Vol 16 No 12, December 2007
1009-1963/2007/16(12)/3605-05

Chinese Physics

© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

Supplement to the previous application of the first junction condition to a highly symmetric spacetime*

Qin Yi-Ping(覃一平)

Center for Astrophysics, Guangzhou University, Guangzhou 510006, *China*
National Astronomical Observatories/Yunnan Observatory, Chinese Academy of Sciences, Kunming 650011, *China*
Physics Department, Guangxi University, Nanning 530004, *China*

(Received 14 August 2006; revised manuscript received 22 April 2007)

Application of the first junction condition to a highly symmetric spacetime was investigated recently in *Chin. Phys. Lett.* B **546** 189 2006, where a partial differential equation arising from the connection of the Robertson–Walker and the Schwarzschild–de Sitter metrics was presented, but no solutions of the equation were provided. Here we provide a proof to the statement that there exist solutions of the equation. In addition, an example of the solution and some analyses associated with this issue are presented. We find that in connecting the two metrics, there are three conditions which should be satisfied. Of these conditions, one condition constrains the place where the two metrics can take the same value for a local system whose mass is provided which marks the boundary of the system, and the other two constrain the transformation form. In realizing the connection of the two metrics, the latter two conditions are required to be satisfied only at the boundary defined by the former condition.

Keywords: general relativity, metrics, junction conditions
PACC: 0420, 0490

Recent observations revealed that type Ia supernovae are 10% to 15% farther out than expected even in a universe with little matter. This together with the measurement of the characteristic angular size of fluctuations in the cosmic microwave background (CMB) suggests that current expansion of the universe is accelerated rather than decelerated and the cosmological constant is nonzero and positive.[1−7] It is generally accepted that a dark energy component with negative pressure is responsible for the current acceleration of the universe. Besides the cosmological constant Λ,[8] there are many other candidates of the dark energy component in the literature, including an evolving scalar field,[9,10] the phantom energy,[11] the so-called 'X-matter',[12−15] the Chaplygin gas,[16−18] the Cardassion model,[19−23] and the brane world model.[24−26]

The suggestion that the cosmological constant exists leads to a solution of the Einstein field equation that is in the form of the so-called Schwarzschild–de Sitter metric rather than the conventional Schwarzschild one according to Gibbons and Hawking.[27] It is known that the metric of the universe would be the Robertson–Walker (R-W) one due to the homogeneous and isotropic property of the spacetime. Recently, an ideal model of spacetime has be considered where the local system is assumed to be highly symmetric and the cosmological matter is kept away from and not disturbed by the local system. It has been proposed that a metric system is used to describe this spacetime: in the part of spacetime beyond the boundary, which separates the local part from global part of spacetime, is the R-W metric while inside the part of spacetime, enclosed by the boundary, is the Schwarzschild–de Sitter metric (see Ref.[28], which, hereafter, is referred to as Paper I). It is obvious that this metric system is a solution of the Einstein field equation on both sides of the boundary, as both metrics are the exact solution of the equation in their corresponding part of spacetime (note that the Einstein field equation is a differential equation). At the boundary, requiring the metrics adopted to be a solution of the field equation leads to the second junction condition.[29] This in turn might provide an independent constraint on the transformation of the two metrics involved (this will be discussed later).

Particle tunnelling radiation from a black hole has been discussed in recent years, in which the property

*Project supported by the National Science Fund for Distinguished Young Scholars (Grant No 10125313), the National Natural Science Foundation of China (Grant Nos 10573005 and 10273019), and the Fund for Top Scholars of Guangdong Province (Grant No Q02114) and the Guangzhou Education Bureau and Guangzhou Science and Technology Bureau, China

of a boundary connecting two parts of spacetime, one outside the black hole and the other inside the black hole, is involved.[30−38] The boundary of black hole, the horizon, is somewhat similar to the boundary discussed in Paper I. It is clear that, at the black hole boundary, junction conditions can be well satisfied.

What is considered in Paper I is the first condition of the junction conditions in the case of the ideal spacetime discussed there, which gives rise to a boundary condition in connecting the two metrics on both sides of the boundary. The requirement for the boundary condition leads to a partial differential equation (Eq.(4) in Paper I). Whether there exist solutions to the equation has not yet been clarified.

Here we provide a proof showing that the partial differential equation is solvable.

Following Paper I, we use coordinates t and r to describe the R-W metric, and $\widetilde{t}$ and $\widetilde{r}$ to describe the Schwarzschild-de Sitter metric. The transformation between the two sets of the coordinates is assumed to be

$$Rr = \widetilde{r} \tag{1}$$

and

$$t = t(\widetilde{t}, \widetilde{r}). \tag{2}$$

To meet the requirement that the two metrics should take the same value at the place they are connected (or, the induced metric should be continuous across the boundary that connects the two parts of spacetime), the following condition should be satisfied:

$$\frac{\partial t}{\partial \widetilde{r}} = \frac{r}{r^2\left(\dfrac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - (1-kr^2)} \frac{\mathrm{d}R}{\mathrm{d}t}. \tag{3}$$

As suggested in Paper I, when $R = R(t)$ is known, there will exist a solution of this partial differential equation.

Proof Equation (3) can be written as

$$\frac{\partial t}{\partial \widetilde{r}} = f(\widetilde{r}, t). \tag{4}$$

For $r = r_0$, one has

$$R|_{r=r_0} = \widetilde{r}/r_0, \tag{5}$$

and

$$t|_{r=r_0} = t_{-1}(R|_{r=r_0}) = t_{-1}(\widetilde{r}/r_0), \tag{6}$$

where $t_{-1}(R)$ is an inverse function of $R(t)$. Assume that within the range concerned there is a one-to-one relationship between R and t; at any time t, there exists a certain $\dfrac{\mathrm{d}R}{\mathrm{d}t}$; $r^2\left(\dfrac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - (1-kr^2) \neq 0$ for any point $(\widetilde{t}, \widetilde{r})$. Under this assumption, $f(\widetilde{r}, t)$ and $t_{-1}(\widetilde{r}/r_0)$ are analytical within the range. According to the Cauchy theorem, there exists a unique analytical solution to Eq.(3). The proof is now completed.

Second, we check whether there indeed exist some solutions to the partial differential equation.

a) The simplest solution to Eq.(3) is obtained when $R = \text{const}$. It leads to $t(\widetilde{t}, \widetilde{r}) = \widetilde{t} + C$, where C is a constant.

b) Assume

$$R(t) = R_0 \mathrm{e}^{t/t_0} \tag{7}$$

and

$$k = 0, \tag{8}$$

where R_0 and t_0 are constants. Under the above assumptions, the universe is flat and expands with an acceleration. Then from Eq.(3) we obtain

$$\frac{\partial t}{\partial \widetilde{r}} = \frac{t_0 \widetilde{r}}{\widetilde{r}^2 - t_0^2}, \tag{9}$$

where the assumptions (7) and (8) have been applied. A solution of Eq.(9) is

$$t(\widetilde{t}, \widetilde{r}) = \frac{t_0}{2} \ln\left[\left(\frac{\widetilde{r}}{t_0}\right)^2 - 1\right] + f_1(\widetilde{t}), \tag{10}$$

where function $f_1(\widetilde{t})$ is a product of the integral and its form is free.

This shows explicitly that there indeed exist solutions of the partial differential equation (3).

Third, we explore in great detail some conclusions expected from the connection of the two metrics.

As analysed in Paper I, applying transformation relations (1) and (2) can change the conventional form of R-W metric into the following from:

$$\begin{aligned} \mathrm{d}s^2 = & \left[1 - \frac{r^2}{1-kr^2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\right]\left(\frac{\partial t}{\partial \widetilde{t}}\right)^2 \mathrm{d}\widetilde{t}^2 + 2\left[\frac{\partial t}{\partial \widetilde{r}} + \frac{r}{1-kr^2}\frac{\mathrm{d}R}{\mathrm{d}t}\left(1 - r\frac{\mathrm{d}R}{\mathrm{d}t}\frac{\partial t}{\partial \widetilde{r}}\right)\right]\frac{\partial t}{\partial \widetilde{t}}\mathrm{d}\widetilde{t}\mathrm{d}\widetilde{r} \\ & - \left[\frac{1}{1-kr^2}\left(1 - r\frac{\mathrm{d}R}{\mathrm{d}t}\frac{\partial t}{\partial \widetilde{r}}\right)^2 - \left(\frac{\partial t}{\partial \widetilde{r}}\right)^2\right]\mathrm{d}\widetilde{r}^2 - \widetilde{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \end{aligned} \tag{11}$$

If Eq.(3) holds, this form of metric would turn to be

$$\mathrm{d}s^2 = \frac{R^2 - \left[k + \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\right]\widetilde{r}^2}{R^2 - k\widetilde{r}^2}\left(\frac{\partial t}{\partial \widetilde{t}}\right)^2 \mathrm{d}\widetilde{t}^2 - \frac{R^2}{R^2 - \left[k + \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\right]\widetilde{r}^2}\mathrm{d}\widetilde{r}^2 - \widetilde{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{12}$$

Comparing it with the Schwarzschild-de Sitter metric and assigning the same value to both metrics yield

$$\left[\frac{k + \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2}{R^2} - \frac{\Lambda}{3}\right]\widetilde{r}^3 = 2GM, \tag{13}$$

and

$$\left(\frac{\partial t}{\partial \widetilde{t}}\right)^2 = 1 - kr^2. \tag{14}$$

According to the above analysis, one should keep in mind that applying transformation relations (1) and (2), and assigning the same value to both the R-W and Schwarzschild-de Sitter metrics lead to conditions (3), (13) and (14). They are necessary conditions. In this way, we are aware that in addition to a solution of Eq.(3), we need also a solution of Eq.(14), and the two solutions should be consistent.

As has been already known, in the framework of the Friedmann–Robertson–Walker (FRW) cosmology, the following relation holds when the cosmological constant is taken into account:

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 + k = \frac{8\pi G}{3}\left(\rho + \frac{\Lambda}{8\pi G}\right)R^2. \tag{15}$$

Applying Eq.(15), one obtains from Eq.(13)

$$\widetilde{r}^3 = \frac{M}{\frac{4\pi}{3}\rho} \tag{16}$$

or

$$M = \frac{4\pi}{3}\rho R^3 r^3. \tag{17}$$

As ρR^3 is a constant in the very late epoch of the universe,[39] expression (17) shows that condition (13) constrains nothing but, for a certain local system (where its total mass M is provided), the space where the two metrics can take the same value which forms the so-called boundary of the local system, or constrains, for a given boundary, the total mass of the local system.

In contrast, conditions (3) and (14) constrain only the transformation relation (2). When condition (13) is satisfied (in this case, a boundary could be defined and is marked by r_{M}), then we only need to require that conditions (3) and (14) be satisfied at r_{M}. This largely relaxes the two conditions, which now can be expressed as

$$\frac{\partial t}{\partial \widetilde{r}} = \frac{r}{r^2\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - (1 - kr^2)}\frac{\mathrm{d}R}{\mathrm{d}t}, \qquad (r = r_{\mathrm{M}}) \tag{18}$$

and

$$\left(\frac{\partial t}{\partial \widetilde{t}}\right)^2 = 1 - kr^2, \quad (r = r_{\mathrm{M}}). \tag{19}$$

As mentioned in Paper I, relations (16) and (17) suggest that under the transformation (1) and (2) that satisfy Eq.(3), assigning the same value to both the R-W and Schwarzschild-de Sitter metrics at time t at position r requires the central mass of the Schwarzschild-de Sitter metric (the total mass of the local system) to be the same as the total mass of the cosmological matter within the region restricted by r, which is a necessary condition of the connection. At time t, position r corresponds to the radial coordinate $\widetilde{r}$ of the Schwarzschild-de Sitter metric, which can be determined by expression (16). As the universe expands, $\widetilde{r}$ increases, which is equivalent to the decreasing of the cosmological density ρ. Expression (17) shows that r is determinate when M is given. We represent it as r_{M} and the corresponding radial coordinate of the Schwarzschild-de Sitter metric as $\widetilde{r}_{\mathrm{M}}$ (note that $\widetilde{r}_{\mathrm{M}}$ can evolve with time, see Paper I).

Fourth, we explore the corresponding results in connecting the two metrics in the case of a flat universe.

For a flat universe, conditions (3) and (14) turn into

$$\frac{\partial t}{\partial \widetilde{r}} = \frac{r}{r^2\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - 1}\frac{\mathrm{d}R}{\mathrm{d}t} \tag{20}$$

and

$$\left(\frac{\partial t}{\partial \widetilde{t}}\right)^2 = 1. \tag{21}$$

A solution of Eq.(21) is

$$t(\widetilde{t},\widetilde{r}) = \widetilde{t} + f_2(\widetilde{r}), \tag{22}$$

where function $f_2(\widetilde{r})$ is in a free form. One thus obtains from Eq.(20)

$$\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}} f_2(\widetilde{r}) = \frac{\widetilde{r}\dfrac{1}{R}\dfrac{\mathrm{d}R}{\mathrm{d}t}}{\widetilde{r}^2\left(\dfrac{1}{R}\dfrac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - 1}. \tag{23}$$

This equation is not valid as long as $(\mathrm{d}R/\mathrm{d}t)/R$ is not a constant. However, as discussed above, it is unnecessary to require condition (20) to hold at any time and any position. In fact, to connect the two metrics, it is enough to require condition (20) to hold at r_{M} (see Eq.(18)). In this way, condition (20) should be replaced by

$$\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}} f_2(\widetilde{r}) = \frac{r}{r^2\left(\dfrac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - 1}\frac{\mathrm{d}R}{\mathrm{d}t}, \qquad (r = r_{\mathrm{M}}). \tag{24}$$

We then come to the conclusion that in the case of a flat universe, the connection of the R-W and Schwarzschild-de Sitter metrics requires the transformation relation (2) to take the form of expression (22), where $f_2(\widetilde{r})$ is constrained by condition (24).

Fifth, we check whether conditions (3), (13) and (14) are satisfied in the example above and show how one can come to this situation.

Under the assumptions of expressions (7) and (8), we obtain a solution of Eq.(3), which is presented by expression (10). In the sample, the universe is assumed to be flat. Thus Eqs.(22) and (24) hold. Comparing expressions (10) and (22) yields

$$t(\widetilde{t},\widetilde{r}) = \widetilde{t} + \frac{t_0}{2}\ln\left[(\frac{\widetilde{r}}{t_0})^2 - 1\right] + C, \tag{25}$$

where C is a constant. Transformation relation (10) is now replaced by expression (25).

One can check that adopting transformation relation (25) can make both conditions (3) and (14) satisfied since the function $t(\widetilde{t},\widetilde{r})$ is a solution to both equations. As condition (3) holds, condition (13) can easily be satisfied when the boundary is found at the place restricted by expression (17) for any local system concerned.

Comparing expression (25) with expression (22) gives

$$f_2(\widetilde{r}) = \frac{t_0}{2}\ln\left[\left(\frac{\widetilde{r}}{t_0}\right)^2 - 1\right] + C, \tag{26}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}\widetilde{r}} f_2(\widetilde{r}) = \frac{\dfrac{\widetilde{r}}{t_0}}{\left(\dfrac{\widetilde{r}}{t_0}\right)^2 - 1} = \frac{r}{r^2\left(\dfrac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - 1}\frac{\mathrm{d}R}{\mathrm{d}t}. \tag{27}$$

This is exactly identical with Eq.(23). It suggests that not only condition (24) but also condition (23) are satisfied. That means that in this situation, condition (3) holds at any time and any position as long as the transformation relation (25) is adopted.

According to the above analysis, we come to the conclusion that there exist solutions of the partial differential equation presented in Paper I, which guarantees the vanishing of the crossing term of space and time when transformation relations (1) and (2) are applied to the conventional R-W metric. In connecting the R-W and Schwarzschild-de Sitter metrics, we find there exist three conditions which should be satisfied. Of the three conditions, one condition constrains the place where the two metrics can take the same value for a local system whose mass is provided which therefore marks the boundary of the system, and the other two constrain the transformation relation (2). In realizing the connection of the two metrics, we only require the latter two conditions to be satisfied at the boundary determined by the former condition.

References

[1] Garnavich P M, Kirshner R P, Challis P, Tonry J, Gilliland R L, Smith R C, Clocchiatti A, Diercks A, Filippenko A V, Hamuy M, Hogan C J, Leibundgut B, Phillips M M, Reiss D, Riess A G, Schmidt B P, Schommer R A, Spyromilio J, Stubbs C, Suntzeff N B and Wells L 1998 *Astrophys. J.* **493** L53

[2] Perlmutter S, Aldering G, della Valle M, Deustua S, Ellis R S, Fabbro S, Fruchter A, Goldhaber G, Groom D E, Hook I M, Kim A G, Kim M Y, Knop R A, Lidman C, McMahon R G, Nugent P, Pain R, Panagia N, Pennypacker C R, Ruiz-Lapuente P, Schaefer B and Walton N 1998 *Nature* **391** 51

[3] Perlmutter S, Aldering G, Goldhaber G, Knop R A, Nugent P, Castro P G, Deustua S, Fabbro S, Goobar A, Groom D E, Hook I M, Kim A G, Kim M Y, Lee J C, Nunes N J, Pain R, Pennypacker C R, Quimby R, Lidman C, Ellis R S, Irwin M, McMahon R G, Ruiz-Lapuente P, Walton N, Schaefer B, Boyle B J, Filippenko A V, Matheson T, Fruchter A S, Panagia N, Newberg H J M and Couch W J 1999 *Astrophys. J.* **517** 565

[4] Riess A G, Filippenko A V, Challis P, Clocchiatti A, Diercks A, Garnavich P M, Gilliland R L, Hogan C J, Jha S, Kirshner R P, Leibundgut B, Phillips M M, Reiss D, Schmidt B P, Schommer R A, Smith R C, Spyromilio J, Stubbs C, Suntzeff N B and Tonry J 1998 *Astron. J.* **116** 1009
[5] Zehavi I and Dekel A 1999 *Nature* **401** 252
[6] Spergel D N, Verde L, Peiris H V, Komatsu E, Nolta M R, Bennett C L, Halpern M, Hinshaw G, Jarosik N, Kogut A, Limon M, Meyer S S, Page L, Tucker G S, Weiland J L, Wollack E and Wright E L 2003 *Astrophys. J. Supp.* **148** 175
[7] Riess A G, Strolger Louis-Gregory, Tonry J, Casertano S, Ferguson H C, Mobasher B, Challis P, Filippenko A V, Jha S, Li Weidong, Chornock R, Kirshner R P, Leibundgut B, Dickinson M, Livio M, Giavalisco M, Steidel C C, Benítez T and Tsvetanov Z 2004 *Astrophys. J.* **607** 665
[8] Carroll S, Press W H and Turner E L 1992 *Annu. Rev. Astron. Astrophys.* **30** 499
[9] Ratra B and Peebles P J E 1988 *Phys. Rev.* D **37** 3406
[10] Caldwell R, Dave R and Steinhardt P J 1998 *Phys. Rev. Lett.* **80** 1582
[11] Caldwell R 2002 *Phys.Lett.* B **545** 23
[12] Turner M S and White M 1997 *Phys. Rev.* D **56** R4439
[13] Zhu Z H 1998 *Astron. Astrophys.* **338** 777
[14] Zhu Z H, Fujimoto M K and Tatsumi D 2001 *Astron. Astrophys.* **372** 377
[15] Zhu Z H, Fujimoto M K and He X T 2004 *Astron. Astrophys.* **417** 833
[16] Kamenshchik A, Moschella U and Pasquier V 2001 *Phys. Lett.* B **511** 265
[17] Bento M C, Bertolami O and Sen A A 2002 *Phys. Rev.* D **66** 043507
[18] Zhu Z H 2004 *Astron. Astrophys.* **423** 421
[19] Freese K and Lewis M 2002 *Phys. Lett.* B **540** 1
[20] Zhu Z H and Fujimoto M K 2002 *Astrophys. J.* **581** 1
[21] Zhu Z H and Fujimoto M K 2003 *Astrophys. J.* **585** 52
[22] Zhu Z H and Fujimoto M K 2004 *Astrophys. J.* **602** 12
[23] Zhu Z H, Fujimoto M K and He X T 2004 *Astrophys. J.* **603** 365
[24] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 3370
[25] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 4690
[26] Deffayet C, Dvali G and Gabadadze G 2002 *Phys. Rev.* D **65** 044023
[27] Gibbons G W and Hawking S W 1977 *Phys. Rev.* D **15** 2738
[28] Qin Y P 2006 *Chin. Phys. Lett.* **23** 758 (Paper I)
[29] Marolf D and Yaida S 2005 *Phys. Rev.* D **72** 044016
[30] Parikh M K and Wilczek F 2000 *Phys. Rev. Lett.* **85** 5042
[31] Parikh M K 2002 *Phys. Lett.* B **546** 189
[32] Zhang J Y and Zhao Z 2005 *J. High Energy Physics JHEP* **10** 055
[33] Zhang J Y and Zhao Z 2005 *Nucl. Phys.* B **725** 173
[34] Zhang J Y and Zhao Z 2006 *Acta Phys. Sin.* **55** 3796 (in Chinese)
[35] Han Y W 2005 *Acta Phys. Sin.* **54** 5018 (in Chinese)
[36] Yang S Z, Jiang Q Q and Li H L 2005 *Chin. Phys.* **14** 2411
[37] Ren J and Zhao Z 2006 *Chin. Phys.* **15** 0292
[38] Han Y W 2007 *Chin. Phys.* **16** 923
[39] Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley) p472