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Introduction of the Coulomb dressed potential into atomic scattering*

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The model of the Coulomb dressed potential is applied to solving the problem of electron scattering for simplifying the calculation in the electrostatic field and the excimer laser field. The introduction and the application of the model are based on the electric dipole approximation, so the contribution of the electric multipole is neglected. In this paper, rigorous analysis and deduction are carried out for the introduction of the dressed Coulomb potential into the laser field. It is found that the introduction of the dressed potential in the fractional form is feasible only when the laser field (not including far ultraviolet field and x-ray) is a weak field, i.e. the quiver radius of the free electron is smaller than the atomic scale. In addition, the necessary analysis is also conducted of the limitation of the application of the Coulomb dressed potential.

Keywords: scattering, coulomb potential, dressed potential, quiver radius
PACC: 3115, 0365, 0365N

1. Introduction

In order to investigate the electron transition between the bound states when the atom lies in a laser field, the model of the bound-state effective potential is established by Henneberger in 1968.[1] The mathematization of the model is improved by Faisal in 1973.[2] Dimou and Faisal[3] developed the model of the Coulomb dressed potential at the end of the 1980s, and the Floquet close-coupling (FCC) method was utilized in the scattering between the charged particles and atoms.[4−9] However, we find that Dimou and Faisal's model was based on the approximate condition of the electric dipole, which is valid only for the case of weak laser field. In the explicit expression for a laser field, the magnetic dipole can be neglected because the rotor $\nabla \times \boldsymbol{A}$ of the vector potential $\boldsymbol{A}$ does not appear in the Hamilton $\hat{H}$, which means the scattering of the incident electron and the Coulomb potential and the unitary transformation of the wavefunction come into being. In addition, considering the wave vector k_l corresponding to the visible light (about $10^{-5}\,\mathrm{cm}^{-1}$) and the scale of the atom (about $10^{-8}\,\mathrm{cm}$), the atomic wave function is close to zero over $10^{-8}\,\mathrm{cm}$; when the atomic wave function is more than zero in the limit of integration, the series expansion of the light-wavefunction takes the first term. So the method of the electric dipole approximation can be accepted. But the scale of the atom is bigger in Rydberg states, and the valence electrons polarize the core. The contribution of the electric multipole must be taken into account when the fine structure of the level in Rydberg State is studied. In Ref.[10] the contribution of the electric octupole is calculated. The magnitude of wave vector k_l is greater than that of the visible light wave vector for the ultraviolet laser field and the kinetic energy of the scattering particles is very big and the magnitudes of their radius vectors are close to infinite ($r \to \infty$) in the Coulomb scattering;[11] hence the higher-order radiation should be taken into consideration. Generally speaking, the scattering correlates not only with the emission and absorption of the corresponding single photon of the electric dipole radiation but also with those of the corresponding multiple photon of the electric multipole radiation,[12−18] and the electric multipole expansion is usually considered in scattering.[12−24] For example, Lambrecht *et al*[4] calculated the scatterings between the electron and H, between the electron and He^+, and between the electron and Li^+ separately,

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and the emission and absorption of two or three photons through the model of the Coulomb dressed potential in the ultraviolet laser field (with wavelengths of 260–10 nm).[4] In this paper, the phase factor including the vector potential of the exponential function is approximated by Taylor polynomial over the electric dipole, and the rigorous analysis and deduction are carried out for the introduction of the Coulomb dressed potential, and the application of the Coulomb dressed potential is also analysed.

2. Coulomb dressed potential and its limitation

When Hydrogen-like atom (ion) and the monochromatic laser field interact with each other, the condition of the transversal wave for the laser field is introduced and the Hartree atomic unit is adopted, then the radial Schrödinger equation can be written as follows:

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t) = \left(-\frac{1}{2}\Delta - \frac{Z}{r} + \frac{\mathrm{i}}{c}\hat{\boldsymbol{p}}\cdot\boldsymbol{A}(t) + \frac{\boldsymbol{A}^2(t)}{2c^2}\right) \times \Psi(\boldsymbol{r},t), \tag{1}$$

where $\hat{\boldsymbol{p}} = -\mathrm{i}\nabla$ is the momentum operator and the potential vector $\boldsymbol{A}(t)$ of the circularly polarized field is

$$\boldsymbol{A}(t) = A_0(\boldsymbol{\varepsilon}_x\cos(\omega t+\delta) - \boldsymbol{\varepsilon}_y\sin(\omega t+\delta)). \tag{2}$$

Here, ω is the frequency, δ is the initial phase, $\boldsymbol{\varepsilon}_x$ and $\boldsymbol{\varepsilon}_y$ are the unit vectors in the x and y directions respectively, and A_0 is the weak vector potential amplitude. In order to separate the interaction term in the Hamiltonian, the wavefunction is processed with the unitary transformation

$$\Psi(\boldsymbol{r},t) = \exp\left(-\mathrm{i}\int^t \mathrm{d}\tau\left(-\frac{\boldsymbol{A}(\tau)}{c}\cdot\hat{\boldsymbol{p}} + \frac{1}{2c^2}\boldsymbol{A}^2(\tau)\right)\right) \times \Phi(\boldsymbol{r},t). \tag{3}$$

References [1–4] all are approximated by the electric dipole, and the radial Schrödinger equation that is equivalent to Eq.(1) is obtained from Eq.(3) as follows:

$$\mathrm{i}\frac{\partial}{\partial t}\Phi(\boldsymbol{r},t) = \left(-\frac{1}{2}\Delta - \frac{Z}{|\boldsymbol{r}-\boldsymbol{\alpha}_0(t)|}\right)\Phi(\boldsymbol{r},t), \tag{4}$$

where

$$\boldsymbol{\alpha}_0(t) = \int^t \mathrm{d}\tau\frac{\boldsymbol{A}(\tau)}{c} \quad \text{with} \quad |\boldsymbol{\alpha}_0(t)| = \frac{A_0}{c\omega} = a. \tag{5}$$

The variable a is the electric quiver radius and the expression $\dfrac{Z}{|\boldsymbol{r}-\boldsymbol{\alpha}_0(t)|}$ is the Coulomb dressed potential. Then the wavefunction $\Phi(\boldsymbol{r},t)$ is expanded by using the Floquet partial wave method as follows:

$$\Phi(\boldsymbol{r},t) = \sum_{n=-\infty}^{\infty}\sum_{lm}\exp[-\mathrm{i}Et + \mathrm{i}n(\omega t+\delta)]\frac{F_{nlm}(r)}{r} \times Y_l^m(\theta,\varphi). \tag{6}$$

The coupled radial system of equations (FCC Equation) are separated from Eq.(6)

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + k_n^2 - \frac{l(l+1)}{r^2} + \frac{2Z}{(r^2+\alpha_0^2)^{1/2}}\right]F_{nlm}(r) = -2Z\sum_{\lambda=1,p=0}^{\infty,\lambda}\sum_{l'm'}\langle Y_{lm}|V_{\lambda_p}(\theta,\varphi)\,|Y_{l'm'}\rangle \times F_{n-\lambda_p,l',m'}(r), \tag{7}$$

where $\lambda_p = \lambda - 2p$, $k_n = [2(E-n\omega)]^{1/2}$, V_{λ_p} is the coupled potential of the electric multipole,[4] and

$$V_{\lambda_p}(\theta,\varphi) = \frac{(2\lambda-1)!!}{2^\lambda\lambda!}\begin{pmatrix}\lambda\\ p\end{pmatrix}\left[\frac{r\alpha_0\sin\theta}{r^2+\alpha_0^2}\right]^\lambda \times\frac{1}{(r^2+\alpha_0^2)^{1/2}}\exp(\mathrm{i}\lambda_p\varphi). \tag{8}$$

There appears a problem of self-consistency, because Eq.(4) is derived from the approximate condition of the electric dipole[1−3] which corresponds to the emission and absorption of the single photon. But in Eqs.(7), (8) adopted is the expansion of the electric multipole which corresponds to the emission and absorption of the multiple photons.[3−4] Obviously they are not consistent with each other. In addition, there are some shortcomings in the solutions of Eqs.(4) and (7) as explained later.

In order to solve the ordinary differential equation which the Floquet partial wavefunction $F_{nlm}(r)$ satisfies, the Coulomb dressed potential $\dfrac{Z}{|\boldsymbol{r}-\alpha_0(t)|}$ is expanded by the Legendre polynomials in Refs.[7–9]

$$\frac{Z}{|\boldsymbol{r}-\boldsymbol{\alpha}_0(t)|} = \begin{cases}\dfrac{Z}{r}\displaystyle\sum_{k=0}^{\infty}\left(\frac{a}{r}\right)^k P_k(\cos\theta'), & r>a,\\ \dfrac{Z}{a}\displaystyle\sum_{k=0}^{\infty}\left(\frac{r}{a}\right)^k P_k(\cos\theta'), & r<a,\end{cases} \tag{9}$$

where $a = A_0/\omega c$, and $\cos\theta' = \boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_a$, with $\Omega = r/r = \Omega(\theta,\varphi)$ and $\Omega_a = \alpha_0(t)/|\boldsymbol{\alpha}_0(t)| = (\pi/2, \pi/2-\omega t-\delta)$. The combined condition is utilized in order to solve

the coupled radial system of equations in Refs.[9–11],

$$F_{nlm}(r)|_{r\to a-0} = F_{nlm}(r)|_{r\to a+0}, \quad \text{and}$$
$$F'_{nlm}(r)|_{r\to a-0} = F'_{nlm}(r)|_{r\to a+0}. \tag{10}$$

In fact, the deduction of Eq.(9), strictly speaking, is not reasonable enough; furthermore, it is invalid under the condition of $r < a$. According to the commutation relation between the momentum operator and the coordinate operator, and the power series expansion of the exponential function, the expression of the equivalent Schrödinger equation (4) involves the first term on the right-hand side of Eq.(9) in the process of the unitary transformation, here the dressed potential on the left-hand side of Eq.(9) is suitable; whereas the second term on the right of Eq.(9) cannot appear, so Eq.(9) is inappropriate for the case of $r < a$. Such an equation as Eq.(10) is not perfect when the Coulomb dressed potential is adopted.

3. The deduction of the dressed potential in the equivalent Schrödinger equation

The cross section $\frac{\mathrm{i}}{c}\hat{\boldsymbol{p}}\cdot\boldsymbol{A}(t)$ of the Hamiltonian does not disappear with the increase of the scattering distance, and it is inconvenient to treat it for the tightly-coupled asymptotic free condition; consequently the unitary transformation like Eq.(3) is necessary. From Eqs.(1) and (3), we have

$$\begin{aligned}\mathrm{i}\frac{\partial}{\partial t}\Psi(\boldsymbol{r},t) &= \exp\left(-\mathrm{i}\int^t \mathrm{d}\tau\left(-\frac{\boldsymbol{A}(\tau)}{c}\cdot\hat{\boldsymbol{p}}+\frac{1}{2c^2}\boldsymbol{A}^2(\tau)\right)\right)\left(-\frac{\boldsymbol{A}(\tau)}{c}\cdot\hat{\boldsymbol{p}}+\frac{1}{2c^2}\boldsymbol{A}^2(\tau)\right)\Phi(\boldsymbol{r},t)\\ &\quad+\exp\left(-\mathrm{i}\int^t \mathrm{d}\tau\left(-\frac{\boldsymbol{A}(\tau)}{c}\cdot\hat{\boldsymbol{p}}+\frac{1}{2c^2}\boldsymbol{A}^2(\tau)\right)\right)\mathrm{i}\frac{\partial}{\partial t}\Phi(\boldsymbol{r},t)\\ &= \left(-\frac{1}{2}\Delta-\frac{\boldsymbol{A}(t)}{c}\cdot\hat{\boldsymbol{p}}+\frac{1}{2c^2}\boldsymbol{A}^2(t)-\frac{Z}{r}\right)\exp\left(-\mathrm{i}\int^t \mathrm{d}\tau\left(-\frac{\boldsymbol{A}(\tau)}{c}\cdot\hat{\boldsymbol{p}}+\frac{1}{2c^2}\boldsymbol{A}^2(\tau)\right)\right)\Phi(\boldsymbol{r},t).\end{aligned} \tag{11}$$

The canonical momentum of the electron is expressed as $\hat{\boldsymbol{p}} = -\mathrm{i}\nabla$ in the coordinate representation, and the vector potential $\boldsymbol{A}(t)$ meets the condition of the transversal wave and commutation relation. So, $\nabla\cdot A(t) = 0$, $\boldsymbol{A}(t)\cdot\hat{\boldsymbol{p}}-\hat{\boldsymbol{p}}\cdot\boldsymbol{A}(t) = \mathrm{i}\nabla\cdot\boldsymbol{A}(t) = 0$ from Eq.(2). Namely

$$\boldsymbol{A}(t)\cdot\hat{\boldsymbol{p}} = \hat{\boldsymbol{p}}\cdot\boldsymbol{A}(t), \tag{12}$$

where the radius vector r and the momentum operator $\hat{p}$ do not satisfy the commutation relation. In order to obtain $\Phi(\boldsymbol{r},t)$ that satisfies the simplified differential equation, the commutation relation between $\left(-\frac{1}{2}\Delta-\frac{\boldsymbol{A}(\tau)}{c}\cdot\hat{\boldsymbol{p}}+\frac{1}{2c^2}\boldsymbol{A}^2(\tau)-\frac{Z}{r}\right)$ and $\exp\left(-\mathrm{i}\int^t \mathrm{d}\tau\left(-\frac{\boldsymbol{A}(\tau)}{c}\cdot\hat{\boldsymbol{p}}+\frac{1}{2c^2}\boldsymbol{A}^2(\tau)\right)\right)$ is considered. In the paper, because of the commutation relation between $\boldsymbol{A}(t)$ and $\hat{\boldsymbol{p}}$ as well as the relation between $\boldsymbol{r}$ and $\boldsymbol{A}(t)$, only the commutation relation between $-\frac{Z}{r}$ and $\exp\left(-\mathrm{i}\int^t \mathrm{d}\tau\left(-\frac{\boldsymbol{A}(\tau)\cdot\hat{\boldsymbol{p}}}{c}\right)\right)$ is taken into account. So that

$$\begin{aligned}&-\frac{Z}{r}\exp\left(-\mathrm{i}\int^t \mathrm{d}\tau\left(-\frac{\boldsymbol{A}}{c}\cdot\hat{\boldsymbol{p}}\right)\right)\\ &= -\frac{Z}{r}\sum_j\frac{\mathrm{i}^j}{j!}\left(\int^t \mathrm{d}\tau\left(\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c^j}\right)\right)^j.\end{aligned} \tag{13}$$

When $j = 0$, then

$$-\frac{Z}{r}\times 1 = 1\times\left(-\frac{Z}{r}\right); \tag{14}$$

when $j = 1$, then

$$-\frac{Z}{r}\frac{\mathrm{i}}{c}\int^t \mathrm{d}\tau\,(\boldsymbol{A}\cdot\hat{\boldsymbol{p}}) = -\frac{\mathrm{i}Z}{c}\int^t \mathrm{d}\tau\frac{1}{r}\boldsymbol{A}\cdot\hat{\boldsymbol{p}}. \tag{15}$$

According to the commutation relation between the momentum operator and the coordinate operator, thus

$$\frac{1}{r}\boldsymbol{A}\cdot\hat{\boldsymbol{p}} = \boldsymbol{A}\cdot\hat{\boldsymbol{p}}\frac{1}{r}-\mathrm{i}\frac{\boldsymbol{A}\cdot\boldsymbol{r}}{r^3}. \tag{16}$$

Likewise, now

$$\frac{\boldsymbol{A}(\tau_1)\cdot\boldsymbol{r}}{r^3}(\boldsymbol{A}(\tau_2)\cdot\hat{\boldsymbol{p}})=(\boldsymbol{A}(\tau_2)\cdot\hat{p})\frac{\boldsymbol{A}(\tau_1)\cdot\boldsymbol{r}}{r^3}+\mathrm{i}\frac{\boldsymbol{A}(\tau_1)\cdot\boldsymbol{A}(\tau_2)}{r^3}-3\mathrm{i}\frac{(\boldsymbol{A}(\tau_1)\cdot\boldsymbol{r})(\boldsymbol{A}(\tau_2)\cdot\boldsymbol{r})}{r^5},$$
$$\frac{\boldsymbol{A}(\tau_1)\cdot\boldsymbol{r}}{r^5}(\boldsymbol{A}(\tau_2)\cdot\hat{\boldsymbol{p}})=(\boldsymbol{A}(\tau_2)\cdot\hat{p})\frac{\boldsymbol{A}(\tau_1)\cdot\boldsymbol{r}}{r^5}+\mathrm{i}\frac{\boldsymbol{A}(\tau_1)\cdot\boldsymbol{A}(\tau_2)}{r^5}-5\mathrm{i}\frac{(\boldsymbol{A}(\tau_1)\cdot\boldsymbol{r})(\boldsymbol{A}(\tau_2)\cdot\boldsymbol{r})}{r^7},$$
$$\cdots$$

when $j=0,1$, and sum up the terms on the right-hand side of Eq.(13), then we will have

$$-\frac{Z}{r}\sum_{j=0}^{1}\frac{\mathrm{i}^j}{j!}\left(\int^t\mathrm{d}\tau\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)^j=\sum_{j=0}^{1}\frac{(-\mathrm{i})^j}{j!}\left(\int^t\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\right)^j\sum_{l=0}^{1}\left(\int^t\mathrm{d}\tau\left(\frac{A}{c}\right)\right)^l P_l(\cos\theta')(-\frac{Z}{r^{l+1}})$$
$$+\mathrm{i}\int^t\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\left|\int^t\mathrm{d}\tau\frac{A}{c}\right|P_l(\cos\theta')(-\frac{Z}{r^2}),\tag{17}$$

when $j=2$ in a similar way, from Eq.(13), we can obtain

$$-\frac{Z}{r}\frac{\mathrm{i}^2}{2!}\int^t\mathrm{d}\tau\left(\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\int^t\mathrm{d}\tau\left(\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)$$
$$=\int^t\mathrm{d}\tau_1\frac{Z}{2c^2}\left(\frac{1}{r}\boldsymbol{A}(\tau_1)\cdot\hat{\boldsymbol{p}}\right)\int^t\mathrm{d}\tau_2(\boldsymbol{A}(\tau_2)\cdot\hat{\boldsymbol{p}})$$
$$=\frac{(-\mathrm{i})^2}{2!}\int^t\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)^2\left(-\frac{Z}{r}\right)-\mathrm{i}\int^t{}\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\int^t\mathrm{d}\tau\left(-\frac{Z\boldsymbol{A}\cdot\boldsymbol{r}}{cr^3}\right)$$
$$+\left|\int^t\mathrm{d}\tau\frac{\boldsymbol{A}(\tau)}{c}\right|^2\left(\frac{Z}{2r^3}\right)+\left(\int^t\mathrm{d}\tau\frac{\boldsymbol{A}\cdot\boldsymbol{r}}{Ar}\right)\left(-\frac{3Z}{2r^3}\right).\tag{18}$$

And then the last term of Eq.(21) is merged into the summation $\sum$ of Eq.(23), which will be seen later similarly, when $j=3$, from Eq.(13), we can have

$$-\frac{Z}{r}\frac{\mathrm{i}^3}{3!}\left(\int^t\mathrm{d}\tau\left(\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c^3}\right)\right)^3=\frac{(-\mathrm{i})^3}{3!}\left(\int^t\mathrm{d}\tau\left(\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c^3}\right)\right)^3\left(-\frac{Z}{r}\right)+\frac{(-\mathrm{i})^2}{2!}\left(\int^t\mathrm{d}\tau\left(\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c^3}\right)\right)^2\int^t\mathrm{d}\tau\left(-\frac{Z\boldsymbol{A}\cdot\boldsymbol{r}}{cr^3}\right)$$
$$+\frac{\mathrm{i}}{2}\int^t\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{p}}{c^3}\right)\left(\int^t\mathrm{d}\tau\frac{\boldsymbol{A}}{c}\right)^2\left(-\frac{Z}{r^3}\right)$$
$$+(-\mathrm{i})\int^t\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\left(\int^t\mathrm{d}\tau(\frac{\boldsymbol{A}\cdot\boldsymbol{r}}{cr^3})\right)^2\left(-\frac{3Z}{2r^5}\right)$$
$$-\left(\int^t\mathrm{d}\tau\frac{A}{c}\right)^2\int^t\mathrm{d}\tau\left(\frac{\boldsymbol{A}\cdot\boldsymbol{r}}{c}\right)\left(-\frac{3Z}{2r^5}\right)+\left(\int^t\mathrm{d}\tau\frac{\boldsymbol{A}\cdot\boldsymbol{r}}{cr}\right)^3\left(-\frac{5Z}{2r^7}\right).\tag{19}$$

Also do the same way, when $j=4,5,\ldots$ from Eq.(13) we can work out the corresponding terms, and the corresponding commutation relation is solved.

Summing up the terms on the right-hand side of Eq.(13) when $j=0,1,2,3$, and 4 yields

$$-\frac{Z}{r}\sum_{j=0}^{4}\frac{\mathrm{i}^j}{j!}\left(\int^t\mathrm{d}\tau\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)^j=\sum_{j=0}^{3}\frac{(-\mathrm{i})^j}{j!}\left(\int^t\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\right)^j\sum_{l=0}^{3}\left|\int^t\mathrm{d}\tau\frac{\boldsymbol{A}}{c}\right|^l P_l(\cos\theta')\left(-\frac{Z}{r^{l+1}}\right)$$
$$+\frac{(-\mathrm{i})^4}{4!}\left(\int^t\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{p}}{c}\right)\right)^4\left(-\frac{Z}{r}\right)-\frac{(-\mathrm{i})^3}{3!}$$
$$\times\left(\int^t\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\right)^3\left|\int^t\mathrm{d}\tau\frac{\boldsymbol{A}}{c}\right|^3P_3(\cos\theta')\left(-\frac{Z}{r^4}\right)+\cdots.\tag{20}$$

Thus terms of $j=0,1,2,3$ each are merged into the summation $\sum$ of Eq.(20). Making use of the step-by-step approximation method and limit definition,[12] we can obtain

$$\lim_{n\to\infty}-\frac{Z}{r}\sum_{j=0}^{n}\frac{\mathrm{i}^j}{j!}\left(\int^t\mathrm{d}\tau\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)^j=\sum_{j=0}^{\infty}\frac{(-\mathrm{i})^j}{j!}\left(\int^t\mathrm{d}\tau\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)^j\sum_{l=0}^{\infty}\left|\int^t\mathrm{d}\tau\frac{\boldsymbol{A}}{c}\right|^l P_l(\cos\theta')\left(-\frac{Z}{r^{l+1}}\right)$$
$$=\exp\left(-\mathrm{i}\int^t\mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\right)\sum_{l=0}^{\infty}\left|\int^t\mathrm{d}\tau\frac{\boldsymbol{A}}{c}\right|^l\left(-\frac{Z}{r^{l+1}}\right)P_l(\cos\theta'),\tag{21}$$

namely

$$-\frac{Z}{r}\exp\left(-\mathrm{i}\int^t \mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\right)=\exp\left(-\mathrm{i}\int^t \mathrm{d}\tau\left(-\frac{\boldsymbol{A}\cdot\hat{\boldsymbol{p}}}{c}\right)\right)\sum_{l=0}^{\infty}\left|\int^t \mathrm{d}\tau\frac{\boldsymbol{A}}{c}\right|^l\left(-\frac{Z}{r^{l+1}}\right). \tag{22}$$

Inserting Eq.(22) into Eq.(11) yields

$$\mathrm{i}\frac{\partial}{\partial t}\Phi(\boldsymbol{r},t)=\left(-\frac{1}{2}\Delta+\sum_{l=0}^{\infty}\left|\int^t \mathrm{d}\tau\frac{\boldsymbol{A}}{c}\right|^l\times\left(-\frac{Z}{r^{l+1}}\right)P_l(\cos\theta')\right)\Phi(\boldsymbol{r},t), \tag{23}$$

namely

$$\mathrm{i}\frac{\partial}{\partial t}\Phi(\boldsymbol{r},t)=\left(-\frac{1}{2}\Delta-\sum_{l=0}^{\infty}\frac{a^l}{l+1}P_l(\cos\theta')\right)\Phi(\boldsymbol{r},t). \tag{24}$$

For $r>a$, from the definition of the generating function for Legendre Polynomials,[25] we have

$$Z\sum_{l=0}^{\infty}\frac{a^l}{l+1}P_l(\cos\theta')=\frac{Z}{|\boldsymbol{r}-\boldsymbol{\alpha}_0(t)|}; \tag{25}$$

for $r<a$, $Z\sum\limits_{l=0}^{\infty}\frac{a^l}{l+1}P_l(\cos\theta')\neq\dfrac{Z}{|\boldsymbol{r}-\boldsymbol{\alpha}_0(t)|}$.

4. Discussion and conclusions

1. In this paper, the analysis and deduction which are more accurate than those of the emission and absorption of the single photon in Refs.[2,3] are carried out for introducing the Coulomb dressed potential which includes the emission and absorption of the four photons, into the laser field; we assume that the laser field (not including far ultraviolet field and x-ray) is only a weak field, and what we want to emphasize is that the introduction of the dressed potential in the fractional form is feasible only when the quiver radius of the free electron is smaller than the atomic scale.

2. Because of the commutation relation between Z/r and $(\int^t \mathrm{d}\tau(\boldsymbol{A}\cdot\hat{\boldsymbol{p}}/c^j))^j$, such a term as $\dfrac{r^k}{k+1}P_k(\cos\theta')$ cannot be produced by the unitary transformation of Eq.(3), thus Eq.(9) is inappropriate for the case of $r<a$.

3. The effect of the factor $\boldsymbol{\alpha}_0$ of the dressed potential is attenuated with the increase of radial coordinate r. When $r\to\infty$, the wavefunction meets the purely asymptotic condition of the Coulomb scattering.

4. In order to obtain FCC equations, the wavefunction $\Phi(\boldsymbol{r},t)=\dfrac{1}{r}\sum\limits_{nlm}F_{nlm}(r)Y_{lm}(\hat{\boldsymbol{r}})\exp(-\mathrm{i}Et+\mathrm{i}n(\omega t+\delta))$ is solved by using the Floquet method. Sometimes, the subsection solution is applied to solving Eq.(9) in some references. Then the undetermined coefficient is solved by the chained conditions $F_{nlm}(r)|_{r\to a-0}=F_{nlm}(r)|_{r\to a+0}$, and $F'_{nlm}(r)|_{r\to a-0}=F'_{nlm}(r)|_{r\to a+0}$. We think that this solution is not accurate and appropriate in terms of the Coulomb dressed potential. Equation (9) is inappropriate under the condition $r<a$. In fact, when the Coulomb dressed potential is not introduced, the chained condition is appropriate for the coupled radial system of Eq.(7). When solving the FCC equations, we should make good use of some conditions and properties, such as the purely asymptotic condition of the Coulomb scattering ($r\to\infty$), the chained condition, the properties of the Whittaker function and the other special functions, etc. The deduction is conducted from the simple coupling to the complex coupling, from low-order approximation to high-order approximation and from elastic scattering to inelastic scattering by the step-by-step method.

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