_2 < 0$;

From Eq.(46) and the expressions (51)–(65), we can write respectively the travelling wave solutions of Eq.(43), but for brevity they are omitted.

Remark: the expressions u_1–u_{11} and u_{12}–u_{15} are respectively solitary wave solutions and triangular periodic wave solutions. To the best of our knowledge, the solutions u_8–u_{15} are new, which are unavailable in the existing literature.

4. Conclusions

By using some new solutions for an auxiliary ordinary differential equation, we have found some new exact travelling wave solutions for two NPDEs that include new solitary wave solutions and triangular periodic wave solutions. This suggests that one can obtain different exact solutions by choosing different solutions of an auxiliary ordinary differential equation or different auxiliary ordinary differential equations, in the auxiliary equation method. Thus the auxiliary ordinary differential equation in the auxiliary equation method is worth further studying and it may be help us to find more new solutions to NPDEs.

References

[1] Wang M L 1995 *Phys. Lett.* A **199** 169
[2] Wang M L, Zhou Y B and Li Z B 1996 *Phys. Lett.* A **216** 67
[3] Yang L, Zhu Z and Wang Y 1999 *Phys. Lett.* A **260** 55
[4] Fan E G and Zhang H Q 1998 *Acta Phys. Sin.* **47** 353 (in Chinese)
[5] Fan E G 2000 *Acta Phys. Sin.* **49** 1409 (in Chinese)
[6] Yan C 1996 *Phys. Lett.* A **224** 77
[7] Liu S K, Fu Z T, Liu S D and Zhao Q 2001 *Phys. Lett.* A **289** 69
[8] Fu Z T, Liu S K, Liu S D and Zhao Q 2001 *Phys. Lett.* A **290** 72
[9] Liu S D, Fu Z T, Liu S K and Zhao Q 2002 *Acta Phys. Sin.* **51** 718 (in Chinese)
[10] Zhang S Q and Li Z B 2003 *Acta Phys. Sin.* **52** 1066 (in Chinese)
[11] Zhu J M, Ma Z Y, Fang J P, Zheng C L and Zhang J F 2004 *Chin. Phys.* **13** 798
[12] Li D S and Zhang H Q 2006 *Acta Phys. Sin.* **55** 1565 (in Chinese)
[13] Parkes E J and Duffy B R 1996 *Comput. Phys. Commun.* **98** 288
[14] Li Z B and Liu Y P 2002 *Comput. Phys. Commun.* **148** 256
[15] Li D S and Zhang H Q 2003 *Acta Phys. Sin.* **52** 1569 (in Chinese)
[16] Zhu J M 2005 *Chin. Phys.* **14** 1290
[17] Sirendaoreji and Sun J 2003 *Phys. Lett.* A **309** 387
[18] Sirendaoreji 2004 *Chaos Solitons Fractals* **19** 147
[19] Zhao Q, Liu S K and Fu Z T 2005 *Commun. Theor. Phys.* **43** 615
[20] Liu C P and Liu X P 2006 *Phys. Lett.* A **348** 222
[21] Fan E G 2003 *Chaos Solitons Fractals* **16** 819
[22] Zhang W G, Chang Q S and Fan E G 2003 *J Math. Anal. Appl.* **287** 1
[23] Huang D J and Zhang H Q 2006 *Chaos Solitons Fractals* **29** 928
[24] M.Wadati 1975 *J.Phys. Soc. Jpn.* **38** 673
[25] M.N.B.Mohamad 1992 *Math. Meth. Appl. Sci.* **15** 73
[26] V.G.Dubrovsky and E.G. Konopelchenko 1994 *J. Phys.* A **27** 4619
[27] Emmanuel Yomba 2005 *Phys. Lett.* A **336** 463
[28] Fang J P and Zheng C L 2005 *Chin. Phys.* **14** 669
[29] Zhi H Y, Wang Q and Zhang H Q 2005 *Acta Phys. Sin.* **54** 1002 (in Chinese)
[30] Lou S Y and Hu X B 1997 *J. Math. Phys.* A **38** 6401
[31] Chen C L and Li Y S 2002 *Commun. Theor. Phys.* **38** 129