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A kind of extended Korteweg–de Vries equation and solitary wave solutions for interfacial waves in a two-fluid system*

Yang Hong-Li(杨红丽)[a)c)†], Song Jin-Bao(宋金宝)[a)], Yang Lian-Gui(杨联贵)[b)], and Liu Yong-Jun(刘永军)[a)c)]

[a)] *Institute of Oceanology, Chinese Academy of Sciences, Qingdao* 266071, *China*
[b)] *Department of Mathematics, Inner Mongolia University, Hohhot* 010021, *China*
[c)] *Graduate School, Chinese Academy of Sciences, Beijing* 100049, *China*

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This paper considers interfacial waves propagating along the interface between a two-dimensional two-fluid with a flat bottom and a rigid upper boundary. There is a light fluid layer overlying a heavier one in the system, and a small density difference exists between the two layers. It just focuses on the weakly non-linear small amplitude waves by introducing two small independent parameters: the nonlinearity ratio ε, represented by the ratio of amplitude to depth, and the dispersion ratio μ, represented by the square of the ratio of depth to wave length, which quantify the relative importance of nonlinearity and dispersion. It derives an extended KdV equation of the interfacial waves using the method adopted by Dullin *et al* in the study of the surface waves when considering the order up to $O(\mu^2)$. As expected, the equation derived from the present work includes, as special cases, those obtained by Dullin *et al* for surface waves when the surface tension is neglected. The equation derived using an alternative method here is the same as the equation presented by Choi and Camassa. Also it solves the equation by borrowing the method presented by Marchant used for surface waves, and obtains its asymptotic solitary wave solutions when the weakly nonlinear and weakly dispersive terms are balanced in the extended KdV equation.

Keywords: two-fluid system, interfacial waves, extended KdV equation, solitary wave solution
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1. Introduction

Weakly nonlinear models for the evolution of internal waves have been extensively studied in the past. Among these, for unidirectional waves, the Korteweg–de Vries (KdV) equation for shallow water (Benjamin[1]) and the Intermediate Long Wave (ILW) equation for a fluid of finite depth are well known (Benjamin[2]; Ono[3]). Models, in the so-called Kadomtsev and Petviashvili[4] class, for two-dimensional waves have been derived both for shallow water and for deep water (Ablowitz and Segur[5]).

For the case of a homogeneous fluid layer, Matsuno[6] derived new types of nonlinear equations that describe finite-amplitude surface waves on a fluid of finite depth for one-dimensional waves, and Choi[7] for two-dimensional waves.

In contrast to surface gravity waves, some investigators have been concerned with interfacial waves in two-layer fluid system where each fluid has a constant density. They derived nonlinear evolution equations (NEEs), such as the higher-order Benjamin–Ono equations describing the time evolution of the interfacial elevation (Matsuno[8], Choi and Camassa[9]). The wave equations describing interfacial internal waves in two-layer or multi-layer fluid are nonlinear wave equations and there are many methods for solving nonlinear wave equations. For example, perturbation method,[10−17] the homogeneous balance principle,[18−20] expansion method about the Jacobi elliptic function,[21] various tanh method,[22] the general projective Riccati method[23] and so on.

Recently, for the case of a homogeneous fluid layer, much progress has been made in this direction. Dullin *et al*[24] developed the Camassa–Holm (CH) equation as a shallow water wave equation with surface tension in an asymptotic expansion, and showed that CH equation is asymptotically equivalent to KdV-5 equation (the fifth-order integrable equation in the KdV hierarchy). In this paper, we extend some

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†Corresponding author. E-mail: songjb@ms.qdio.ac.cn

of Dullin *et al*'s work[24] to the case of two-layer fluid system, without considering the surface tension, and have derived a kind of extended KdV equation governing the interfacial waves in two-layer fluid system. We consider a one-dimensional system in which a layer of light fluid overlies a layer of a heavier one resting on a flat bottom (see Fig.1). The upper boundary is assumed to be rigid, for when there is a small density difference between the two layers, the variation of the free surface caused by the internal wave propagation has no effect on the internal wave itself (Patrick and Philip[25]).

The present paper is organized as follows. Starting with the basic equations presented in §2, in §3 we derive the equation for the interfacial waves adopting the method introduced by Dullin *et al*[24] in the study of surface waves. Finally, we obtain its asymptotic solitary wave solutions of our equation by making use of the method presented by Marchant.[26]

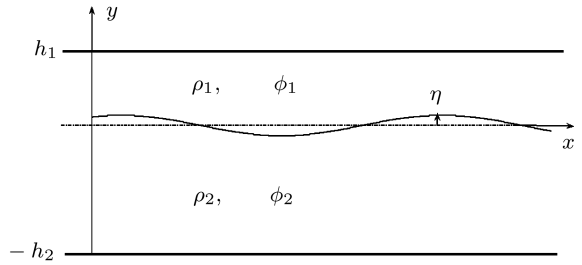


Fig.1. Sketches of a two-fluid system.

2. Basic equations

We assume that the fluid in each layer is homogeneous, inviscid, incompressible, and irrotational. The nondimensional quantities, with asterisks, are related to the corresponding dimensional ones by the relations

$$x^* = \frac{x}{\lambda}, \quad y^* = \frac{y}{h_1}, \quad t^* = \frac{\sqrt{gh_1}}{\lambda}t,$$
$$\eta^* = \frac{\eta}{a}, \quad \phi_i^* = \frac{\sqrt{gh_1}}{ga\lambda}\,\phi_i, \tag{1}$$

where the Cartesian coordinates (x, y) are introduced with origin at the interface of two fluids of different densities, ρ_1 for the upper fluid and ρ_2 for the lower fluid; h_1 and $-h_2$ are the vertical coordinates of the upper and lower boundaries respectively; $R = \rho_2/\rho_1\ (>1)$, $H = h_2/h_1$ are the density and depth ratios respectively; t is time, and g is the acceleration due to gravity; λ and a are characteristic scales of wavelength and amplitude of the wave, respectively, $\phi_1 = \phi_1(x, y, t)$ and $\phi_2 = \phi_2(x, y, t)$ are the velocity potentials for the upper and the lower fluids, respectively; $\eta = \eta(x, t)$ is the interfacial elevation.

The equations governing the motion of the fluid and the boundary conditions are written in dimensionless form, dropping asterisks, as follows:

(i) Continuity equations

$$\mu\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0 \quad (\varepsilon\eta < y < 1), \tag{2}$$
$$\mu\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial y^2} = 0 \quad (-H < y < \varepsilon\eta). \tag{3}$$

(ii) Kinematic boundary conditions at the fluid interface

$$\frac{\partial\eta}{\partial t} + \varepsilon\frac{\partial\eta}{\partial x}\frac{\partial\phi_1}{\partial x} = \frac{1}{\mu}\frac{\partial\phi_1}{\partial y}, \quad y = \varepsilon\eta, \tag{4}$$
$$\frac{\partial\eta}{\partial t} + \varepsilon\frac{\partial\eta}{\partial x}\frac{\partial\phi_2}{\partial x} = \frac{1}{\mu}\frac{\partial\phi_2}{\partial y}, \quad y = \varepsilon\eta. \tag{5}$$

(iii) Dynamic boundary condition at the fluid interface

$$\frac{\partial\phi_1}{\partial t} + \frac{\varepsilon}{2}\left[\left(\frac{\partial\phi_1}{\partial x}\right)^2 + \frac{1}{\mu}\left(\frac{\partial\phi_1}{\partial y}\right)^2\right] + \eta$$
$$= R\left\{\frac{\partial\phi_2}{\partial t} + \frac{\varepsilon}{2}\left[\left(\frac{\partial\phi_2}{\partial x}\right)^2 + \frac{1}{\mu}\left(\frac{\partial\phi_2}{\partial y}\right)^2\right] + \eta\right\},$$
$$y = \varepsilon\eta. \tag{6}$$

(iv) Upper and lower boundary conditions

$$\frac{\partial\phi_1}{\partial y} = 0, \quad y = 1, \tag{7}$$
$$\frac{\partial\phi_2}{\partial y} = 0, \quad y = -H. \tag{8}$$

We are just interesting in weakly nonlinear small amplitude waves, and define the small parameters ε and μ as follows

$$\varepsilon = \frac{a}{h_1}, \quad \mu = \left(\frac{h_1}{\lambda}\right)^2, \tag{9}$$

here ε and μ measure the magnitudes of the nonlinearity and dispersion respectively, and we only require $\varepsilon \geq \mu > \varepsilon^2 \geq \varepsilon\mu \geq \mu^2$.

3. Derivation of the equation of the interfacial waves

Following Whitham,[27] based on the continuity equations, upper and lower boundary conditions, we can expand the upper and lower velocity potentials at the height $y = 1$, $y = -H$ respectively, up to order $O(\mu^3)$ as

$$\phi_1 = \Phi_1 - \frac{\mu}{2}(y-1)^2\frac{\partial^2\Phi_1}{\partial x^2} + \frac{\mu^2}{4!}(y-1)^4\frac{\partial^4\Phi_1}{\partial x^4} - \frac{\mu^3}{6!}(y-1)^6\frac{\partial^6\Phi_1}{\partial x^6} + o(\mu^3), \tag{10}$$

$$\phi_2 = \Phi_2 - \frac{\mu}{2}(y+H)^2\frac{\partial^2\Phi_2}{\partial x^2} + \frac{\mu^2}{4!}(y+H)^4\frac{\partial^4\Phi_2}{\partial x^4} - \frac{\mu^3}{6!}(y+H)^6\frac{\partial^6\Phi_2}{\partial x^6} + o(\mu^3). \tag{11}$$

Substituting Eqs.(10) and (11) into the boundary conditions at the fluid interface Eqs.(4)–(6), up to order $O(\mu^2)$, we get

$$\frac{\partial\eta}{\partial t} - \frac{\partial^2\Phi_1}{\partial x^2} + \varepsilon\left[\eta\frac{\partial\Phi_1}{\partial x}\right]_x + \frac{\mu}{3!}\frac{\partial^4\Phi_1}{\partial x^4} - \frac{\varepsilon\mu}{2}\left[\eta\frac{\partial^3\Phi_1}{\partial x^3}\right]_x - \frac{\mu^2}{5!}\frac{\partial^6\Phi_1}{\partial x^6} = 0, \tag{12}$$

$$\frac{\partial\eta}{\partial t} + H\frac{\partial^2\Phi_2}{\partial x^2} + \varepsilon\left[\eta\frac{\partial\Phi_2}{\partial x}\right]_x - \frac{\mu}{6}H^3\frac{\partial^4\Phi_2}{\partial x^4} - \frac{\varepsilon\mu}{2}H^2\left[\eta\frac{\partial^3\Phi_2}{\partial x^3}\right]_x + \frac{\mu^2}{5!}H^5\frac{\partial^6\Phi_2}{\partial x^6} = 0, \tag{13}$$

$$\begin{aligned}&\frac{\partial\Phi_1}{\partial t} + \eta + \frac{\varepsilon}{2}\left(\frac{\partial\Phi_1}{\partial x}\right)^2 - \frac{\mu}{2}\frac{\partial^3\Phi_1}{\partial t\partial x^2} + \varepsilon\mu\left[\eta\frac{\partial^3\Phi_1}{\partial t\partial x^2} - \frac{1}{2}\frac{\partial\Phi_1}{\partial x}\frac{\partial^3\Phi_1}{\partial x^3} + \frac{1}{2}\left(\frac{\partial^2\Phi_1}{\partial x^2}\right)^2\right] + \frac{\mu^2}{4!}\frac{\partial^5\Phi_1}{\partial t\partial x^4}\\ =\ & R\left\{\frac{\partial\Phi_2}{\partial t} + \eta + \frac{\varepsilon}{2}\left(\frac{\partial\Phi_2}{\partial x}\right)^2 - \frac{\mu}{2}H^2\frac{\partial^3\Phi_2}{\partial t\partial x^2} - \varepsilon\mu\left[H\eta\frac{\partial^3\Phi_2}{\partial t\partial x^2} + \frac{1}{2}H^2\frac{\partial\Phi_2}{\partial x}\frac{\partial^3\Phi_2}{\partial x^3} - \frac{1}{2}H^2\left(\frac{\partial^2\Phi_2}{\partial x^2}\right)^2\right]\right.\\ &\left. + \frac{\mu^2}{4!}H^4\frac{\partial^5\Phi_2}{\partial t\partial x^4}\right\}.\end{aligned} \tag{14}$$

Differentiating Eq.(14) with respect to x and introducing the horizontal velocities $\frac{\partial\Phi_1}{\partial x} = u_1$, $\frac{\partial\Phi_2}{\partial x} = u_2$ at the height $y = 1$, $y = -H$, we can rewrite Eqs.(12), (13) and (14) as

$$\frac{\partial\eta}{\partial t} - \frac{\partial u_1}{\partial x} + \varepsilon[\eta u_1]_x + \frac{\mu}{3!}\frac{\partial^3 u_1}{\partial x^3} - \frac{\varepsilon\mu}{2}\left[\eta\frac{\partial^2 u_1}{\partial x^2}\right]_x - \frac{\mu^2}{5!}\frac{\partial^5 u_1}{\partial x^5} = 0, \tag{15}$$

$$\frac{\partial\eta}{\partial t} + H\frac{\partial u_2}{\partial x} + \varepsilon[\eta u_2]_x - \frac{\mu}{6}H^3\frac{\partial^3 u_2}{\partial x^3} - \frac{\varepsilon\mu}{2}H^2\left[\eta\frac{\partial^2 u_2}{\partial x^2}\right]_x + \frac{\mu^2}{5!}H^5\frac{\partial^5 u_2}{\partial x^5} = 0, \tag{16}$$

$$\begin{aligned}&\frac{\partial u_1}{\partial t} + \frac{\partial\eta}{\partial x} + \frac{\varepsilon}{2}(u_1^2)_x - \frac{\mu}{2}\frac{\partial^3 u_1}{\partial t\partial x^2} + \varepsilon\mu\left[\left(\eta\frac{\partial^2 u_1}{\partial t\partial x}\right)_x - \frac{1}{2}\left(u_1\frac{\partial^2 u_1}{\partial x^2}\right)_x + \frac{1}{2}\left(\frac{\partial u_1}{\partial x}\right)_x^2\right] + \frac{\mu^2}{4!}\frac{\partial^5 u_1}{\partial t\partial x^4}\\ =\ & R\left\{\frac{\partial u_2}{\partial t} + \frac{\partial\eta}{\partial x} + \frac{\varepsilon}{2}(u_2^2)_x - \frac{\mu}{2}H^2\frac{\partial^3 u_2}{\partial t\partial x^2} - \varepsilon\mu\left[H\left(\eta\frac{\partial^2 u_2}{\partial t\partial x}\right)_x + \frac{1}{2}H^2\left(u_2\frac{\partial^2 u_2}{\partial x^2}\right)_x\right.\right.\\ &\left.\left. - \frac{1}{2}H^2\left(\frac{\partial u_2}{\partial x}\right)_x^2\right] + \frac{\mu^2}{4!}H^4\frac{\partial^5 u_2}{\partial t\partial x^4}\right\}.\end{aligned} \tag{17}$$

From Eqs.(15) and (16), we can obtain the relationship between velocities u_1 and u_2

$$\begin{aligned}u_1 =\ & -Hu_2 - \varepsilon(1+H)\eta u_2 + \frac{\mu}{6}(H^3 - H)\frac{\partial^2 u_2}{\partial x^2} - \varepsilon^2(1+H)\eta^2 u_2\\ &+ \frac{\varepsilon\mu}{6}\left[(H^3 + 3H^2 + 2H)\eta\frac{\partial^2 u_2}{\partial x^2} - (1+H)(\eta u_2)_{xx}\right] - \frac{\mu^2}{360}(3H^5 - 10H^3 + 7H)\frac{\partial^4 u_2}{\partial x^4} + C,\end{aligned}$$

here C is integration constant.

By using the above expression, we have a set of equations for velocity u_2 and interfacial elevation η:

$$\frac{\partial\eta}{\partial t} + H\frac{\partial u_2}{\partial x} + \varepsilon[\eta u_2]_x - \frac{\mu}{6}H^3\frac{\partial^3 u_2}{\partial x^3} - \frac{\varepsilon\mu}{2}H^2\left[\eta\frac{\partial^2 u_2}{\partial x^2}\right]_x + \frac{\mu^2}{5!}H^5\frac{\partial^5 u_2}{\partial x^5} = 0, \tag{18}$$

$$\begin{aligned}&(R+H)\frac{\partial u_2}{\partial t} + (R-1)\frac{\partial\eta}{\partial x} + \varepsilon\left[(1+H)(\eta u_2)_t + \frac{1}{2}(R - H^2)(u_2^2)_x\right] - \frac{\mu}{6}(H^3 + 3RH^2 + 2H)\frac{\partial^3 u_2}{\partial t\partial x^2}\\ &+\varepsilon^2[(1+H)(\eta^2 u_2)_t - (H^2+H)(\eta u_2^2)_x] - \varepsilon\mu\left[H(R-1)\left(\eta\frac{\partial^2 u_2}{\partial t\partial x}\right)_x - \frac{H^4 - 3RH^2 + 2H^2}{6}\left(u_2\frac{\partial^2 u_2}{\partial x^2}\right)_x\right.\end{aligned}$$

$$-\frac{H^2(R-1)}{2}\left(\frac{\partial u_2}{\partial x}\right)_x^2+\frac{H^3+3H^2+2H}{6}\left(\eta\frac{\partial^2 u_2}{\partial x^2}\right)_t+\frac{1+H}{3}(\eta u_2)_{xxt}\Bigg]$$
$$+\frac{\mu^2}{360}(3H^5+15RH^4+20H^3-8H)\frac{\partial^5 u_2}{\partial t\partial x^4}=0. \tag{19}$$

As in the derivation of the KdV-5 equation for surface waves in Dullin *et al*,[24] we now restrict our discussion to unidirectional waves by assuming a relationship

$$u_2=\sqrt{\frac{R-1}{H(R+H)}}\,\eta+\varepsilon\,A(\eta)+\mu\,B(\eta)+\varepsilon^2C(\eta)+\varepsilon\mu\,D(\eta)+\mu^2E(\eta). \tag{20}$$

Upon ordering in powers of the small parameters, we can determine the coefficients in the transformation (20) by requiring that the both boundary conditions of Eqs.(18) and (19) are satisfied simultaneously. We find

$$A=-\frac{3H^2+4H+R}{4H^2(R+H)}\sqrt{\frac{H(R-1)}{R+H}}\eta^2, \tag{21}$$
$$B=\frac{H^2+2RH+1}{6(R+H)}\sqrt{\frac{H(R-1)}{R+H}}\eta_{xx}, \tag{22}$$
$$C=\frac{-H^4+6H^3-8RH^2+2RH+R^2+8H^2-8RH^3}{8H^3(R+H)^2}\sqrt{\frac{H(R-1)}{R+H}}\eta^3, \tag{23}$$
$$D=\left[\frac{-12H^4-24RH^2-29RH^3-16H^3-13H^2+9R^2H+17R+4RH}{48H(1+RH)^2}\eta_x^2\right.$$
$$\left.+\frac{-RH^2-10RH^3-3H^4+6R^2H-5H^2+5R-2H+2RH}{12H(1+RH)^2}\eta\eta_{xx}\right]\sqrt{\frac{H(1-R)}{1+RH}}, \tag{24}$$
$$E=\frac{7H^5+28RH^4+10H^3+36R^2H^3+40RH^2+19H+4R}{360(R+H)^2}\sqrt{\frac{H(R-1)}{R+H}}\eta_{xxxx}. \tag{25}$$

So we have a single decoupled equation for the interfacial elevation η:

$$\frac{\partial\eta}{\partial t}+c_0\frac{\partial\eta}{\partial x}+\varepsilon\,c_1\eta\,\eta_x+\mu\,c_2\eta_{xxx}+\varepsilon^2c_3(\eta^3)_x+\varepsilon\mu\,[c_4(\eta_x^2)_x+c_5(\eta\eta_{xx})_x]+\mu^2c_6\eta_{xxxxx}=0, \tag{26}$$

where

$$c_0=\sqrt{\frac{H(R-1)}{R+H}},\quad c_1=\frac{3(R-H^2)}{2H(R+H)}c_0,$$
$$c_2=\frac{H(1+RH)}{6(R+H)}c_0,$$
$$c_3=\left[\frac{7}{18}\frac{c_1^2}{c_0}-\frac{R+H^3}{H^2(R+H)}\,c_0\right],$$
$$c_4=\left[\frac{17}{12}\,\frac{c_1c_2}{c_0}-\frac{RH+3H+4}{12(R+H)}\,c_0\right],$$
$$c_5=\left[\frac{7}{3}\,\frac{c_1c_2}{c_0}-\frac{H(R-1)}{6(R+H)}\,c_0\right],$$
$$c_6=\left[\frac{3}{2}\frac{c_2^2}{c_0}+\frac{H(1+RH^3)}{90(R+H)}\,c_0\right]. \tag{27}$$

For the extreme case $\rho_1=0$, Eq.(26) reduces to the equation derived by Dullin *et al*[24] for surface waves when the surface tension is neglected. That is, the equations derived from the present work include, as special cases, those obtained by Dullin *et al*[24] for surface waves.

The derived equation (26) is a kind of extended KdV equation for interfacial elevation in two-fluid system. Compared with Choi and Camassa's work,[28] the present paper adopted an alternative method: we expanded the velocity potentials at the upper and lower boundaries respectively, while Choi and Camassa[28] used a depth-averaged velocity, but the derived equation (26) has the same form as the equation (A9) with (A10) presented by Choi and Camassa.[28]

4. The asymptotic solitons of the interfacial waves

Let the weakly nonlinear and weakly dispersive terms be balanced in the extended KdV equation, that is, $\varepsilon = O(\mu)$, and we introduce transformation, still using variable x and t for convenience, $x = x - c_0 t, \quad t = \varepsilon t$.

Substituting the above transformation into Eq.(26), we have

$$\begin{aligned}&\eta_t + c_1\eta\,\eta_x + c_2\eta_{xxx} + \varepsilon\,[3c_3\eta^2\eta_x\\&+(2c_4+c_5)\eta_x\eta_{xx} + c_5\eta\eta_{xxx} + c_6\eta_{xxxxx}]\\&= 0.\end{aligned} \tag{28}$$

Assuming the following formula in brief

$$\begin{aligned}&\eta_t + c_1\eta\,\eta_x + c_2\eta_{xxx} + \varepsilon\,[C_1\eta^2\eta_x\\&+ C_2\eta_x\eta_{xx} + C_3\eta\eta_{xxx} + C_4\eta_{xxxxx}] = 0,\end{aligned} \tag{29}$$

where $C_1 = 3c_3,\ C_2 = 2c_4 + c_5,\ C_3 = c_5,\ C_4 = c_6$.

It is obvious that Eq.(29) has the same form as the extended KdV equation presented by Marchant[26] for surface waves. Therefore, we can use the method of Marchant[26] to obtain the asymptotic solitary wave solutions for Eq.(29). In order to do so, we introduce the transformation as following

$$\begin{aligned}\eta =\ & u + \varepsilon\left(\frac{C_3}{6} - \frac{C_1}{6} + \frac{2C_4}{3}\right)u^2\\&+\varepsilon\left(\frac{C_2}{12} - \frac{C_4}{2} - \frac{C_1}{12}\right)u_{xx},\end{aligned} \tag{30}$$

$$\tau = t + \varepsilon\frac{C_4}{3}x, \quad \xi = x + \varepsilon\,C_5\int_{-\infty}^{x} u(p,t)\mathrm{d}p,$$

where $C_5 = (8C_4 - C_3)/3$ and $u(x,t) \to 0$, as $x \to \pm\infty$.

Then $u(\xi,\tau)$ is a solution of the KdV equation

$$u_\tau + c_1 u\,u_\xi + c_2 u_{\xi\xi\xi} = 0. \tag{31}$$

The soliton solution of the KdV equation (31) is

$$u = A^*\mathrm{sech}^2\theta. \tag{32}$$

where $\theta = k^*(\xi - V^*\tau - s)$, A^* is the amplitude, k^* and A^* are related by $k^{*2} = \dfrac{c_1}{12c_2}A^*$, the velocity $V^* = \dfrac{c_1}{3}A^*$, and s gives the position of the solitons at $t = 0$. Using the KdV soliton solution (32) in the transformation, from (30) we get

$$\begin{aligned}\eta =\ & A^*\mathrm{sech}^2\theta + \varepsilon A^{*2}\left(\frac{C_2}{6} + \frac{C_3}{6} - \frac{C_1}{3} - \frac{C_4}{3}\right)\mathrm{sech}^2\theta\\&+\varepsilon A^{*2}\left(\frac{3C_4}{2} + \frac{C_1}{4} - \frac{C_2}{4}\right)\mathrm{sech}^4\theta + \cdots,\end{aligned} \tag{33}$$

where

$$\begin{aligned}\theta =\ & k^*\left[\left(1 - \varepsilon\frac{C_4}{3}A^*\right)x\right.\\&\left.+\varepsilon 2k^*C_5(\tanh\theta + 1) - s - V^*t\right].\end{aligned}$$

The phase θ is defined implicitly. It can be made explicit by expanding it in a Taylor series.

Expanding the phase and scaling the amplitude by using

$$A = A^*\left(1 - \varepsilon\frac{4C_4}{3}A^*\right), \tag{34}$$

we can rewrite the higher-order solitary wave (33) as

$$\eta = A\mathrm{sech}^2\theta + \varepsilon A^2 C_6\mathrm{sech}^2\theta + \varepsilon A^2 C_7\mathrm{sech}^4\theta + \cdots, \tag{35}$$

where

$$C_6 = \frac{2C_3}{3} - \frac{C_1}{6} + \frac{C_2}{6} - 5C_4,$$

$$C_7 = \frac{15C_4}{2} - \frac{C_3}{2} + \frac{C_1}{12} - \frac{C_2}{4},$$

$$\theta = k[x - s(1 + \varepsilon\frac{2C_4}{3}A) + 2\varepsilon C_5 k - Vt],$$

$$V = \frac{c_1}{3}A + 4\varepsilon C_4 A^2 + \cdots, \quad k^2 = \frac{c_1}{12c_2}A.$$

5. Conclusions

In this paper, we have derived the wave elevation equation for the interfacial waves in a two-fluid system with rigid upper boundary and flat bottom boundary. The elevation of the interfacial wave is assumed to be small compared with the thickness of the upper layer, and the thickness of the upper layer is small compared with the wavelength. The new equation is an extended KdV equation when we consider up to $O(\mu^2)$, and has the same form as the equation presented by Choi and Camassa.[28] It is noted that the equations derived from the present work include, as special cases, those obtained by Dullin *et al*[24] for surface waves when the surface tension is neglected. Obviously the method we adopted here is effective in solving such problems as interfacial waves. When the weakly nonlinear and weakly dispersive terms are in balance in the extended KdV equation, we obtained the asymptotic solitary wave solutions of this equation.

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