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Lie symmetry analysis and reduction of a new integrable coupled KdV system*

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In this paper, Lie symmetry is investigated for a new integrable coupled Korteweg–de Vries (KdV) equation system. Using some symmetry subalgebra of the equation system, we obtain five types of the significant similarity reductions. Abundant solutions of the coupled KdV equation system, such as the solitary wave solution, exponential solution, rational solution and polynomial solution, etc. are obtained from the reduced equations. Especially, one type of group-invariant solution of reduced equations can be acquired by means of the Painlevé I transcendent function.

Keywords: the coupled KdV equations, symmetry reduction, group-invariant solutions
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1. Introduction

Many coupled systems have been proposed since the soliton theory came into being in the last century. Because of the rich structures of most coupled systems, both the mathematicians and physicists have paid much attention to them. For example, an infinite number of conservation laws, multi-Hamiltonian structures, symmetry reductions, Darboux transformations and Painlevè property can be found in the coupled systems.[1−6] A class of nonlinearly coupled evolution equation in higher-dimensional spaces is reduced to a simple nonlinear evolution equation in lower-dimensional spaces by using a simple transformation.[7] New explicit exact solutions of the coupled Schrödinger Korteweg-de Vries (KdV) equations are constructed by using the special solutions of a class of Riccati equations.[8] The Painlevé classification of the coupled KdV equations was made recently by Lou *et al.*[9] Especially, the following new Painlevè integrable system (1) is isolated there:

$$\begin{aligned} &u_t + (u_{xx} + u^2)_x + 2vu_x = 0, \\ &v_t + (v_{xx} + v^2)_x + 2uv_x = 0. \end{aligned} \tag{1}$$

In Ref.[9], by using a long-wave approximation, Lou *et al* have derived some new types of coupled KdV equation systems with some arbitrary parameters from a two-layer fluid model which is used to describe the atmospheric and oceanic phenomena such as the atmospheric blockings, the interactions between the atmospheric and oceanic phenomena, the oceanic circulations and even hurricanes or typhoons.

To find symmetries for a given nonlinear system, one can use the standard Lie group approach.[10−13] The purpose of this paper is to give a systematic investigation of Lie symmetry analysis and invariance of the coupled KdV equation (1). This enables us to obtain similarity reductions and abundant particular solutions, such as the solitary wave solution, exponential solution, rational solution and polynomial solution *etc* for the coupled KdV system. Especially the group-invariant solution of reduction equations of type 1 can be solved by means of the Painlevè I transcendent function.

2. Symmetry groups

In this section, we discuss the Lie symmetry analysis for the coupled KdV equation system (1). Consider a one-parameter Lie group of infinitesimal transformations in the following form:

$$\begin{aligned} x \to \bar{x} = x + \varepsilon\xi^1(x,t,u,v), \\ t \to \bar{t} = t + \varepsilon\xi^2(x,t,u,v), \end{aligned}$$

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$$u \to \bar{u} = u + \varepsilon\phi_1(x,t,u,v),$$
$$v \to \bar{v} = v + \varepsilon\phi_2(x,t,u,v). \tag{2}$$

The vector field associated with the above group of transformations can be written as

$$v = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial t} + \phi_1\frac{\partial}{\partial u} + \phi_2\frac{\partial}{\partial v}. \tag{3}$$

We wish to determine all possible coefficient functions ξ^1, ξ^2 and ϕ_1, ϕ_2 so that the corresponding one-parameter group $\exp(\varepsilon v)$ is a symmetry group of the coupled KdV equations (1). According to Theorem 2.31 in Ref.[2], one needs to know the third prolongation of v

$$pr^{(3)}v = v + \sum_J \phi_1^J(x,t,u,v)\frac{\partial}{\partial u_J} + \sum_J \phi_2^J(x,t,u,v)\frac{\partial}{\partial v_J}. \tag{4}$$

Applying $pr^{(3)}v$ to Eq.(1), we find the infinitesimal criterion of Theorem 2.31 in Ref.[2] to be

$$\begin{aligned}&\phi_1^t + \phi_1^{xxx} + 2(u+v)\phi_1^x + 2u_x\phi_1 + 2u_x\phi_2 = 0,\\ &\phi_2^t + \phi_2^{xxx} + 2(u+v)\phi_2^x + 2v_x\phi_1 + 2v_x\phi_2 = 0,\end{aligned} \tag{5}$$

which must be satisfied whenever Eq.(1) holds. According to the general prolongation formula, replacing u_t, v_t by $u, v, u_x, v_x, u_{xx}, v_{xx}, \ldots$ whenever it occurs, and equating the coefficients of the various monomials in the first- and second- and third-order partial derivatives of u and v, we find the determining equations for the symmetry group of the coupled KdV equations. Solving these determining equations, we can conclude that the most general infinitesimal symmetry of the Eq.(1) has coefficient functions as follows:

$$\begin{aligned}&\xi^1 = c_1x + 4c_2t + c_3,\\ &\xi^2 = 3c_1t + c_4,\\ &\phi_1 = -(2c_5t + c_6 + c_1)u - (2c_5t + c_7 + c_1)v + c_5x - c_8 + c_2,\\ &\phi_2 = (2c_5t + c_6 - c_1)u + (2c_5t + c_7 - c_1)v - c_5x + c_8 + c_2,\end{aligned} \tag{6}$$

where $c_1, \ldots, c_8$ are arbitrary parameters.

Thus the Lie algebra of infinitesimal symmetry of the coupled KdV equation system (1) is spanned by the eight vector fields

$$\begin{aligned}&v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = 4t\partial_x + \partial_u + \partial_v,\\ &v_4 = x\partial_x + 3t\partial_t - (u+v)\partial_u - (u+v)\partial_v,\\ &v_5 = -(2tu + 2tv - x)\partial_u + (2tu + 2tv - x)\partial_v,\\ &v_6 = -u\partial_u + u\partial_v, \quad v_7 = -v\partial_u + v\partial_v, v_8 = -\partial_u + \partial_v.\end{aligned} \tag{7}$$

The non-zero commutation relations between these vector fields are given as follows:

$$\begin{aligned}&[v_1,v_4] = v_1, \quad [v_1,v_5] = -v_8, \quad [v_2,v_3] = 4v_1,\\ &[v_2,v_4] = 3v_2, \quad [v_2,v_5] = 2(v_6+v_7),\\ &[v_3,v_4] = -2v_3, \quad [v_3,v_6] = v_8, [v_4,v_5] = v_5,\\ &[v_4,v_6] = -(v_6+v_7), \quad [v_5,v_6] = -v_5, \quad [v_5,v_7] = v_5,\\ &[v_6,v_7] = v_6+v_7, \quad [v_6,v_8] = v_8, \quad [v_7,v_8] = -v_8.\end{aligned}$$

The corresponding group transformations for the eight vector fields are easily found. The one-parameter groups g_i generated by the v_i are given in the following table: (The entries give the transformed point $\exp(\varepsilon v_i)(x,t,u,v) = (\bar{x},\bar{t},\bar{u},\bar{v})$)

$$\begin{aligned}&g_1 : (x+\varepsilon, t, u, v),\\ &g_2 : (x, t+\varepsilon, u, v),\\ &g_3 : (x+4\varepsilon t, t, u+\varepsilon, v+\varepsilon),\\ &g_4 : (\mathrm{e}^{\varepsilon}x, \mathrm{e}^{3\varepsilon}t, (u+v)\mathrm{e}^{-\varepsilon} - v, (u+v)\mathrm{e}^{-\varepsilon} - u),\\ &g_5 : (x, t, [(2tu+2tv-x)\mathrm{e}^{-2t\varepsilon} - 2tv + x]/(2t),\\ &\qquad [(2tu+2tv-x)\mathrm{e}^{2t\varepsilon} - 2tu + x]/(2t)),\\ &g_6 : (x, t, \mathrm{e}^{-\varepsilon}u, u + v - \mathrm{e}^{-\varepsilon}u),\\ &g_7 : (x, t, u + v - \mathrm{e}^{\varepsilon}v, \mathrm{e}^{\varepsilon}v),\\ &g_8 : (x, t, u - \varepsilon, v + \varepsilon).\end{aligned} \tag{8}$$

In Eq.(8), g_1, g_2, and g_8 are related to the space $\boldsymbol{x}$, time t and the fields $\boldsymbol{u}$ and $\boldsymbol{v}$'s translations, g_3 is the Galilean boost, g_4 and g_6 are the scaling invariance

for the space–time and the fields respectively, while g_5 is related to a time-dependent scaling for the fields $\boldsymbol{u}$ and $\boldsymbol{v}$.

Each group g_i above is a symmetry group, which means that if $(u,v)=(f(x,t),g(x,t))$ is a solution of the coupled KdV equations (1), so are the functions

$$
\begin{aligned}
&(u^{(1)},v^{(1)})=(f(x-\varepsilon,t),g(x-\varepsilon,t)),\\
&(u^{(2)},v^{(2)})=(f(x,t-\varepsilon),g(x,t-\varepsilon)),\\
&(u^{(3)},v^{(3)})=(f(x-4\varepsilon t,t)+\varepsilon,g(x-4\varepsilon t,t)+\varepsilon),\\
&(u^{(4)},v^{(4)})=(\mathrm{e}^{-\varepsilon}f(\mathrm{e}^{-\varepsilon}x,\mathrm{e}^{-3\varepsilon}t)+(\mathrm{e}^{-\varepsilon}-1)g(\mathrm{e}^{-\varepsilon}x,\mathrm{e}^{-3\varepsilon}t),\mathrm{e}^{-\varepsilon}g(\mathrm{e}^{-\varepsilon}x,\mathrm{e}^{-3\varepsilon}t)\\
&\qquad\qquad +(\mathrm{e}^{-\varepsilon}-1)f(\mathrm{e}^{-\varepsilon}x,\mathrm{e}^{-3\varepsilon}t)),\\
&(u^{(5)},v^{(5)})=\Big(\Big(f(x,t)+g(x,t)-\frac{x}{2t}\Big)\mathrm{e}^{-2t\varepsilon}-g(x,t)+\frac{x}{2t},(f(x,t)+g(x,t)\\
&\qquad\qquad -\frac{x}{2t}\Big)\mathrm{e}^{2t\varepsilon}-f(x,t)\Big)+\frac{x}{2t},\\
&(u^{(6)},v^{(6)})=(\mathrm{e}^{-\varepsilon}f(x,t),g(x,t)+(1-\mathrm{e}^{-\varepsilon})f(x,t)),\\
&(u^{(7)},v^{(7)})=(f(x,t)+(1-\mathrm{e}^{-\varepsilon})g(x,t),\mathrm{e}^{-\varepsilon}g(x,t)),\\
&(u^{(8)},v^{(8)})=(f(x,t)-\varepsilon,g(x,t)+\varepsilon),
\end{aligned}
$$

where ε is any real number.

Using one-parameter invariant groups, one can obtain some solutions depending on one-parameter from a known special solution. For example, if let $(u,v)=(c_1,c_2)$ be just a constant solution, then we immediately conclude that so is (u_1,v_1) with the following form:

$$
\begin{aligned}
u_1&=[(2tc_1+2tc_2-x)\mathrm{e}^{-2t\varepsilon}-2tc_2+x]/(2t),\\
v_1&=[(2tc_1+2tc_2-x)\mathrm{e}^{2t\varepsilon}-2tc_1+x]/(2t).
\end{aligned}
$$

3. Group-invariant solutions and symmetry reduction

Consider a system of partial differential equations Δ defined over an open subset $M\subset X\times U$ of the space of independent and dependent variables: $X=\{(x_1,x_2,\ldots,x_p)\}$, and $U=\{(u_1,u_2,\ldots,u_q)\}$. Let G be a local group of transformations acting on M. If G is a symmetry group of equations Δ, then under some additional regularity assumptions of the action of G, we can find all the group invariant solutions of Δ by solving a reduced system of differential equations, denoted by Δ/G, which will involve fewer independent variables than the original system Δ. Now we list the steps in finding the group invariant solutions of Δ.

The first step: find the invariants of a given symmetry group G. A local invariant $\xi(\boldsymbol{x},\boldsymbol{u})$ of G is a solution of the linear, homogeneous first-order partial differential equation $\boldsymbol{v}(\xi)=0$, where $\boldsymbol{v}$ is the vector field of G, $\boldsymbol{x}\in X,\boldsymbol{u}\in U$. The general solution of equation $\boldsymbol{v}(\xi)=0$ can be found by integrating the corresponding characteristic system of ordinary differential equations. In this way, one can obtain two classes of G-invariants:

$$
\begin{aligned}
y_1=\eta_1(\boldsymbol{x},\boldsymbol{u}),\ldots,y_{p-s}=\eta_{p-s}(\boldsymbol{x},\boldsymbol{u});\\
v_1=\zeta_1(\boldsymbol{x},\boldsymbol{u}),\ldots,v_q=\zeta_q(\boldsymbol{x},\boldsymbol{u}),
\end{aligned}
$$

where $1\leqslant s\leqslant p$ corresponds to the dimensionality of the orbits of some subgroup of the full symmetry group.

The second step: solve global invariants of a given symmetry group G to obtain the expressions of the $\boldsymbol{x}$ and $\boldsymbol{u}$. Substituting these expressions into the equations Δ, one can obtain the reduced system Δ/G for the group invariant solutions.

The last step: solve the reduced system Δ/G=0. For each solution $\boldsymbol{v}=\boldsymbol{h}(\boldsymbol{y})$ of Δ/G there exists a corresponding group invariant solution $\boldsymbol{u}=\boldsymbol{f}(\boldsymbol{x})$ of the original system, which is given implicitly by the relation $\boldsymbol{\zeta}(\boldsymbol{x},\boldsymbol{u})=\boldsymbol{h}[\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{u})]$.

For the given system (1), all vector fields of Lie symmetry group can be any linear combinations of the generators given in Eq.(7).

We are now interested in finding the group-invariants of the coupled KdV equation system (1) in some different cases.

Case 1. $c_1\neq 0$

Solving the characteristic equation associated with the infinitesimal symmetry Eq.(6), one obtains global invariants of this group as follows:

$$\begin{aligned}
\xi =& \ \frac{3c_1^2x-2c_2T+3c_1c_3-4c_2c_4}{3c_1^2T^{1/3}},\\
f(\xi) =& \ \frac{2\xi}{3c_1A(1+A)}-\frac{4T^{2/3}(c_1u+c_1v-c_2)}{9c_1^3A(1+A)},\\
g(\xi) =& \ (-\frac{2c_5T^{5/3+A}}{9c_1^2(1+A)}+\frac{BT^{2/3+A}}{9c_1^2A})u+(-\frac{2c_5T^{5/3+A}}{9c_1^2(1+A)}+\frac{BT^{2/3+A}}{9c_1^2A}+T^{2/3+A})v\\
&+\frac{c_5\xi T^{1+A}}{3c_1(1+A)}+\frac{2c_2c_5T^{5/3+A}}{9c_1^3(1+A)}-\frac{T^{2/3+A}[2c_2B+3A(3c_1^2c_2+3c_1^2c_8-4c_2c_4c_5+3c_1c_3c_5)]}{9c_1^3A(2+3A)},
\end{aligned}$$

where $T=3c_1t+c_4, A=-(2c_1-c_6+c_7)/(3c_1), B=3c_1^2+2c_4c_5-3c_1c_6$. Under this similarity transformation one can reduce the partial differential equation (PDE) system (1) to an ordinary differential equation (ODE) system in the following form:

$$2f'''+[(c_1+c_6-c_7)(2c_1-c_6+c_7)f+4c_1\xi]f'+2c_1f=0, \tag{9a}$$

$$2g'''+[(c_1+c_6-c_7)(2c_1-c_6+c_7)f+4c_1\xi]g'+2(c_6-c_7)g=0, \tag{9b}$$

where prime denotes the differentiation with respect to ξ. If (f,g) is a solution of Eqs.(9a) and (9b), then the group-invariant solution of coupled equation system (1) takes the following form:

$$\begin{aligned}
u =& \ \frac{(3c_1^2-2c_4c_5+3c_1c_7)(c_1^2x-2c_1c_2t+c_1c_3-2c_2c_4)}{2c_1^2(2c_1+c_7-c_6)(3c_1t+c_4)}\\
&-\frac{2c_5(2c_1+c_7-c_6)t+c_7^2-c_1^2+2c_4c_5-c_6c_7-c_1c_6}{4(3c_1t+c_4)^{2/3}}f(\frac{c_1^2x-2c_1c_2t+c_1c_3-2c_2c_4}{c_1^2(3c_1t+c_4)^{1/3}})\\
&-(3c_1t+c_4)^{\frac{c_7-c_6}{3c_1}}g(\frac{c_1^2x-2c_1c_2t+c_1c_3-2c_2c_4}{c_1^2(3c_1t+c_4)^{1/3}})+\frac{c_1c_2c_7+c_1c_3c_5-2c_2c_4c_5+c_1^2c_8}{c_1^2(c_7-c_6)},\\
v =& \ \frac{(3c_1^2+2c_4c_5-3c_1c_6)(c_1^2x-2c_1c_2t+c_1c_3-2c_2c_4)}{2c_1^2(2c_1+c_7-c_6)(3c_1t+c_4)}\\
&+\frac{2c_5(2c_1+c_7-c_6)t+c_1^2-c_6^2+2c_4c_5+c_6c_7-c_1c_7}{4(3c_1t+c_4)^{2/3}}f(\frac{c_1^2x-2c_1c_2t+c_1c_3-2c_2c_4}{c_1^2(3c_1t+c_4)^{1/3}})\\
&+(3c_1t+c_4)^{\frac{c_7-c_6}{3c_1}}g(\frac{c_1^2x-2c_1c_2t+c_1c_3-2c_2c_4}{c_1^2(3c_1t+c_4)^{1/3}})-\frac{c_1c_2c_6+c_1c_3c_5-2c_2c_4c_5+c_1^2c_8}{c_1^2(c_7-c_6)}.
\end{aligned}$$

For Eq.(9a), let $f(\xi)=-\dfrac{w(\xi)}{(c_1+c_6-c_7)(2c_1+c_7-c_6)}$, and integrate once, then

$$6ww''-3(w')^2-w^3+6c_1\xi w^2+C=0,$$

where C is an integration constant. This reduced equation is just the P34 (i.e. the 34th equation in the Painlevé/Gambier classification) which is the Miura transformation of the Painlevé II equation.[14]

In the same way, for the one-parameter group of scaling transformation generated by $v=x\partial_x+3t\partial_t-2u\partial_u-2v\partial_v$, invariants are given by

$$\xi=xt^{-1/3}, \quad f(\xi)=t^{2/3}u, \quad g(\xi)=t^{2/3}v,$$

where f and g satisfies the following ODEs:

$$\begin{aligned}
3f'''+6ff'+6gf'-\xi f'-2f=0,\\
3g'''+6gg'+6fg'-\xi g'-2g=0,
\end{aligned} \tag{10}$$

and prime denotes the differentiation with respect to ξ. If (f,g) is a solution of Eqs.(10), $(u,v)=(t^{-2/3}f(t^{-1/3}x),t^{-2/3}g(t^{-1/3}x))$ is a group-invariant solution of coupled KdV equation system (1).

Case 2 $c_4 = 1, \quad c_3 = k, \quad c_1 = c_2 = c_5 = c_6 = c_7 = c_8 = 0$

In general, travelling wave solutions of a PDE arise as special group invariant solutions in which the group under consideration is a translation group in the space of independent variables. Consider the symmetry group generated by $v = k\partial_x + \partial_t$. Global invariants of this group are

$$\xi = x - kt, \quad f(\xi) = u, \quad g(\xi) = v, \tag{11}$$

in which k is a fixed constant, which will determine the speed of the waves. So that $x = \xi + kt, \quad u = f(\xi)$, and $v = g(\xi)$. Substituting x, u, and v into the coupled KdV equations (1), we obtain the reduced equations of usual travelling wave

$$\begin{aligned} &f''' + 2ff' + 2gf' - kf' = 0, \\ &g''' + 2gg' + 2fg' - kg' = 0. \end{aligned} \tag{12}$$

Let $P(\xi) = f(\xi) + g(\xi)$, and $Q(\xi) = f(\xi) - g(\xi)$, then from Eq.(12) we will find that

$$\begin{aligned} &P''' + 2PP' - kP' = 0, \\ &Q''' + 2PQ' - kQ' = 0. \end{aligned}$$

The reduced equation $P''' + 2PP' - kP' = 0$ can be immediately integrated once, then we have

$$P'' + P^2 - kP = C_1,$$

and a second integration is performed after multiplying it by P' and given as

$$(P')^2/2 = -P^3/3 + kP^2/2 + C_1P + C_2,$$

where C_1 and C_2 are integration constants. If let $C_1 = C_2 = 0$, then this equation has real solutions as follows:

$$P = \frac{3k}{2}\sec h^2\left(\frac{\sqrt{k}}{2}\xi + \delta\right), \quad k > 0$$

or

$$P = \frac{3k}{2}\sec^2\left(\frac{\sqrt{-k}}{2}\xi + \delta\right), \quad k < 0,$$

where δ is an arbitrary phase shift.

Let $Q = 2C$, the coupled equations(11) have real solutions as follows:

$$(f, g) = \left(\frac{3k}{4}\sec h^2\left(\frac{\sqrt{k}}{2}\xi + \delta\right) + C, \frac{3k}{4}\sec h^2\left(\frac{\sqrt{k}}{2}\xi + \delta\right) - C\right), k > 0,$$

and

$$(f, g) = \left(\frac{3k}{4}\sec^2\left(\frac{\sqrt{-k}}{2}\xi + \delta\right) + C, \frac{3k}{4}\sec^2\left(\frac{\sqrt{-k}}{2}\xi + \delta\right) - C\right), k < 0. \tag{13}$$

where C is an arbitrary constant. Substituting Eq.(13) into Eq.(11) yields the general travelling wave solutions of the coupled KdV equation system (1), which are bell-shaped soliton solutions i.e. singular periodic wave solutions

$$(u, v) = \left(\frac{3k}{4}\sec h^2\left(\frac{\sqrt{k}}{2}(x - kt) + \delta\right) + C, \frac{3k}{4}\sec h^2\left(\frac{\sqrt{k}}{2}(x - kt) + \delta\right) - C\right), k > 0,$$

and

$$(u, v) = \left(\frac{3k}{4}\sec^2\left(\frac{\sqrt{-k}}{2}(x - kt) + \delta\right) + C, \frac{3k}{4}\sec^2\left(\frac{\sqrt{-k}}{2}(x - kt) + \delta\right) - C\right), k < 0.$$

Case 3 $c_2 = 1, \quad c_1 = c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = 0$

The one-parameter group of Galilean boosts, generated by $v = 4t\partial_x + \partial_u + \partial_v$, has global invariants $\xi = t, \quad f(\xi) = u - x/(4t)$, and $g(\xi) = v - x/(4t)$, so that $t = \xi, \quad u = f(\xi) + x/(4t), \quad v = g(\xi) + x/(4t)$. Substituting t, u, v into the coupled KdV equations(1), we obtain that the reduced equations are simply

$$2tf' + f + g = 0,$$

$$2tg' + f + g = 0.$$

so the general Galilean-invariant solution of coupled KdV equation system (1) is

$$u = x/(4t) + C_1/t - C_2,$$
$$v = x/(4t) + C_1/t + C_2.$$

where C_1, C_2 are arbitrary constants.

Remark: A more interesting class of solution with Galilean-like invariance can be found by adding a time translational component to this group. The generator $v = 4t\partial_x + \partial_t + \partial_u + \partial_v \neq 0$ has global invariants

$$\xi = -x/2 + t^2, \quad f(\xi) = u - t, \quad g(\xi) = v - t,$$

so the reduced equations of the coupled KdV equation system (1) are

$$f''' + 8ff' + 8gf' - 8 = 0, \tag{14a}$$
$$g''' + 8gg' + 8fg' - 8 = 0. \tag{14b}$$

Let $P(\xi) = f(\xi) + g(\xi)$ and $Q(\xi) = f(\xi) - g(\xi)$, from Eq.(13) we find that

$$P''' + 8PP' - 16 = 0, \tag{15a}$$
$$Q''' + 8PQ' = 0. \tag{15b}$$

Integrating Eq.(14a) once, we obtain a second-order equation

$$P'' + 4P^2 - 16\xi + C_1 = 0,$$

which is known as the first Painlevè transcendent. Its solutions $P = w(\xi)$ are meromorphic in the entire complex plane, but are essentially new functions not expressible in terms of standard special functions. Now let $Q = 2C$, the corresponding solutions of coupled KdV equation system(1) take the form

$$u = w(-x^2/2 + t^2)/2 + t + C,$$
$$v = w(-x^2/2 + t^2)/2 + t - C,$$

where C is an arbitrary constant.

Case 4 $c_2 = c_6 = 1, \quad c_1 = c_3 = c_4 = c_5 = c_7 = c_8 = 0$

The symmetry group invariants generated by $v = 4t\partial_x - (u-1)\partial_u + (u+1)\partial_v$ turn out to be the similarity variables

$$\xi = t, \quad f(\xi) = (u-1)\mathrm{e}^{\frac{x}{4t}}, \quad g(\xi) = (u+v) - \frac{x}{2t} - 1.$$

Thus $t = \xi, \quad u = \mathrm{e}^{-\frac{x}{4t}} f(\xi) + 1, \quad v = -\mathrm{e}^{-\frac{x}{4t}} f(\xi) + g(\xi) + \dfrac{x}{2t}$. So we obtain the reduced equations of the coupled KdV equation system (1) as follows:

$$\begin{aligned} &64\xi^3 f' - f - 32\xi^2 f - 32\xi^2 fg = 0, \\ &\xi g' + g + 1 = 0. \end{aligned} \tag{16}$$

The general solutions of coupled equation system(16) are

$$f(\xi) = C_1 \mathrm{e}^{-(64C_2\xi+1)/(128\xi^2)},$$
$$g(\xi) = C_2/\xi - 1,$$

where C_1 and C_2 are integration constants. Thus the coupled KdV equation system (1) possesses the exact solutions

$$u = C_1 \mathrm{e}^{\frac{-32xt-64C_2t-1}{128t^2}} + 1,$$
$$v = -C_1 \mathrm{e}^{\frac{-32xt-64C_2t-1}{128t^2}} + \frac{x}{2t} + \frac{C_2}{t} - 1.$$

Case 5 $c_3 = 1, \quad c_6 = k, \quad c_1 = c_2 = c_4 = c_5 = c_7 = c_8 = 0$

For the one-parameter group generated by $v = \partial_x - ku\partial_u + ku\partial_v$, where k is an arbitrary constant, one can have the similarity variables of the form

$$\xi = t, \quad f(\xi) = \mathrm{e}^{kx}u, \quad g(\xi) = u + v$$

so the reduced equations of coupled KdV equations (1) are

$$\begin{aligned} &f' - k^3 f - 2kfg = 0, \\ &g' = 0. \end{aligned} \tag{17}$$

The solution of Eq.(17) is $(f, g) = (C_1 \mathrm{e}^{(k^3+2C_2k)t}, C_2)$. So the coupled KdV equation system (1) possesses exponential solutions as follows:

$$\begin{aligned} &u = C_1 \mathrm{e}^{-k[x-(k^2+2C_2)t]}, \\ &v = -C_1 \mathrm{e}^{-k[x-(k^2+2C_2)t]} + C_2. \end{aligned} \tag{18}$$

where C_1 and C_2 are the integration constants.

One can also investigate some other sub-cases by assuming some of the other arbitrary constants in the vector fields Eq.(9) to be zero. However, they all cannot lead to any simple similarity reductions.

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