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A Birkhoff–Noether method of solving differential equations*

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In this paper, a Birkhoff–Noether method of solving ordinary differential equations is presented. The differential equations can be expressed in terms of Birkhoff's equations. The first integrals for differential equations can be found by using the Noether theory for Birkhoffian systems. Two examples are given to illustrate the application of the method.

Keywords: differential equation, Birkhoffian system, Noether theory, first integral
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1. Introduction

The key and essential issue for differential equations is how to find their solutions. Any general analytical solution of the differential equations contains n constants of integration that are usually evaluated from the n initial conditions. A method of expressing the general solution is to obtain n first integrals of differential equations. It is usually not possible to obtain all the first integrals through any direct process. Symmetry method for differential equations can be used to find first integrals for higher-order differential equations and reduce their orders. In principle, symmetry reduction can be accomplished in three different ways: Noether symmetry,[1−4] Lie symmetry[2−16] and form invariance.[17−20] In this paper, we introduce a symmetry method named Birkhoff–Noether method to solve differential equations. The main idea of this method is to find first integrals for differential equations by following the procedures: give the representation of differential equations in terms of Birkhoff's equations; and then use Noether symmetry for Birkhoffian systems to find first integrals for differential equations. If one could find all integrals of differential equations, their solutions could be found. The notion of 'representation' of a given differential equation in terms of Birkhoff's equations is defined in detail in Ref.[21]. There have been some developments in symmetry method for differential equations as evidenced by research papers.[22,23]

2. Birkhoffian representations of differential equations

We study the following equations:

$$\dot{a}^{\mu} - \sigma_{\mu}(t, \boldsymbol{a}) = 0, \ (\mu = 1, \cdots, 2n). \tag{1}$$

Suppose that Eqs.(1) possess a representation in terms of Birkhoff's equations

$$\dot{a}^{\mu} - \sigma_{\mu} = \left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial t} = 0, \tag{2}$$

which are equivalent to

$$\left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\sigma_{\nu} = \frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t}. \tag{3}$$

If Birkhoffian B is given, Eqs.(3) are of Cauchy–Kovalevski form. The solutions of Eqs.(3) exist according to Cauchy–Kovalevski theorem. Equations (1) can be rewritten in the following form:

$$(\Omega_{\mu\nu}\dot{a}^{\nu} + \Gamma_{\mu})_{SA} = 0, \tag{4}$$

where the subscript SA denotes self-adjoint, and

$$\Omega_{\mu\nu} = \left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right). \tag{5}$$

The method of computing the Birkhoffian functions R_{μ} and B is given by Santilli.[21] According to the

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rules

$$R_\mu = \int_0^1 [\mathrm{d}\tau\tau\Omega_{\nu\mu}(t,\tau\boldsymbol{a})]a^\nu,$$
$$B = -\left[\int_0^1 \mathrm{d}\tau\left(\Gamma_\mu + \frac{\partial R_\mu}{\partial t}\right)(t,\tau\boldsymbol{a})\right]a^\mu, \quad (6)$$

Eqs.(1) can be rewritten in terms of Birkhoff's equations (2). It is easy to express Eqs.(1) in the self-adjoint form.

3. First integrals and reduction through Noether theory

If the differential equations possess a Birkhoffian representation, one can find their first integrals by using the Noether theorem for Birkhoff's equations.[24] If the infinitesimal generators $\xi_0 = \xi_0(t,\boldsymbol{a}), \xi_\mu = \xi_\mu(t,\boldsymbol{a})$ and gauge function $G_N = G_N(t,\boldsymbol{a})$ satisfy the Noether identity

$$\left(\frac{\partial R_\mu}{\partial t}\dot{a}^\mu - \frac{\partial B}{\partial t}\right)\xi_0 + \left(\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu}\right)\xi_\mu - B\dot{\xi}_0 + R_\mu\dot{\xi}_\mu + \dot{G}_N = 0, \quad (7)$$

then, there exists the following Noether conserved quantity for Birkhoff's equations (2)

$$I_N = R_\mu\xi_\mu - B\xi_0 + G_N = \text{const.} \quad (8)$$

4. Examples

To illustrate the Birkhoff–Noether method, two examples are given below.

Example 1. Find the solution of the following second-order differential equation:

$$\ddot{q} = 2\dot{q}^2\cot q + \sin q\cos q. \quad (9)$$

Equation (9) is non-self-adjoint. Multiplying each side of Eq.(9) by the factor $\sin^{-4}q$ we have

$$\ddot{q}\sin^{-4}q = 2\dot{q}^2\sin^{-5}q\cos q + \sin^{-3}q\cos q. \quad (10)$$

Letting

$$a^1 = q, \text{ and } a^2 = \dot{q}\sin^{-4}q, \quad (11)$$

Eq.(10) is equivalent to

$$\begin{aligned} -\dot{a}^2 = &\ 2(a^2)^2\cos a^1\sin^3 a^1 - \sin^{-3}a^1\cos a^1, \\ \dot{a}^1 = &\ a^2\sin^4 a^1. \end{aligned} \quad (12)$$

Equations (12) are self-adjoint. According to the expressions (6) we have

$$\begin{aligned} R_1 = &\ \frac{1}{2}a^2,\ R_2 = -\frac{1}{2}a^1, \\ B = &\ \frac{1}{2}(a^2)^2\sin^4 a^1 + \frac{1}{2}\sin^{-2}a^1. \end{aligned} \quad (13)$$

We can obtain the following equation by using the Noether identity (7):

$$\begin{aligned} &\left\{-\frac{1}{2}\dot{a}^2 - 2(a^2)^2\sin^3 a^1\cos a^1 + \sin^{-3}a^1\cos a^1\right\}\xi_1 + \left(\frac{1}{2}\dot{a}^1 - a^2\sin^4 a^1\right)\xi_2 \\ &-B\dot{\xi}_0 + \frac{1}{2}a^2\dot{\xi}_1 - \frac{1}{2}a^1\dot{\xi}_2 + \dot{G}_N = 0. \end{aligned} \quad (14)$$

Its solutions are as follows:

$$\xi_0 = 0,\ \xi_1 = -2\cos t\sin^2 a^1,\ \xi_2 = 0,\ G_N = \sin t\cot a^1; \quad (15)$$
$$\xi_0 = 0,\ \xi_1 = 2\sin t\sin^2 a^1,\ \xi_2 = 0,\ G_N = \cos t\cot a^1. \quad (16)$$

Substituting the expressions (15) and (16) into the expression (8), we can obtain the first integrals

$$I_{N_1} = -a^2\cos t\sin^2 a^1 + \sin t\cot a^1 = C_1, \quad (17)$$
$$I_{N_2} = a^2\sin t\sin^2 a^1 + \cos t\cot a^1 = C_2. \quad (18)$$

Eliminating a^2 from the expressions (17) and (18), we have

$$\cot a^1 = C_1 \sin t + C_2 \cos t, \tag{19}$$

i.e.

$$\cot q = C_1 \sin t + C_2 \cos t. \tag{20}$$

The expression (20) is a general solution of Eq.(9).

Example 2. Find the solution of Hojman–Urrutia equations[21,25] by using Birkhoff–Noether method

$$\ddot{x} + \dot{y} = 0,\ \ddot{y} + y = 0. \tag{21}$$

Letting

$$a^1 = x,\ a^2 = y,\ a^3 = \dot{x},\ a^4 = \dot{y}, \tag{22}$$

we have

$$\dot{a}^1 = a^3,\ \dot{a}^2 = a^4,\ \dot{a}^3 = -a^4,\ \dot{a}^4 = -a^2. \tag{23}$$

The expressions (23) are not in the Hamilton form. They are regrouped as

$$\begin{aligned} &-\dot{a}^2 - \dot{a}^3 = 0,\ \dot{a}^1 - a^3 = 0, \\ &\dot{a}^1 - \dot{a}^4 - a^2 - a^3 = 0,\ \dot{a}^3 + a^4 = 0. \end{aligned} \tag{24}$$

Equations (24) are non-self-adjoint. Computing the Birkhoffian functions R_μ and B by using the expressions (6), we obtain

$$\begin{aligned} &R_1 = \frac{1}{2}(a^2 + a^3),\ R_2 = -\frac{1}{2}a^1, \\ &R_3 = \frac{1}{2}(-a^1 + a^4), R_4 = -\frac{1}{2}a^3, \\ &B = \frac{1}{2}\left\{(a^3)^2 + 2a^2a^3 - (a^4)^2\right\}. \end{aligned} \tag{25}$$

According to the Noether identity (7), we have

$$\begin{aligned} &\left(-\frac{1}{2}\dot{a}^2 - \frac{1}{2}\dot{a}^3\right)\xi_1 + \left(\frac{1}{2}\dot{a}^1 - a^3\right)\xi_2 + \left(\frac{1}{2}\dot{a}^1 - \frac{1}{2}\dot{a}^4 - a^2 - a^3\right)\xi_3 + \left(\frac{1}{2}\dot{a}^3 + a^4\right)\xi_4 \\ &+\frac{1}{2}(a^2 + a^3)\dot{\xi}_1 - \frac{1}{2}a^1\dot{\xi}_2 + \frac{1}{2}(-a^1 + a^4)\dot{\xi}_3 - \frac{1}{2}a^3\dot{\xi}_4 - B\dot{\xi}_0 + \dot{G}_N = 0. \end{aligned} \tag{26}$$

The solutions of Eq. (26) are as follows:

$$\xi_0 = 0,\ \xi_1 = 1,\ \xi_2 = \xi_3 = \xi_4 = 0,\ G_N = \frac{1}{2}(a^2 + a^3); \tag{27}$$

$$\xi_0 = 0,\ \xi_1 = -t,\ \xi_2 = 0,\ \xi_3 = -1,\ \xi_4 = 0,\ G_N = \frac{1}{2}(a^1 - a^4) - \frac{1}{2}t(a^2 + a^3); \tag{28}$$

$$\xi_0 = 0,\ \xi_1 = \cos t,\ \xi_2 = \sin t,\ \xi_3 = -\sin t,\ \xi_4 = \cos t,\ G_N = \frac{1}{2}(a^2 \cos t - a^4 \sin t); \tag{29}$$

$$\xi_0 = 0,\ \xi_1 = \sin t,\ \xi_2 = -\cos t,\ \xi_3 = \cos t,\ \xi_4 = \sin t,\ G_N = \frac{1}{2}(a^2 \sin t + a^4 \cos t). \tag{30}$$

Substituting the expressions (27), (28), (29) and (30) into the expression (8), we can obtain the first integrals

$$\begin{aligned} &I_{N_1} = a^2 + a^3 = C_3, \\ &I_{N_2} = a^1 - a^4 - t(a^2 + a^3) = C_4, \\ &I_{N_3} = a^2 \cos t - a^4 \sin t = C_1, \\ &I_{N_4} = a^2 \sin t + a^4 \cos t = C_2. \end{aligned} \tag{31}$$

The expressions (31) lead to

$$\begin{aligned} &a^1 = -c_1 \sin t + c_2 \cos t + c_3 t + c_4, \\ &a^2 = c_1 \cos t + c_2 \sin t, \\ &a^3 = -c_1 \cos t - c_2 \sin t + c_3, \\ &a^4 = -c_1 \sin t + c_2 \cos t. \end{aligned} \tag{32}$$

The general solutions of the problem can be derived from the first two expressions of the expressions (32) as follows:

$$\begin{aligned} &x = -c_1 \sin t + c_2 \cos t + c_3 t + c_4, \\ &y = c_1 \cos t + c_2 \sin t. \end{aligned} \tag{33}$$

5. Summary

We have discussed the idea of Birkhoff–Noether method of solving the differential equations. The main procedure is as follows: give the representation of differential equations in terms of Birkhoff's equations,

and then use the Noether symmetry for Birkhoffian systems to find first integrals for differential equations. This method can be viewed as a method of analytical mechanics.

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