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Entanglement in a generalized Jaynes–Cummings model*

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We investigate the pairwise entanglement and global entanglement in a generalized Jaynes–Cummings model, which can be used to realize Greenberger–Horne–Zeilinger (GHZ) entangled state (Zheng S B 2001 *Phys. Rev. Lett.* **87** 230404). Our results show that the W-type entangled states cannot be generated based on the model. The dependences of entanglement on Rabi frequency λ and dipole–dipole coupling strength Ω are given. It is found that there exists the quantum phase transition when $\lambda = \Omega$. For typical experimental data, the critical temperature for pairwise entanglement is on the order of 10^{-6} K. Based on these results, two strategies that overcome decoherence are proposed.

Keywords: thermal entanglement, dipole–dipole coupling strength, decoherence
PACC: 0365, 4250

1. Introduction

Entanglement is one of the most striking features of quantum mechanics. It plays a fundamental role in almost all efficient protocols of quantum computation (QC) and quantum information processing (QIP).[1−4] In the past years, many schemes[5−11] have been proposed for generating two or many particle entanglement. For pure states, we know almost all the information about it. It is convenient to measure the pure-state entanglement by calculating the entropy of its subsystems. Unfortunately, the pure state is only a special case of the mixed states. In practice, a pure state will change into a mixed state under non-unitary evolution. For any further use, we must know all the properties of the mixed state, through the theory of mixed-state entanglement is more complicated and less well understood. Much interesting work in the field of mixed-state entanglement has been done already,[12−14] but there is still plenty of work to be explored in mixed-state entanglement. Recently, a scheme[15,16] has been proposed for generating entangled atomic states. In this scheme, atoms are simultaneously sent through a nonresonant cavity in a vacuum state, two-atom maximally entangled states and N-atom Greenberger–Horne–Zeilinger (GHZ) states can be generated by adjusting the interaction time. In the present paper, the scheme[16] is extended and the dipole–dipole coupling strength Ω between the atoms are considered. The pairwise entanglement and global entanglement are investigated. The results show that the W-type entangled states cannot be generated in this scheme. The entanglement evolution with respect to the temperature, the Rabi frequency and dipole–dipole coupling strength is given for typical experimental data. Based on the results, two strategies that overcome decoherence are proposed. We divide the remainder of the present paper into five sections. In Section 2 the model and the ground state entanglement are present. In Section 3 we discuss the thermal entanglement. In Section 4 two strategies that overcome decoherence are proposed. The conclusions will be given in Section 5.

2. The model and the ground state entanglement

2.1. The model and solutions

When the detuning between the atomic transition frequency and cavity frequency is far higher than the atom–cavity coupling strength ($(\delta = \omega_0 - \omega \gg g\sqrt{n+1})$, with $\bar{n}$ being the mean photon number of the cavity field), there is no energy exchange between the atomic system and the cavity. In Zheng's scheme,[16] he has considered the interaction between atom and cavity when the cavity mode is initially in a

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vacuum state. Here we extend the model and consider dipole–dipole coupling strength Ω between the atoms. Our model can be described by the following effective Hamiltonian:

$$H = \hbar\lambda \sum_{j=1}^{3} |e_j\rangle\langle e_j| + \hbar\Omega \sum_{j,k=1}^{3} S_j^{\dagger} S_k^{-}, \quad j \neq k, \tag{1}$$

where the Rabi frequency $\lambda = g^2/\delta$; g is the atom-cavity coupling strength; δ is the detuning; $S_j^{\dagger} \equiv |e_j\rangle\langle g_j|$ and $S_j^{-} \equiv |g_j\rangle\langle e_j|$, with e_j (g_j) being the excited (ground) states of the jth atom. In the following, we will write $\hbar = 1$ unless it is specially stressed. The basis for the atom subsystem is chosen as

$$|g\rangle = |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad |e\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2}$$

In order to study thermal entanglement, the first step is to obtain all the eigenvalues and the corresponding eigenstates of Hamiltonian Eq.(1). The eigenvalues themselves do not suffice to calculate the entanglement. The eigenvalues are more simply obtained as

$$\begin{aligned} E_1 &= 0, \\ E_2 &= E_3 = \lambda - \Omega, \\ E_4 &= E_5 = 2\lambda - \Omega, \\ E_6 &= \lambda + 2\Omega, \\ E_7 &= 3\lambda, \\ E_8 &= 2\lambda + 2\Omega, \end{aligned} \tag{3}$$

and the corresponding eigenstates are explicitly given by

$$\begin{aligned} |\Psi_1\rangle &= |000\rangle, \\ |\Psi_2\rangle &= \frac{1}{\sqrt{2}}(|001\rangle - |100\rangle), \\ |\Psi_3\rangle &= \frac{1}{\sqrt{2}}(|010\rangle - |100\rangle), \\ |\Psi_4\rangle &= \frac{1}{\sqrt{2}}(|011\rangle - |110\rangle), \\ |\Psi_5\rangle &= \frac{1}{\sqrt{2}}(|101\rangle - |110\rangle), \\ |\Psi_6\rangle &= \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle), \\ |\Psi_7\rangle &= |111\rangle, \\ |\Psi_8\rangle &= \frac{1}{\sqrt{3}}(|110\rangle + |101\rangle + |011\rangle). \end{aligned} \tag{4}$$

Because $\lambda > 0$ and $\Omega > 0$, it is readily found that the ground-state energy is equal to

$$\begin{aligned} E_1 = 0, &\qquad \text{if } \lambda > \Omega; \\ E_1 = E_2 = E_3 = 0, &\qquad \text{if } \lambda = \Omega; \\ E_2 = E_3 = \lambda - \Omega, &\qquad \text{if } \lambda < \Omega. \end{aligned} \tag{5}$$

For $\lambda = \Omega$ ($\lambda < \Omega$), the ground state is three- (two-) fold degenerate. We will see that the degeneracy of the system can influence thermal entanglement greatly.

2.2. The ground state entanglement

Apparently, the states $|\Psi_1\rangle$ and $|\Psi_7\rangle$ are unentangled states. We can rewrite $|\Psi_i\rangle (i = 2, 3, 4, 5)$ as follows:

$$\begin{aligned} |\Psi_2\rangle &= \frac{1}{\sqrt{2}}(|01\rangle_{13} - |10\rangle_{13}) \otimes |0\rangle_2, \\ |\Psi_3\rangle &= \frac{1}{\sqrt{2}}(|01\rangle_{12} - |10\rangle_{12}) \otimes |0\rangle_3, \\ |\Psi_4\rangle &= \frac{1}{\sqrt{2}}(|01\rangle_{13} - |10\rangle_{13}) \otimes |1\rangle_2, \\ |\Psi_5\rangle &= \frac{1}{\sqrt{2}}(|01\rangle_{23} - |10\rangle_{23}) \otimes |1\rangle_1. \end{aligned} \tag{6}$$

It is easily found that the first atom and the third one are in the maximally entangled states for $|\Psi_2\rangle$ and $|\Psi_4\rangle$, but they are not entangled with the second one. Similarly, $|\Psi_3\rangle$ ($|\Psi_5\rangle$) is the maximally entangled state for the first (second) and the second (third) atoms.

The entanglement of two qubits can be measured by concurrence C which is written as $C = \max\left[0, 2\max[\lambda_i] - \sum_i^4 \lambda_i\right]$,[17] where λ_i are the square roots of the eigenvalues of the matrix $R = \rho S \rho^* S$, ρ is the density matrix, $S = \sigma_1^y \otimes \sigma_2^y$ and $*$ represents the complex conjugate. The original measure Q of a many qubit pure state $|\Psi\rangle$ was introduced by Meyer and Wallach.[18] It was later shown by Brennen[19] that this kind of global entanglement could be written as

$$Q(|\psi\rangle) = 2\left[1 - \frac{1}{n}\sum_{l=1}^{n} \mathrm{Tr}(\rho_l^2)\right], \tag{7}$$

with ρ_l, the density matrix, reduced to a single qubit l. $|\Psi_6\rangle = |W\rangle$ and $|\Psi_8\rangle$ have the form of the W-state, and the three concurrences are identical for the two states. After a straight calculation, the global entanglement is as follows:

$$\begin{aligned} Q &= 0, \quad \text{for } |\Psi_1\rangle, |\Psi_7\rangle; \\ Q &= \frac{2}{3}, \quad \text{for } |\Psi_2\rangle, |\Psi_3\rangle, |\Psi_4\rangle, |\Psi_5\rangle; \\ Q &= \frac{8}{9}, \quad \text{for } |\Psi_6\rangle, |\Psi_8\rangle. \end{aligned} \tag{8}$$

3. Thermal entanglement

For a system in equilibrium at a temperature T, the density matrix is $\rho = \mathrm{e}^{-\beta H}/Z$, where $\beta = 1/(kT)$, k is the Boltzmann constant and $Z = \mathrm{tr}\mathrm{e}^{-\beta H}$ is the partition function. For simplicity, we take $k = 1$. As the thermal fluctuation is introduced into the system, the entangled ground states will be mixed with the untangled excited states. This effect will make the entanglement decrease. At the same time, the disentangled ground state mixes with entangled excited states. To see the change of the entanglement, we calculate the entanglement of the thermal state $\rho = \mathrm{e}^{-\beta H}/Z$.

From expression (3), the partition function is obtained as

$$\begin{aligned} Z = {} & 1 + 2\mathrm{e}^{-\beta(\lambda-\Omega)} + 2\mathrm{e}^{-\beta(2\lambda-\Omega)} \\ & + \mathrm{e}^{-2\beta(\lambda+\Omega)} + \mathrm{e}^{-\beta(\lambda+2\Omega)} + \mathrm{e}^{-3\beta\lambda}. \end{aligned} \tag{9}$$

And the density matrix can be written as

$$\begin{aligned} \rho(T) = {} & \frac{1}{Z}[|\Psi_1\rangle\langle\Psi_1| + \mathrm{e}^{-3\beta\lambda}|\Psi_7\rangle\langle\Psi_7| \\ & + \mathrm{e}^{-\beta(\lambda-\Omega)}(|\Psi_2\rangle\langle\Psi_2| + |\Psi_3\rangle\langle\Psi_3|) \\ & + \mathrm{e}^{-\beta(2\lambda-\Omega)}(|\Psi_4\rangle\langle\Psi_4| + |\Psi_5\rangle\langle\Psi_5|) \\ & + \mathrm{e}^{-\beta(\lambda+2\Omega)}|\Psi_6\rangle\langle\Psi_6| \\ & + \mathrm{e}^{-2\beta(\lambda+\Omega)}|\Psi_8\rangle\langle\Psi_8|]. \end{aligned} \tag{10}$$

In this paper, we consider pairwise thermal entanglement, so we need to calculate the reduced density matrix $\rho_{12} = \mathrm{tr}_3[\rho(T)]$. The reduced density matrix $\mathrm{tr}_3[|\Psi_i\rangle\langle\Psi_i| + |\Psi_j\rangle\langle\Psi_j|]$ is denoted by $\rho_{12}^{(ij)}$ and $\mathrm{tr}_3[|\Psi_k\rangle\langle\Psi_k|]$ by $\rho_{12}^{(k)}$. ρ_{13} and ρ_{23} have similar definitions. From expression (4), we obtain

$$\rho_{12}^{(1)} = \rho_{13}^{(1)} = \rho_{23}^{(1)} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix}, \tag{11}$$

$$\rho_{12}^{(23)} = \rho_{13}^{(23)} = \begin{pmatrix} 0&0&0&0 \\ 0&1&-\frac{1}{2}&0 \\ 0&-\frac{1}{2}&\frac{1}{2}&0 \\ 0&0&0&\frac{1}{2} \end{pmatrix}, \tag{12}$$

$$\rho_{12}^{(45)} = \begin{pmatrix} 1&0&0&0 \\ 0&\frac{1}{2}&0&0 \\ 0&0&\frac{1}{2}&0 \\ 0&0&0&0 \end{pmatrix}, \tag{13}$$

$$\rho_{12}^{(6)} = \rho_{13}^{(6)} = \rho_{23}^{(6)} = \begin{pmatrix} 0&0&0&0 \\ 0&\frac{1}{3}&\frac{1}{3}&0 \\ 0&\frac{1}{3}&\frac{1}{3}&0 \\ 0&0&0&\frac{1}{3} \end{pmatrix}, \tag{14}$$

$$\rho_{12}^{(7)} = \rho_{13}^{(7)} = \rho_{23}^{(7)} = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \tag{15}$$

$$\rho_{12}^{(8)} = \rho_{13}^{(8)} = \rho_{23}^{(8)} = \begin{pmatrix} \frac{1}{3}&0&0&0 \\ 0&\frac{1}{3}&\frac{1}{3}&0 \\ 0&\frac{1}{3}&\frac{1}{3}&0 \\ 0&0&0&0 \end{pmatrix}, \tag{16}$$

$$\rho_{13}^{(45)} = \rho_{23}^{(45)} = \begin{pmatrix} \frac{1}{2}&0&0&0 \\ 0&1&-\frac{1}{2}&0 \\ 0&-\frac{1}{2}&\frac{1}{2}&0 \\ 0&0&0&0 \end{pmatrix}, \tag{17}$$

$$\rho_{23}^{(23)} = \begin{pmatrix} 0&0&0&0 \\ 0&\frac{1}{2}&0&0 \\ 0&0&\frac{1}{2}&0 \\ 0&0&0&1 \end{pmatrix}. \tag{18}$$

From expressions (10–18), we obtain

$$\begin{aligned} \rho_{12} = {} & \frac{1}{Z}\Big(\rho_{12}^{(1)} + \mathrm{e}^{-\beta(\lambda-\Omega)}\rho_{12}^{(23)} + \mathrm{e}^{-\beta(2\lambda-\Omega)}\rho_{12}^{(45)} \\ & + \mathrm{e}^{-\beta(\lambda+2\Omega)}\rho_{12}^{(6)} + \mathrm{e}^{-3\beta\lambda}\rho_{12}^{(7)} + \mathrm{e}^{-2\beta(\lambda+\Omega)}\rho_{12}^{(8)}\Big) \\ = {} & \frac{1}{Z}\begin{pmatrix} u&0&0&0 \\ 0&w_1&z&0 \\ 0&z&w_2&0 \\ 0&0&0&v \end{pmatrix} \end{aligned} \tag{19}$$

with

$$\begin{aligned} u &= \mathrm{e}^{-3\beta\lambda} + \mathrm{e}^{\beta(-2\lambda+\Omega)} + \frac{1}{3}\mathrm{e}^{-2\beta(\lambda+\Omega)}, \\ w_1 &= \left[2 + 2\mathrm{e}^{\beta\lambda} + 3\mathrm{e}^{3\beta\Omega} + 6\mathrm{e}^{\beta(\lambda+3\Omega)}\right]\epsilon, \\ w_2 &= \left(1 + \mathrm{e}^{\beta\lambda}\right)\left(2 + 3\mathrm{e}^{3\beta\Omega}\right)\epsilon, \\ z &= \left[2 + \mathrm{e}^{\beta\lambda}\left(2 - 3\mathrm{e}^{3\beta\Omega}\right)\right]\epsilon, \end{aligned}$$

$$v = 1 + \frac{1}{6}\mathrm{e}^{-\beta(\lambda+2\Omega)}\left(2 + 3\mathrm{e}^{3\beta\Omega}\right),$$
$$\epsilon = \frac{1}{6}\mathrm{e}^{-2\beta(\lambda+\Omega)}. \tag{20}$$

The concurrence quantifying the entanglement is readily obtained as[14]

$$C_{12} = \frac{2}{Z}\max[0, |z| - \sqrt{uv}]. \tag{21}$$

We can also obtain C_{13} and C_{23} in the same way.

C_{12} and C_{13} as functions of temperature T and Rabi frequency λ for different values of atomic dipole–dipole coupling strength Ω are given in the left panels of Figs.1–4. Accordingly, the contours about them are plotted in the right panels of Figs.1–4. The left panels of these figures show that the values of critical temperature T_c are different for C_{12} and C_{13}. For $\Omega = 1.5\,\lambda$, when the temperature is low and close to zero, $C_{12} = C_{13} = \frac{1}{2}$. This is very easily understood since $|\Psi_2\rangle$ and $|\Psi_3\rangle$ constitute a doubly-degenerate ground state. Here the value $\frac{1}{2}$ appears due to the

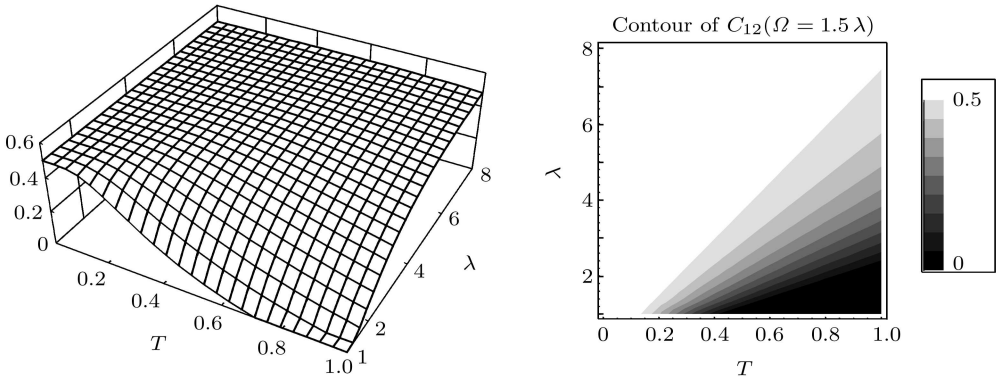


Fig.1. C_{12} is plotted vs T and λ for $\Omega = 1.5\,\lambda$ in the left panel, and the right panel corresponds to the contour. T is in units of the Boltzmann constant; k, λ and Ω are in units of $\hbar$.

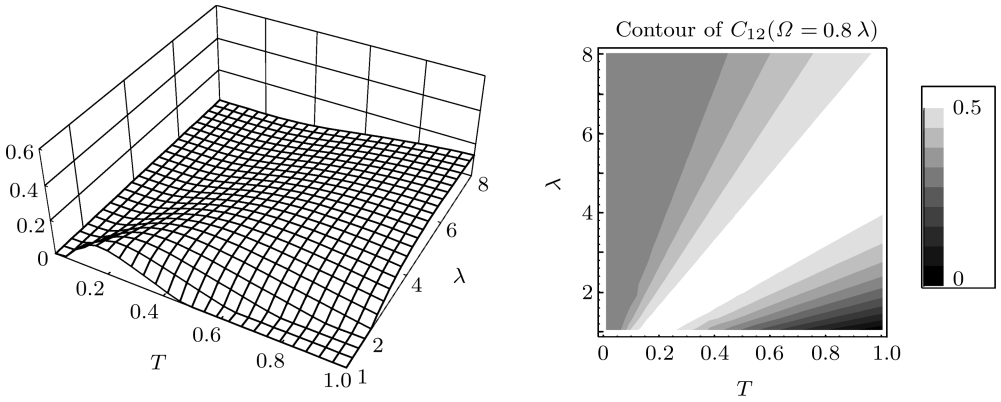


Fig.2. The same legend as that of Fig.1, but $\Omega = 0.8\lambda$.

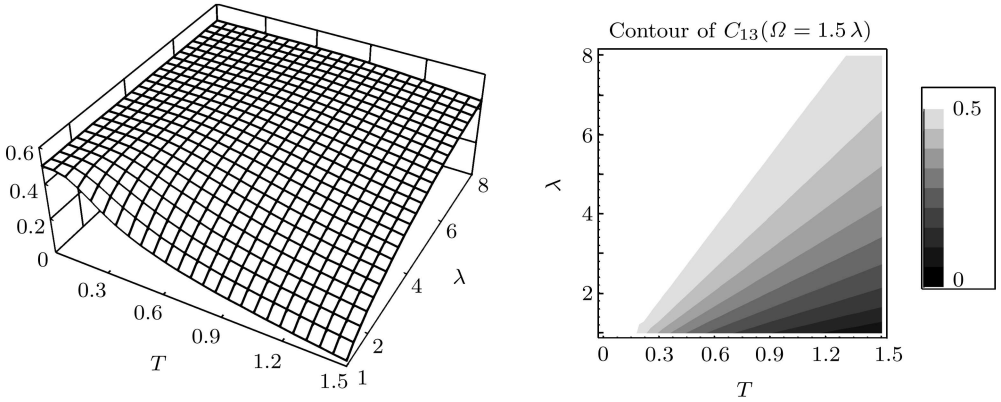


Fig.3. C_{13} is plotted vs T and λ for $\Omega = 1.5\,\lambda$ in the left panel, and the right panel corresponds to the contour. T is in units of the Boltzmann constant; k, λ and Ω are in units of $\hbar$.

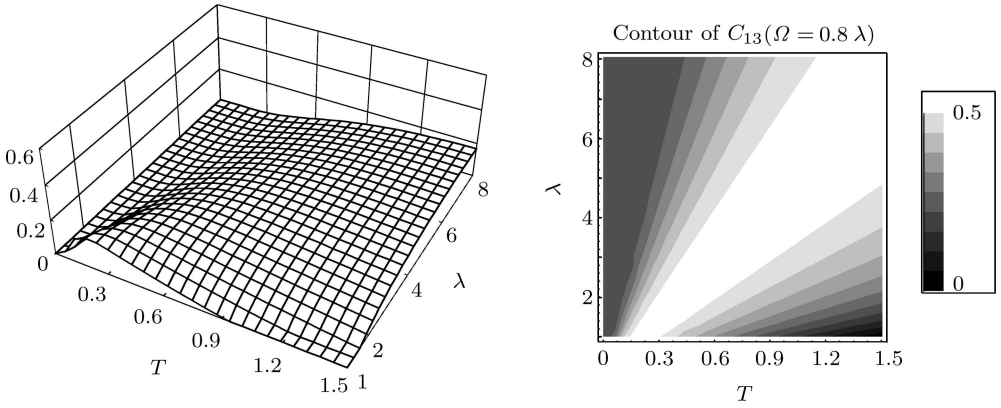


Fig.4. The same legend as that of Fig.3, but $\Omega = 0.8\lambda$.

degeneracy. In fact, at zero temperature, the thermal entanglement can be calculated from $\rho_{12}^{(23)} = \rho_{13}^{(23)}$. It is easy to obtain the concurrence, which is just $\frac{1}{2}$. When $\Omega = \lambda$ and $T \to 0$, $C_{12} = C_{13} = \frac{1}{3}$ because $|\Psi_1\rangle$, $|\Psi_2\rangle$ and $|\Psi_3\rangle$ constitute a three-degenerate ground state. As the temperature increases, the entanglement between qubits drops off quickly until they reach zero at a critical temperature, thus the two-atom entangled state in the system disappears. This result is obvious since the quantum coherence of the atomic subsystem will drop off as T increases. $\Omega = \frac{4}{5}\,\lambda$, $|\Psi_1\rangle$ is the ground state, which is an unentangled state and $C_{12} = C_{13} = 0$. The entanglement will increase as the temperature goes up and becomes zero until at a critical point. We may obtain a much higher entanglement value and critical temperature by increasing Ω. It is noticed that there is no entanglement between atom 2 and atom 3. Because $\lambda > 0$ and $\Omega > 0$, it is impossible that the ground-state is $|\Psi_5\rangle$, $|\Psi_6\rangle$ or $|\Psi_8\rangle$, for which $C_{23} \neq 0$. Ω and T cannot help to increase the value of C_{23}.

4. Two strategies that overcome decoherence

Cavity-quantum electrodynamics (QED) is one of the candidates for designing quantum logic gate, quantum communication and quantum computer. How to keep the coherence of qubits is the principal task. Up to now, three ways for avoiding decoherence are proposed: quantum error-correcting codes,[20] dynamical decoupling theory[21] and pairing quantum bits.[22] But these ways do not consider the real physics systems. Based on the real experiment data, in Fig.5, we give the plot of C_{12} as a function of temperature T for different values of λ and Ω. The atomic dipole–dipole coupling strength $\Omega = 10\,g$, $g = 2\pi \times 24\,\mathrm{kHz}$.[15,16] From Fig.5, we can see that the critical temperature for C_{12} is on the order of $10^{-6}\,\mathrm{K}$. We may improve T_c and entanglement by increasing Ω. That is to say, the atomic subsystem quantum coherence will be stronger at higher temperatures as we increase Ω. Comparing the figures in Fig.5 shows that the entanglement turns smaller as λ increases for both $\Omega = 10\,g$ cases.

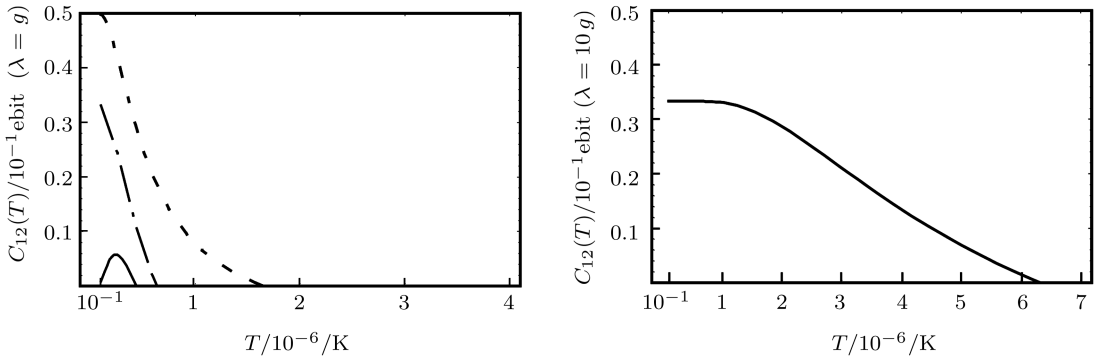


Fig.5. C_{12} is plotted vs T for different experimental data. The left panel corresponds to $\lambda = g$ case, $\Omega = 0.6\,g$ (solid); $\Omega = g$ (dash-dotted); $\Omega = 10\,g$ (dotted). The right panel corresponds to $\Omega = 10\,g$ case.

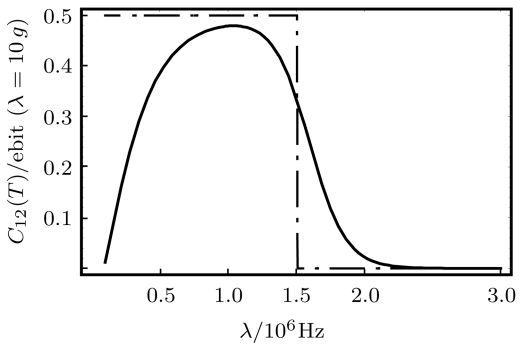


Fig.6. Pairwise concurrence C_{12} is plotted vs λ for different system temperatures when $\Omega = 10\,g$. $T = 10^{-6}$ (solid) and $T = 10^{-9}$ (dash-dotted).

We can also keep the coherence between qubits by lowering temperature based on the existing cooling technique. When $\Omega = 10g$ and $\lambda = g$, $T = 10^{-9}$, $C_{12} = 0.5000$ ebit but $T = 10^{-6}$, $C_{12} = 0.0779$ ebit. These results can be seen from Fig.6. We can see that there exists the quantum phase transition in the system for $\lambda = \Omega$.

5. Conclusions

In this paper, atom pairwise thermal entanglement and the global entanglement are investigated. We study the influences of temperature, Rabi frequency and atomic dipole–dipole coupling strength on entanglement. The results show that the W-type entangled states cannot be generated in this scheme. The entanglement evolution with respect to the temperature, the Rabi frequency and dipole–dipole coupling strength is studied. For typical experimental data, the critical temperature for C_{12} is on the order of 10^{-6} K. In addition, there exists a quantum phase transition when $\lambda = \Omega$. Based on the results, two strategies that overcome decoherence are proposed.

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