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A new high-order spectral problem of the mKdV and its associated integrable decomposition*

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A new approach to formulizing a new high-order matrix spectral problem from a normal 2×2 matrix modified Korteweg–de Vries (mKdV) spectral problem is presented. It is found that the isospectral evolution equation hierarchy of this new higher-order matrix spectral problem turns out to be the well-known mKdV equation hierarchy. By using the binary nonlinearization method, a new integrable decomposition of the mKdV equation is obtained in the sense of Liouville. The proof of the integrability shows that r-matrix structure is very interesting.

Keywords: spectral problem, integrable decomposition, mKdV equation hierarchy
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1. Introduction

Formulizing high-order matrix spectral problems and searching for new integrable hierarchies of their corresponding soliton equations are important and interesting topics in soliton theory. As is known, Ma and Zhou[1] studied a kind of multicomponent Ablowitz–Kaup–Newell–Segur (AKNS) equations that are associated with an arbitrary order degenerate matrix spectral problem in the binary non-linearization approach, and Ma[2] discussed the integrable coupling of this multicomponent AKNS equations; Ma and Zhou[3] discussed the binary symmetry constraints of N-wave interaction equations in 1+1 and 2+1 dimensions; Guo and Zhang[4] presented a new loop algebra and a corresponding integrable hierarchy; Xia and Fan[5] discussed the multicomponent Kaup–Newell hierarchy; Ma *et al*[6,7] obtained new completely integrable Neumann systems related to the perturbation Korteweg–de Vries hierarchy and AKNS hierarchy, respectively. Many other studies are devoted to obtaining coupling integrable systems and coupling soliton equations by formulating new 4×4 spectral problem.[8−11]

Especially, Ma *et al*[12] presented a 3×3 matrix spectral problem for AKNS hierarchy; Zhu *et al*[13] formulized a 3×3 matrix spectral problem for Kaup–Newell hierarchy; Blaszak and Ma[14] offered a way to formulize a new 3×3 matrix spectral problem from a normal 2×2 matrix spectral problem.

The mKdV equation[15] is an important equation in mathematical physics. The nonlinearization method[1,7,9,12−22] presented a way to obtain the integrable Hamiltonian systems and the solutions of a partial differential equation by decomposing it into two ordinary differential equations. Hence, we can obtain new solutions of mKdV equation from new high-order matrix spectral problems associated constraints.

In this paper, we formulize a kind of new higher-order spectral problem from a normal 2×2 mKdV spectral problem and obtain the corresponding integrable decomposition of the mKdV equation. In Section 2, we recall some results of the hierarchy of the mKdV equations obtained from the mKdV spectral problem in 2×2 matrix representation as a preparatory knowledge for further consideration. In Section 3, we formulize a kind of $m \times m$ spectral problem from a normal 2×2 MKdV spectral problem. It is found that the isospectral evolution equation hierarchy of the new $m \times m$ spectral problem turns out to be the well-known mKdV equation hierarchy. The binary restricted mKdV flows and its Lax representation are presented. The r-matrix formulation is developed,

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and an involutive system of mN (number of integral of motion) functionally independent integrals of motion generated from the Lax representation is deduced.

2. The hierarchy of the mKdV equation

Consider a spectral problem described in the following form:

$$\Phi_x = U(\lambda, u)\Phi, \tag{1}$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad U(\lambda, u) = \begin{pmatrix} u & \lambda \\ -1 & -u \end{pmatrix}.$$

Taking

$$V = \begin{pmatrix} a & b\lambda \\ c & -a \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i\lambda \\ c_i & -a_i \end{pmatrix} \lambda^{-i} \tag{2}$$

we find that the adjoint representation equation $V_x = [U, V] = UV - VU$ can yield

$$b_0 + c_0 = 0, \quad a_{ix} = c_{i+1} + b_{i+1}, \quad b_{ix} = 2(ub_i - a_i), \quad c_{ix} = -2(a_i + c_i), \quad i \geq 0. \tag{3}$$

Choosing $a_0 = u,\ b_0 = 1$, and $c_0 = -1$ yields

$$\begin{aligned} a_1 &= -\frac{1}{4}u_{xx} + \frac{1}{2}u^3, \quad b_1 = \frac{1}{2}u_x + \frac{1}{2}u^2, \quad c_1 = \frac{1}{2}u_x - \frac{1}{2}u^2, \\ a_2 &= \frac{1}{16}u_{xxxx} - \frac{5}{8}u^2u_{xx} - \frac{5}{8}uu_x^2 + \frac{3}{8}u^5, \quad b_2 = -\frac{1}{8}u_{xxx} + \frac{3}{4}u^3, \\ c_2 &= -\frac{1}{8}u_{xxx} + \frac{3}{4}u^2u_x + \frac{1}{4}uu_{xx} - \frac{1}{8}u_x^2 - \frac{3}{8}u^4. \end{aligned}$$

We consider an auxiliary spectral problem described as follows:

$$\Phi_{t_n} = V^{(n)}\Phi, \tag{4}$$

where

$$\begin{aligned} V^{(n)} &= (\lambda^n V)_+ - \begin{pmatrix} 0 & b_{n+1} \\ 0 & 0 \end{pmatrix} \\ &= \sum_{i=0}^{n} \begin{pmatrix} (a\lambda^n)_+ & (b\lambda^n)_+\lambda \\ (c\lambda^n)_+ & -(a\lambda^n)_+ \end{pmatrix}, \quad n \geq 0. \end{aligned}$$

And the zero curvature equation $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0$ gives the isospectral mKdV equation hierarchy

$$u_{t_n} = a_{n,x}, \quad n \geq 0. \tag{5}$$

For $n = 1$, we obtain the well-known mKdV equation

$$u_{t_n} = -\frac{1}{4}u_{xxx} + \frac{3}{2}u^2u_x, \tag{6}$$

and the corresponding Lax operators

$$\begin{aligned} U(\lambda, u) &= \begin{pmatrix} u & \lambda \\ -1 & -u \end{pmatrix}, \\ V^{(1)} &= \begin{pmatrix} u\lambda - \frac{1}{4}u_{xx} + \frac{1}{2}u^3 & \lambda^2 + \frac{1}{2}(u_x + u^2)\lambda \\ -\lambda + \frac{1}{2}(u_x - u^2) & \lambda \end{pmatrix}. \end{aligned} \tag{7}$$

3. A new kind of high-order mKdV spectral problem and its associated integrable decomposition

3.1. A new $m \times m$ spectral problem

Recently, Yu *et al*[23] have formulized a new 3×3 spectral problem from a normal 2×2 mKdV spectral problem. As usual, here we formulize a new $m \times m$ spectral problem from a normal 2×2 mKdV spectral problem.

Assume

$$(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) = (\alpha\phi_1\phi_2, \phi_2^2, \phi_1^2), \tag{8}$$

$$(\bar{\phi}_1, \bar{\phi}_2, \cdots, \bar{\phi}_{m-1}, \bar{\phi}_m)\cdots, \bar{\phi}_{m-1}, \bar{\phi}_m) = (\phi_1^{m-2}\phi_2, \phi_1^{m-3}\phi_2^2, \cdots, \phi_2^{m-1}, \phi_1^{m-1}), \quad m \geq 4. \tag{9}$$

When $\alpha = 1$, Eq.(8) can be deduced from Eq.(9). Thus a new spectral problem can be obtained as follows:

$$\begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_3 \end{pmatrix}_x = \begin{pmatrix} 0 & \alpha\lambda & -\alpha \\ -\dfrac{2}{\alpha} & -2u & 0 \\ \dfrac{2}{\alpha}\lambda & 0 & 2u \end{pmatrix} \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \\ \bar{\phi}_3 \end{pmatrix},$$

when $\alpha = \sqrt{2}$ and performing the simple transformation

$$(\varphi_1, \varphi_2, \varphi_3) = (\bar{\phi}_3, \bar{\phi}_1, \bar{\phi}_2),$$

we obtain the spectral problem as shown in Ref.[23].

When $m \geq 4$, the new $m \times m$ matrix mKdV spectral problem is obtained as follows:

$$\begin{aligned} &\bar{\Phi}_x = \widehat{U}(u, \lambda)\bar{\Phi}, \\ &\bar{\Phi} = (\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3, \ldots, \bar{\phi}_{m-2}, \bar{\phi}_{m-1}, \bar{\phi}_m)^{\mathrm{T}}, \end{aligned} \tag{10}$$

where

$$\widehat{U}(u,\lambda) = \begin{pmatrix} (m-3)u & (m-2)\lambda & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\ -2 & (m-5)u & (m-3)\lambda & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -3 & (m-7)u & (m-4)\lambda & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -(m-2) & (m-3)u & \lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -(m-1) & -(m-1)u & 0 \\ (m-1)\lambda & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (m-1)u \end{pmatrix}.$$

Take

$$\widehat{V}(u,\lambda) = \begin{pmatrix} (m-3)a & (m-2)b\lambda & 0 & 0 & \cdots & 0 & 0 & 0 & c \\ 2c & (m-5)a & (m-3)b\lambda & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 3c & (m-7)a & (m-4)b\lambda & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (m-2)c & -(m-3)a & b\lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & (m-1)c & -(m-1)a & 0 \\ (m-1)b\lambda & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (m-1)a \end{pmatrix}.$$

In a way similar to that in Section 2, we can find a, b and c satisfying Eq.(3), hence, we can obtain the mKdV equation hierarchy (5).

Under N distinct constants $\lambda_1, \ldots, \lambda_N$, we consider the spectral problem (10) and its corresponding adjoint spectral problem

$$\begin{aligned} &\bar{\Phi}_{j,x} = \widehat{U}(\widehat{u}, \lambda_j)\bar{\Phi}_j, \bar{\Phi} = (\bar{\phi}_{1j}, \bar{\phi}_{2j}, \bar{\phi}_{3j}, \cdots, \bar{\phi}_{m-2,j}, \bar{\phi}_{m-1,j}, \bar{\phi}_{mj})^{\mathrm{T}}, \quad 1 \leq j \leq N, \\ &\bar{\Psi}_{j,x} = -\widehat{U}^T(\widehat{u}, \lambda_j)\bar{\Psi}_j, \bar{\Psi} = (\bar{\psi}_{1j}, \bar{\psi}_{2j}, \bar{\psi}_{3j}, \cdots, \bar{\psi}_{m-2,j}, \bar{\psi}_{m-1,j}, \bar{\psi}_{mj})^{\mathrm{T}}, \quad 1 \leq j \leq N, \quad 1 \leq j \leq N. \end{aligned} \tag{11}$$

From Ref.[12] we have

$$\begin{aligned} \frac{\delta\lambda_j}{\delta u} = {} & (m-3) < \bar{\phi}_{1j}, \bar{\psi}_{1j} > + (m-5) < \bar{\phi}_{2j}, \bar{\psi}_{2j} > + (m-7) < \bar{\phi}_{3j}, \bar{\psi}_{3j} > \\ & + \cdots - (m-3) < \bar{\phi}_{m-2,j}, \bar{\psi}_{m-2,j} > - (m-1) < \bar{\phi}_{m-1,j}, \bar{\psi}_{m-1,j} > + (m-1) < \bar{\phi}_{mj}, \bar{\psi}_{mj} > . \end{aligned}$$

As a conventional approach to binary nonlinearization of Lax pairs, we consider the constraints

$$\begin{aligned} u &= \sum_{i=1}^{N} \frac{\delta \lambda_i}{\delta u} \\ &= (m-3) < \bar{Q}_1, \bar{P}_1 > +(m-5) < \bar{Q}_2, \bar{P}_2 > +(m-7) < \bar{Q}_3, \bar{P}_3 > \\ &\quad + \cdots - (m-3) < \bar{Q}_{m-2}, \bar{P}_{m-2} > -(m-1) < \bar{Q}_{m-1}, \bar{P}_{m-1} > +(m-1) < \bar{Q}_m, \bar{P}_m > . \end{aligned} \tag{12}$$

Under the constraint (12) and

$$\begin{aligned} \bar{\Phi}_{j,t_n} &= \widehat{V}^{(n)}(\widehat{u}, \lambda_j)\bar{\Phi}_j, \bar{\Phi} = (\bar{\phi}_{1j}, \bar{\phi}_{2j}, \bar{\phi}_{3j}, \cdots, \bar{\phi}_{m-2,j}, \bar{\phi}_{m-1,j}, \bar{\phi}_{mj})^{\mathrm{T}}, \quad 1 \le j \le N, \\ \bar{\Psi}_{j,t_n} &= -(\widehat{V}^{(n)})^T(\widehat{u}, \lambda_j)\bar{\Psi}_j, \bar{\Psi} = (\bar{\psi}_{1j}, \bar{\psi}_{2j}, \bar{\psi}_{3j}, \cdots, \bar{\psi}_{m-2,j}, \bar{\psi}_{m-1,j}, \bar{\psi}_{m,j})^{\mathrm{T}}, \quad 1 \le j \le N, \end{aligned} \tag{13}$$

we can obtain a hierarchy of finite-dimensional Hamiltonian systems.

For $n = 0$, the system (11) is nonlinearized into the following Hamiltonian system or x-flow:

$$\bar{Q}_{i_x} = \frac{\partial \bar{H}_1}{\partial \bar{P}_i}, \quad \bar{P}_{i_x} = -\frac{\partial \bar{H}_1}{\partial \bar{Q}_i}, \quad i = 1, 2, \ldots, m, \tag{14}$$

where

$$\begin{aligned} \bar{H}_1 &= \frac{1}{2}[(m-3) < \bar{Q}_1, \bar{P}_1 > +(m-5) < \bar{Q}_2, \bar{P}_2 > +(m-7) < \bar{Q}_3, \bar{P}_3 > +\cdots \\ &\quad -(m-3) < \bar{Q}_{m-2}, \bar{P}_{m-2} > -(m-1) < \bar{Q}_{m-1}, \bar{P}_{m-1} > +(m-1) < \bar{Q}_m, \bar{P}_m >]^2 \\ &\quad +((m-1) < \Lambda\bar{Q}_1, \bar{P}_m > +(m-2) < \Lambda\bar{Q}_2, \bar{P}_1 > +(m-3) < \Lambda\bar{Q}_3, \bar{P}_2 > +\cdots + < \Lambda\bar{Q}_{m-1}, \bar{P}_{m-2} >) \\ &\quad -(2 < \bar{Q}_1, \bar{P}_2 > +3 < \bar{Q}_2, \bar{P}_3 > +\cdots + (m-1) < \bar{Q}_{m-2}, \bar{P}_{m-1} > + < \bar{Q}_m, \bar{P}_1 >). \end{aligned}$$

For $n = 1$, the system (13) is nonlinearized into the following Hamiltonian system or t-flows:

$$\bar{Q}_{it_1} = \frac{\partial \bar{H}_2}{\partial \bar{P}_i}, \quad \bar{P}_{it_1} = -\frac{\partial \bar{H}_2}{\partial \bar{Q}_i}, \quad i = 1, 2, \ \cdots, \ m, \tag{15}$$

where

$$\begin{aligned} \bar{H}_2 &= \frac{3}{4}[(m-3) < \Lambda\bar{Q}_1, \bar{P}_1 > +(m-5) < \Lambda\bar{Q}_2, \bar{P}_2 > +(m-7) < \Lambda\bar{Q}_3, \bar{P}_3 > +\cdots \\ &\quad -(m-3) < \Lambda\bar{Q}_{m-2}, \bar{P}_{m-2} > -(m-1) < \Lambda\bar{Q}_{m-1}, \bar{P}_{m-1} > +(m-1) < \Lambda\bar{Q}_m, \bar{P}_m >]^2 \\ &\quad +\frac{1}{4}[(m-3) < \bar{Q}_1, \bar{P}_1 > +(m-5) < \bar{Q}_2, \bar{P}_2 > +(m-7) < \bar{Q}_3, \bar{P}_3 > +\cdots \\ &\quad -(m-3) < \bar{Q}_{m-2}, \bar{P}_{m-2} > -(m-1) < \bar{Q}_{m-1}, \bar{P}_{m-1} > +(m-1) < \bar{Q}_m, \bar{P}_m >]^2 \\ &\quad \frac{1}{8}[(m-3) < \bar{Q}_1, \bar{P}_1 > +(m-5) < \bar{Q}_2, \bar{P}_2 > +(m-7) < \bar{Q}_3, \bar{P}_3 > +\cdots \\ &\quad -(m-3) < \bar{Q}_{m-2}, \bar{P}_{m-2} > -(m-1) < \bar{Q}_{m-1}, \bar{P}_{m-1} > +(m-1) < \bar{Q}_m, \bar{P}_m >]^4 \\ &\quad -\frac{1}{2}[(m-3) < \bar{Q}_1, \bar{P}_1 > +(m-5) < \bar{Q}_2, \bar{P}_2 > +(m-7) < \bar{Q}_3, \bar{P}_3 > +\cdots \\ &\quad -(m-3) < \bar{Q}_{m-2}, \bar{P}_{m-2} > -(m-1) < \bar{Q}_{m-1}, \bar{P}_{m-1} > +(m-1) < \bar{Q}_m, \bar{P}_m >]^2 \\ &\quad [(2 < \bar{Q}_1, \bar{P}_2 > +3 < \bar{Q}_2, \bar{P}_3 > +\cdots + (m-1) < \bar{Q}_{m-2}, \bar{P}_{m-1} > + < \bar{Q}_m, \bar{P}_1 >) \\ &\quad -((m-1) < \Lambda\bar{Q}_1, \bar{P}_m > +(m-2) < \Lambda\bar{Q}_2, \bar{P}_1 > +(m-3) < \Lambda\bar{Q}_3, \bar{P}_2 > +\cdots \\ &\quad + < \Lambda\bar{Q}_{m-1}, \bar{P}_{m-2} >)] - \frac{1}{2}[((m-1) < \Lambda\bar{Q}_1, \bar{P}_m > +(m-2) < \Lambda\bar{Q}_2, \bar{P}_1 > \\ &\quad +(m-3) < \Lambda\bar{Q}_3, \bar{P}_2 > +\cdots + < \Lambda\bar{Q}_{m-1}, \bar{P}_{m-2} >) + (2 < \bar{Q}_1, \bar{P}_2 > +3 < \bar{Q}_2, \bar{P}_3 > +\cdots \\ &\quad +(m-1) < \bar{Q}_{m-2}, \bar{P}_{m-1} > + < \bar{Q}_m, \bar{P}_1 >)]^2 + ((m-1) < \Lambda^2\bar{Q}_1, \bar{P}_m > \\ &\quad +(m-2) < \Lambda^2\bar{Q}_2, \bar{P}_1 > +(m-3) < \Lambda^2\bar{Q}_3, \bar{P}_2 > +\cdots + < \Lambda^2\bar{Q}_{m-1}, \bar{P}_{m-2} >) \\ &\quad -(2 < \Lambda\bar{Q}_1, \bar{P}_2 > +3 < \Lambda\bar{Q}_2, \bar{P}_3 > +\cdots + (m-1) < \Lambda\bar{Q}_{m-2}, \bar{P}_{m-1} > + < \Lambda\bar{Q}_m, \bar{P}_1 >). \end{aligned}$$

We can prove that if $\bar{u}$ satisfies Hamiltonian systems (14) and (15) then $\bar{u}$ determined by the constraint (12) becomes the solution of the mKdV equation (5). Thus, a new decomposition of the mKdV equation (5) is presented.

3.2. Lax representation and r-matrix formula

Through a lengthy but direct calculation, we can obtain the following propositions.

Proposition 1. Hamiltonian systems (14) and (15) have the following Lax representations respectively:

$$L(\lambda)_x = [\tilde{\widehat{U}}(\lambda), L(\lambda)] \quad \text{and} \quad L(\lambda)_t = [\tilde{\widehat{V}^{(1)}}(\lambda), L(\lambda)] \tag{16}$$

where

$$L(\Lambda) = \begin{pmatrix} 0 & \frac{1}{2}(m-2) & 0 & 0 & \cdots & 0 & 0 & 0 & C_0 \\ 2C_0 & 0 & \frac{1}{2}(m-3) & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 3C_0 & 0 & \frac{1}{2}(m-4) & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (m-2)C_0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & (m-1)C_0 & 0 & 0 \\ \frac{1}{2}(m-1) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \widehat{V}, \tag{17}$$

and

$$\widehat{V} = \begin{pmatrix} (m-3)A_i & (m-2)B_i & 0 & 0 & \cdots & 0 & 0 & 0 & C_i \\ 2C_i & (m-5)A_i & (m-3)B_i & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 3C_i & (m-7)A_i & (m-4)B_i & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (m-2)C_i & -(m-3)A_i & B_i & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & (m-1)C_i & -(m-1)A_i & 0 \\ (m-1)B_i & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (m-1)A_i \end{pmatrix}.$$

Here

$$\begin{aligned} C_0 &= -\frac{1}{\lambda}\left[\frac{1}{2} + (m-1) < \bar{Q}_1, \bar{P}_m > + (m-2) < \bar{Q}_2, \bar{P}_1 > + \cdots + < \bar{Q}_{m-1}, \bar{P}_{m-2} >\right] \\ 2A_j &= (m-3)\bar{\phi}_{1j}\bar{\psi}_{1j} + (m-5)\bar{\phi}_{2j}\bar{\psi}_{2j} + (m-7)\bar{\phi}_{3j}\bar{\psi}_{3j} + \cdots \\ &\quad -(m-3)\bar{\phi}_{(m-2)j}\bar{\psi}_{(m-2)j} - (m-1)\bar{\phi}_{(m-1)j}\bar{\psi}_{(m-1)j} + (m-1)\bar{\phi}_{mj}\bar{\psi}_{mj}, \\ B_j &= 2\bar{\phi}_{1j}\bar{\psi}_{2j} + 3\bar{\phi}_{2j}\bar{\psi}_{3j} + \cdots + (m-1)\bar{\phi}_{(m-2)j}\bar{\psi}_{(m-1)j} + \bar{\phi}_{mj}\bar{\psi}_{1j}, \\ C_j &= (m-2)\bar{\phi}_{2j}\bar{\psi}_{1j} + (m-3)\bar{\phi}_{3j}\bar{\psi}_{2j} + \cdots + \bar{\phi}_{(m-1)j}\bar{\psi}_{(m-2)j} + (m-1)\bar{\phi}_{1j}\bar{\psi}_{mj}, \end{aligned} \tag{18}$$

and $\tilde{\widehat{U}}$ and $\tilde{\widehat{V}^{(1)}}$ are two constrained spectral matrices generated from $\widehat{U}$ and $\widehat{V}^{(1)}$ under the constraints (12).

It is convenient to obtain

$$L(\lambda) = \begin{pmatrix} (m-3)A & (m-2)B & 0 & 0 & \cdots & 0 & 0 & 0 & C \\ 2C & -(m-5)A & (m-3)B & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 3C & -(m-7)A & (m-4)B & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (m-2)C & (m-3)A & B & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & (m-1)C & (m-1)A & 0 \\ (m-1)B & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -(m-1)A \end{pmatrix},$$

where

$$A(\lambda) = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} A_j,$$
$$B(\lambda) = \frac{1}{2} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} B_j,$$
$$C(\lambda) = C_0 + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} C_j.$$

After a lengthy but direct calculation, we arrive at

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0,$$
$$\{A(\lambda), B(\mu)\} = \frac{1}{2(\mu - \lambda)}(B(\mu) - B(\lambda)),$$
$$\{A(\lambda), C(\mu)\} = \frac{1}{2(\mu - \lambda)}(C(\lambda) - C(\mu)) - \frac{1}{2\mu} C(\lambda),$$
$$\{B(\lambda), C(\mu)\} = \frac{2}{\mu - \lambda}(A(\mu) - A(\lambda)) + \frac{2}{\mu} A(\lambda).$$

From the above relations, we can obtain the following result.

Proposition 2. With the standard Poisson bracket, the Lax matrix $L(\lambda)$ admits the r-matrix representation

$$\{L_1(\lambda) \otimes L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] + [r_{21}(\lambda, \mu), L_2(\mu)], \tag{19}$$

where

$$r_{12}(\lambda, \mu) = \frac{1}{2(\mu - \lambda)}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_3 + 2\sigma_3 \otimes \sigma_2) - \frac{1}{2\mu}\sigma_2 \otimes \sigma_3,$$
$$r_{21}(\lambda, \mu) = \frac{1}{2(\mu - \lambda)}(\sigma_1 \otimes \sigma_1 + 2\sigma_2 \otimes \sigma_3 + \sigma_3 \otimes \sigma_2) + \frac{1}{2\lambda}\sigma_3 \otimes \sigma_2,$$
$$L_1(\lambda) = L(\lambda) \otimes E, L_2(\lambda) = E \otimes L(\lambda),$$

and

$$\sigma_1 = \begin{pmatrix} (m-3) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & (m-5) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & (m-7) & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -(m-3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -(m-1) & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (m-1) \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & m-2 & 0 & 0 & \cdots & 0\,0\,0\,0 \\ 0 & 0 & m-3 & 0 & \cdots & 0\,0\,0\,0 \\ 0 & 0 & 0 & m-4 & \cdots & 0\,0\,0\,0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\,\vdots\,\vdots\,\vdots \\ 0 & 0 & 0 & 0 & \cdots & 0\;\;1\,0 \\ 0 & 0 & 0 & 0 & \cdots & 0\,0\,0\,0 \\ m-1 & 0 & 0 & 0 & \cdots & 0\,0\,0\,0 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 0\,0\,0\,0 & \cdots & 0 & 0 & 0\,1 \\ 2\,0\,0\,0 & \cdots & 0 & 0 & 0\,0 \\ 0\,0\,3\,0 & \cdots & 0 & 0 & 0\,0 \\ \vdots\,\vdots\,\vdots\,\vdots & \cdots & \vdots & \vdots & \vdots\,\vdots \\ 0\,0\,0\,0 & \cdots & m-2 & 0 & 0\,0 \\ 0\,0\,0\,0 & \cdots & 0 & m-1 & 0\,0 \\ 0\,0\,0\,0 & \cdots & 0 & 0 & 0\,0 \end{pmatrix}, \qquad E = \begin{pmatrix} 1\,0\,0\,0 & \cdots & 0\,0\,0\,0 \\ 0\,1\,0\,0 & \cdots & 0\,0\,0\,0 \\ 0\,0\,1\,0 & \cdots & 0\,0\,0\,0 \\ \vdots\,\vdots\,\vdots\,\vdots & \cdots & \vdots\,\vdots\,\vdots\,\vdots \\ 0\,0\,0\,0 & \cdots & 0\,1\,0\,0 \\ 0\,0\,0\,0 & \cdots & 0\,0\,1\,0 \\ 0\,0\,0\,0 & \cdots & 0\,0\,0\,1 \end{pmatrix},$$

It is interesting to find $\widehat{U}(\widehat{u},\lambda_) = u\sigma_1 + \Lambda\sigma_2 - \sigma_3$. From the r-matrix theory,[24] we have

$$mn\{\mathrm{tr}L(\lambda)\otimes\mathrm{tr}L(\mu)\} = \mathrm{tr}\{L^m(\lambda)\otimes L^n(\mu)\} = 0. \tag{20}$$

Therefore, $\mathrm{tr}L^k(\lambda)$ can generate the involutive integrals of motion. We remark that $\mathrm{tr}L(\lambda) = \mathrm{tr}L^3(\lambda) = 0$, and the involutive integrals of motion can be generated only from $\mathrm{tr}L^2(\lambda)$.

Assuming

$$\frac{1}{2\sum\limits_{l=1,3,\cdots}(m-l)^2}\mathrm{tr}L^2(\lambda) = A^2 + BC = \sum_{k=1}^{\infty}\bar{F}_k\lambda^{-k+1}, \tag{21}$$

we can prove that $\bar{F}_i (i = 1, 2, \cdots, (m-1)N)$ and integrals of motion $f_i = \sum_{k=1}^{m}\phi_{ki}\psi_{ki} (j = 1, 2, \cdots, N)$ are involutive[24] and functionally independent in a certain region[3] of $\mathbb{R}^{2mN}$.

Therefore, we obtain the following theorem.

Theorem 1 The spatial constrained flow (14) and the temporal constrained flow (15) of the mKdV equation (5) are completely integrable in the sense of Liouville, and the decompositions of the mKdV equation (5): Hamiltonian systems (14) and (15) are integrable.

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