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New algorithms for constrained dynamics based on Faddeev–Jackiw approach*

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In this paper, the Faddeev–Jackiw approach is improved by the Wu elimination method, so a great many complicated computations in solving constraints for the finite-dimensional polynomial-type constrained dynamics can be executed easily by using computers. Moreover, based on the Faddeev–Jackiw approach, a new algorithm of solving the constrained dynamics is presented. The new algorithm is simpler and stricter than the Faddeev–Jackiw approach. Using the new algorithm, the second Cawley counterexample is solved.

Keywords: constrained, characteristic chain, symplectic matrix
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1. Introduction

In the dynamical system with a finite number of degrees of freedom, the action equation can be described with generalized coordinates $q = (q_1, \ldots, q_n)$, generalized speeds $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_n)$ and time t as follows:

$$A = \int_{t_2}^{t_1} L\left(t, q, \dot{q}\right) \mathrm{d}t, \tag{1}$$

where $L\left(t, q, \dot{q}\right)$ is the Lagrangian of the system. If the Hessian matrix

$$[w_{ij}] = \left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right]. \tag{2}$$

is nonsingular, then the system is regular, otherwise it is degenerate. For the degenerate case, the system can contain constrains, and it is called constrained dynamics. The constrained dynamics is one of cornerstones of modern physics. A lot of important physical systems belong to constrained dynamics, for instance, relativistic theory, modern gauge theories, Einstein's theory of the gravitational field, the theory of the neutral vector meson field, and supersymmetry. Hence many methods of treating the constrained dynamics have been developed.[1−7] The Faddeev–Jackiw approach[8] is one of treatments of constrained dynamics. In constrained dynamics, the Dirac bracket and quantization play a very important role. In the Faddeev–Jackiw approach, the Dirac bracket can be obtained in a simple way, and the quantization is also easy, so it has been extensively studied,[5,6,9−17] and developed[10,18] and improved.[16]

In the 1980s, based on one-form Lagrangian and geometric structures, the Faddeev–Jackiw approach is presented, and it is also called symplectic formalism. Dirac bracket and the quantization are obtained easily from the Faddeev–Jackiw approach, but there are some weaknesses in the process of solving constraints. In the Faddeev–Jackiw approach, Lagrangian is transformed into one-form through introducing the auxiliary variables. By one-form Lagrangian, the Euler-Lagrange equations of motion are obtained. Euler-Lagrange equations are linear equation systems in unknown $\dot{\xi}$, and the coefficient matrices are symplectic matrices, where ξ is symplectic variable, and $\dot{\xi}$ is one-degree derivative of time. Constraints are obtained by solving the Euler-Lagrange equations. Although Euler-Lagrange equations are linear equations, the entries of symplectic matrix equation are always functions, and with the constraints appearing, the scales of Euler-Lagrange equations are larger and larger, so it is difficult to solve them. The difficulty is mentioned in Ref.[8] also, and Faddeev and Jackiw recommended applying the Dirac method.[3,4,6] In order to obtain all of the constraints, the multipliers are introduced when the constraints appear, and the new constraints can appear after the multipliers have been introduced, and the steps continue until no new

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constraints appear. However, to our knowledge, it is not strictly proved that the Faddeev–Jackiw approach can terminate in finite steps, and only the explanation is given that the Faddeev–Jackiw approach can terminate in finite steps because the constrains are finite.[9,10] In addition, some constraints can appear as the auxiliary variables are introduced. For example, primary constraints occur as the auxiliary variables, which are momenta, are introduced. But in the Faddeev–Jackiw approach, these constrains are not included. Thus applying the Faddeev–Jackiw approach to solving some constrained dynamic systems, for instance Cawley counterexamples,[18,19] can yield incorrect results.

With the development of computer, a great many complicated computations can be implemented in computers, and hence the constrained dynamics can be studied with the help of computers. In constrained dynamics, in order to obtain all the constraints, it is necessary to solve the Euler-Lagrange equations, which are linear. But in general, the entries of coefficient matrices of Euler-lagrange equation are functions, so it is very difficult to solve constraints. In Ref.[20], Groebner base is applied to solving the constrains of constrained Hamiltonian systems in Dirac method. The combination of the involution with the Dirac method[21] and with the Faddeev–Jackiw approach[11] is used to study the constrained dynamics. In Ref.[22], the Wu elimination method[23,24] was applied to solving the constrains in a constrained Lagrange system and constrained Hamilton system. The above methods are all implemented by computers.

In this paper, firstly, for the polynomial-type dynamics, the Wu elimination method is applied to solving constrains from Euler-Lagrange equations in the Faddeev–Jackiw approach. In the process of solving constrains, we do not need to calculate the ranks and inverses of the coefficient matrices, so the method is simple. On the symbolic computation platform, the Wu elimination method can be executed in computers, so a great many complicated calculations of Faddeev–Jackiw approach can be implemented by computers. Our algorithm is easier to execute than the involution method, because the algorithm deals with the equations as algebraic equations, which is not as differential equations involution[11] does. In addition, according to the Wu elimination, the result obtained is in the trapezium form, and the constrained dynamics is very easy to study further.

Although it is improved by the Wu elimination or involution, the Faddeev–Jackiw approach is still very complicated. With the new constraints appearing, new multipliers are introduced, and the one-form Lagrangian and the Euler–Lagrange equations are computed once more. According to the theory of linear equation system and by referring to the constrained Lagrange system, a new algorithm is presented based on one-form Lagrangian. On the one hand, the new algorithm is simpler than Faddeev–Jackiw approach, because multipliers do not need introducing and calculating the one-form Lagrangian and Euler-Lagrange equations once more with new constraints appearing. On the other hand, the process of deduction accords with the theory of linear equation systems, and the constraints which appear as the auxiliary variables are introduced are included, so the new algorithm is stricter than the Faddeev–Jackiw approach. Furthermore, it is proved that the new algorithm can terminate in finite steps. For the polynomial type, the Wu elimination method can be applied to solving constraints. Using the new algorithm, the second Cawley's counterexample[19] is solved.

2. Wu elimination method

Let k be a field of characteristic zero and $k[u_1, \ldots, u_s, x_1, \ldots, x_n]$ a polynomial ring on k. The variables of the polynomial ring are divided into two groups:

$$u_1, \ldots, u_s; \quad x_1, \ldots, x_n;$$

where $u_1, \ldots, u_s$ are parametric variables and $x_1, \ldots, x_n$ are main variables. Let p be a polynomial. In $p(u_1, \ldots, u_s, x_1, \ldots, x_n)$, the variable x_p with the largest subscript is called the principle variable of p, and the subscript of principle variable is called the class of p, denoted by $CLS(p)$. The class of p is zero if no x_i $(i = 1, \ldots, n)$ appears in p, and then p is called trivial. The largest power of an variable x_l in p is denoted by $\deg_{x_l} p = d_l$. Letting $CLS(p) = l$ and $\deg_{x_l} = d$, then (l, d) is called the rank of p. The coefficient of the largest power of principal variable is called initial, denoted by $I(p)$. Every polynomial can be written in descending powers of the principal variable.

Definition 1 Suppose that p and q are two nonzero polynomials. If either

(1) $CLS(p) < CLS(q)$, or

(2) $CLS(p) = CLS(q) = l > 0$, and $\deg_{x_l} p < \deg_{x_l} q$,

then p is said to be lower than p, i.e. $p < q$; if p and q have the same ranks, p is said to be equivalent to q, i.e. $p \sim q$.

Definition 2 Suppose that p and q are two nontrivial polynomials with $CLS\,(p) = l > 0$. q is said to be reduced with respect to p, if $\deg_{x_l} q < \deg_{x_l} p$.

Theorem 1[23,24] Suppose that p and q are two polynomials with $CLS\,(p) = l > 0$,

$$p = c_m x_l^m + c_{m-1} x_l^{m-1} + \cdots + c_1 x_l + c_0,$$

$$q = d_n x_l^n + d_{n-1} x_l^{n-1} + \cdots + d_1 x_l + d_0,$$

then there will exist $r, g \in k[u_1, \ldots, u_s, x_1, \ldots, x_n]$ satisfying $\deg_{x_l} r < m$, so that

$$I\,(p)^s \cdot q = g \cdot p + r. \tag{3}$$

Equation (3) is the polynomial division of q with respect to p with remainder. The r_1 is a remainder, denoted by $\mathrm{Re}(q, p)$. Suppose that $P = \{p_1, \ldots, p_k\}$ is a polynomial set and q is a nonzero polynomial and then let

$$\begin{aligned} r_k &= \mathrm{Re}\,(q, p_k)\,, \quad r_{k-1} = \mathrm{Re}\,(r_k, p_{k-1})\,, \ldots, \\ r_1 &= \mathrm{Re}\,(r_2, p_1)\,. \end{aligned} \tag{4}$$

Expressions (4) are the polynomial divisions of q with respect to P with remainder. r is a remainder and denoted by $\mathrm{Re}(q, P)$. For two polynomial sets P and Q, they are denoted as

$$\mathrm{Re}\,(Q, P) = \{\mathrm{Re}\,(q, P)\,|\mathrm{Re}\,(q, P) \neq 0, q \in Q\}. \tag{5}$$

Definition 3 Suppose that $p_1, p_2, \ldots, p_s$ are polynomials. If

$$p_1 < p_2 < \cdots < p_s, \tag{6}$$

then $p_1, p_2, \ldots, p_s$ is a ascending chain. A nonzero constant in k constructs a special ascending chain and is called contradiction ascending chain. Expressions (4) form a reduced ascending chain if p_i reduces with respect to p_j for all $j < i$.

P and Q are ascending reduced chains,

$$P : p_1, \ldots, p_k; \qquad Q : q_1, \ldots, q_l.$$

If there exists either $j \leq \min\{k, l\}$ so that

$$p_1 \sim q_1, \ldots, p_{j-1} \sim q_{j-1}, p_j < q_j,$$

or $l < k$, and

$$p_1 \sim q_1, \ldots, p_l \sim q_l,$$

then P is smaller than Q, i.e. $P < Q$; P is said to be equivalent to Q, i.e. $P \sim Q$, if $l = k$ and $p_j \sim q_j$ for all $j \leq l$.

Theorem 2[23,24] Let P be a non-empty polynomial set. Then P must have the lowest reduced ascending chain according to the order above. Such a lowest reduced ascending chain is called the basic chain of P. A reduced ascending chain $G \subset P$ is the basic chain of P iff $P \setminus G$ does not contain nonzero polynomial reduced with respect to G.

By theorem 2, there exists an algorithm of finding a basic chain for a finite polynomial set. Our description for the algorithm is presented as a procedure called Algorithm 1.

Algorithm 1

Input Polynomial set Q;

Output Basic chain Bs of Q;

Step 1 Let $Bs = \emptyset$. Find out the lowest rank polynomial f; if $CLS\,(f) = 0$, f is the basic chain, and $Bs = Bs \cup f$, so output Bs; otherwise, jump to Step 2;

Step 2 $Q = Q \backslash Bs$, if $Q = \emptyset$ or the polynomials in Q are all non-reduced with Bs, output Bs; otherwise, in Q, find out the lowest polynomial f which is reduced with Bs; $Bs = Bs \cup f$, and return Step 2.

Remark This algorithm can terminate in finite steps because the set Q is finite.

Definition 4 For a finite polynomial set P, the polynomial set C is the characteristic chain of P, if C satisfies the following conditions:

(1) C is a noncontradiction ascending chain;

(2) $\mathrm{Zero}(P) \subset \mathrm{Zero}(C)$;

(3) $\mathrm{Re}(P, C) = \emptyset$.

Here the $Zero(P) = \{v \in k^n \mid p(v) = 0, \forall p \in P\}$.

The algorithm of computing the characteristic chain is presented as follows:[23,24]

Algorithm 2

Input Polynomial set PS;

Output Characteristic chain As of Ps

$$\begin{array}{rcccc} PS = & PS_1 & PS_2 & \cdots & PS_k \\ & BS_1 & BS_2 & \cdots & BS_k = AS \\ & RS_1 & RS_2 & \cdots & RS_k = \emptyset. \end{array}$$

BS_i is the basic chain of PS_i. $RS_i = \mathrm{Re}\,(PS_i \setminus BS_i, BS_i)$. $PS_i = PS_{i-1} \cup BS_{i-1}$. If an BS_i is trivial, stop the loop and output BS.

Theorem 3 (Zero Decomposition Theorem[23,24] Suppose that $P = \{p_1, \ldots, p_n\}$ is a finite polynomial set, $F = \{f_1, \ldots, f_s\}$ is a characteristic chain of P,

and I_i is the initial of f_i, $i = 1, \ldots, s$, then there will exist

$$\mathrm{Zero}(P) = \mathrm{Zero}(F/J) \cup \bigcup_{i=1}^{s} \mathrm{Zero}\left(P \cup I_i\right), \tag{7}$$

where J is the product of I_i; for $P \cup I_i$, its characteristic chain can be obtained through Algorithm 2 too, and the procedure is continuously carried on, the finally result is obtained as

$$\mathrm{Zero}(P) = \bigcup_{j=1} \mathrm{Zero}\left(F_j/J_j\right), \tag{8}$$

where J_j is the product of all initials of F_j .

3. Faddeev–Jackiw approach

The Faddeev–Jackiw approach is of an iterative algorithm, and the superscript (k) refers to the iteration number. In constrained dynamics, the Lagrangian is a function of generalized coordinates $q = (q_1(t), \ldots, q_n(t))$, generalized speeds $\dot{q} = (\dot{q}_1(t), \ldots, \dot{q}_n(t))$ and the time t. Hereafter, the repeated subscripts denote the sum without further declaration later. After introducing the auxiliary variables p, the symplectic variables are $\xi^{(0)} = (q, p)$, and the Lagrangian can be transformed into one-form as follows:

$$L^{(0)}\mathrm{d}t = a^{(0)}\left(\xi^{(0)}\right)\dot{\xi}^{(0)} - V^{(0)}\left(\xi^{(0)}\right)\mathrm{d}t, \tag{9}$$

where $V^{(0)}\left(\xi^{(0)}\right)\mathrm{d}t$ is symplectic potential. The exterior derivative of the one-form is of symplectic two-form, i.e.

$$f^{(0)} = \mathrm{d}a^{(0)} = \frac{1}{2} f_{ij}^{(0)} \mathrm{d}\xi_i^{(0)} \wedge \mathrm{d}\xi_j^{(0)}, \tag{10}$$

where $f_{ij}^{(0)}$ is symplectic matrix

$$f_{ij}^{(0)} = \frac{\partial a_j^{(0)}}{\partial \xi_i^{(0)}} - \frac{\partial a_i^{(0)}}{\partial \xi_j^{(0)}}. \tag{11}$$

Using the variation principle, the Euler-Lagrange equations of motion are

$$f_{ij}^{(0)}\dot{\xi}_j^{(0)} = \frac{\partial V^{(0)}\left(\xi^{(0)}\right)}{\partial \xi_i^{(0)}}. \tag{12}$$

The solutions of Eqs.(12) are determined by the coefficient matrix $f_{ij}^{(0)}$. If $f_{ij}^{(0)}$ is singular, the constraints can appear. In this paper, only the case with constraints is considered. Supposing that the rank of matrix $\left(f_{ij}^{(0)}\right)$ is $n - m$, then there exist annihilator vectors $g_i, i = 1, \ldots, m$, such that $g_i\left(f_{ij}\right) = 0$. Multiplying the both sides of Eqs.(12) by g_i, we obtain the constraints

$$g_i^{(0)} \frac{\partial V^{(0)}\left(\xi^{(0)}\right)}{\partial \xi_i^{(0)}} = \Omega_i^{(0)} = 0, \quad i = 1, \ldots, m. \tag{13}$$

On the one hand, constraints (13) must satisfy the consistent conditions, i.e.

$$\dot{\Omega}_i^{(0)} = \frac{\partial \Omega_i^{(0)}}{\partial \xi_j^{(0)}} \dot{\xi}_j^{(0)} = 0, \tag{14}$$

on the other hand, according to the constraints (13), the Lagrange multipliers $\eta^{(0)} = \left(\eta_1^{(0)}, \ldots, \eta_m^{(0)}\right)$ are introduced. Using conditions (14) and the multipliers $\eta^{(0)}$, one can obtain the once-iterated one-form Lagrangian

$$L^{(1)} = a_i^{(1)}\left(\xi^{(1)}\right)\dot{\xi}_i^{(1)} - V^{1}\left(\xi^{(1)}\right), \tag{15}$$

where $\xi^{(1)} = \left(\xi^{(0)}, \eta^{(0)}\right)$, $a_i^{(1)} = a_i^{(0)} + \eta_j^{(0)} \dfrac{\partial \Omega_j^{(0)}}{\partial \xi_i^{(0)}}$, and $V^{(1)}\left(\xi^{(1)}\right) = V^{(1)}\left(\xi^{(1)}\right)|_{\Omega_j^{(0)}=0}$. From the once-iterated one-form Lagrangian, the corresponding once-iterated symplectic two-form is obtained as

$$f^{(1)} = \mathrm{d}a^{(1)} = \frac{1}{2} f_{ij}^{(1)} \mathrm{d}\xi_i^{(1)} \wedge \mathrm{d}\xi_j^{(1)}, \tag{16}$$

with

$$f_{ij}^{(1)} = \frac{\partial a_j^{(1)}}{\partial \xi_i^{(1)}} - \frac{\partial a_i^{(0)}}{\partial \xi_j^{(0)}}. \tag{17}$$

Then the once-iterated Euler-Lagrange equations of motion are

$$f_{ij}^{(1)}\dot{\xi}_j^{(1)} = \frac{\partial V^{(1)}\left(\xi^{(1)}\right)}{\partial \xi_i^{(1)}}. \tag{18}$$

For Eqs.(18), one repeats the above procedure until no new constraints occur. After all of the constraints are obtained, one can study the property of the dynamics, calculate the Dirac bracket, or perform quantization.

4. Faddeev–Jackiw approach based on Wu eliminate method

As indicated in the previous section, solving the Euler-Lagrange equations is the core of the Faddeev–Jackiw approach. The constraints occur whether they depend on the solution of the Euler-Lagrange equation or not. Although the Euler-Lagrange equations are a linear equation system, the entries of coefficient matrices are functions in general, so it is very difficult

to solve them. In addition, with the order of the coefficient matrix increasing, the equation system becomes much more complicated, because the constraints occur and the multipliers are induced.

In this section, for the polynomial-type dynamics, the Wu elimination method is applied to solving the Euler-Lagrange equations. Actually, in the Faddeev–Jackiw approach, symplectic variables and their one-degree derivative with respect to time t are regarded as independent variables, so Euler-Lagrange equations are algebraic equations, i.e. polynomials. The Wu elimination method is a very good method of treating polynomials. In the following, the Faddeev–Jackiw approach based on the Wu eliminate Method is presented as algorithm formalism. Assume that auxiliary variables have been introduced and Lagrangian has been transformed into one-form.

Algorithm 3

Input Lagrangian $L^{(0)}$

Output Motion equations Ms, Constraints Ss

Step1 Set $L = L^{(0)}$, $\xi = \xi^{(0)}$, $V = V^{(0)}$, $Ss = \phi$;

Step2 Calculate the Euler-Lagrange equation Ms from L;

Step3 Give an order satisfying

$$\xi = \xi_1 < \cdots < \xi_n < \dot{\xi}_1 < \cdots < \dot{\xi}_n; \tag{19}$$

Step4 Calculate the characteristic chain Cs of Ms, and choose the constraints from Cs, denoted as Ds;

Step5 If Ds are empty, no new constraints occur, and jump to Step6; otherwise, according to the constraints introduce the multipliers $\xi_{n+1}, \ldots, \xi_{n+m}$, $\xi = \xi \cup \{\xi_{n+1}, \ldots, \xi_{n+m}\}$, $Ss = Ss \cup Ds$, calculate one-form Lagrangian L, and jump to Step2;

Step6 Output L, Ms and Ss.

Here we explain the correctness of algorithm 3. In Step2, Euler-Lagrange equations are obtained from one-form Lagrangian. In the Wu elimination method, the order should be given first, so in Step3 the order is given. But as new constraints appear and the multipliers are introduced, the symplectic variables are changed, so that the order must be given again. In the Faddeev–Jackiw approach, constraints are obtained by solving Euler-Lagrange equations. In Step4, according to the order in Step3, we calculate the characteristic chain of Euler-Lagrange equations. The constraints are obtained very easily from characteristic chain because the results are of trapezium form. If the new constraints appear, in Step5, multipliers are introduced, one-form Lagrangian is calculated once more, and return to Step2. Algorithm 3 can terminate in finite steps because the constraints are finite.[9,10] From the description above, in algorithm 3, except the procedure of solving Euler-Lagrange equations, other steps are the same as those of Faddeev–Jackiw approach. So algorithm 3 is correct. The Euler–Lagrange equations can be solved by using other methods, such as the Gauss elimination method. But the entries of coefficient matrices are functions in general, so it is very difficult to solve them. For the polynomial-type constrained dynamics, in algorithm 3, the constraints are obtained by the Wu elimination. The Wu elimination method is a very good method of dealing with polynomials. So algorithm 3 is a very valid method of treating polynomial-type constrained dynamics. In addition, it is easy to study further from the result of the algorithm 3, because the result is of trapezium form. The Wu elimination method can be executed in computers. So for polynomial-type dynamics, the Faddeev–Jackiw approach can be implemented with the help of computers. Algorithm 3 has been executed by Mathematica symbolic computation software, and using the program, some examples are solved. The following example is one of the examples.

Example 1 The Lagrangian of a non-relativistic point-particle moving on a surface is

$$L = \frac{1}{2}\left(\dot{q_1}^2 + \dot{q_2}^2 + \dot{q_3}^2\right) + \lambda\left(q_1^2 + q_2^2 + q_3^2 - 1\right). \tag{20}$$

Introducing auxiliary variables p, which are momentum here, the one-form Lagrangian is

$$L = p_1\dot{q}_1 + p_2\dot{q}_2 + p_3\dot{q}_3 - V_0, \tag{21}$$

where the symplectic potential is

$$V_0 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) - \lambda\left(q_1^2 + q_2^2 + q_3^2 - 1\right). \tag{22}$$

Using algorithm 3, the equations of motion are

$$\begin{aligned}
&q_1\left(-q_2^2 p_1 - q_3^2 + q_1 q_2 p_2 + q_1 q_3 p_3 + q^2\dot{q}_1\right) = 0,\\
&q_1\left(-q_1 q_2 p_1 + q_1^2 p_2 + q_3^2 p_2 - q_1 q_2 p_3 - q^2\dot{q}_2\right) = 0,\\
&q_1\left(-q_1 q_3 p_1 - q_2 q_3 p_2 + q_1^2 p_3 + q_2^2 p_3 - q^2\dot{q}_3\right) = 0,\\
&q_1\left(-q_2 p_1 p_2 + q_1 p_2^2 - q_3 p_1 p_3 + q_1 p_3^2 + q^2\dot{p}_1\right) = 0,\\
&q_1\left(q_2 p_1^2 - q_1 p_1 p_2 - q_3 p_2 p_3 + q_2 p_3^2 + q^2\dot{p}_2\right) = 0,\\
&q_1 q_2\left(q_3 p_1 + q_3 p_2^2 - q_1 p_1 p_3 - q_2 p_2 p_3 + q^2\dot{p}_3\right) = 0,\\
&q_1 q_2 q^2\left(p^2 + q^2\dot{\eta}\right) = 0,\\
&q_1\left(-q_1 p_1 - q_2 p_2 - q_3 p_3 + q^2\dot{\rho}\right) = 0,
\end{aligned}$$

where η and ρ are multipliers, and $p^2 = p_1^2 + p_2^2 + p_3^2$, $q^2 = q_1^2 + q_2^2 + q_3^2$.

The dimension of Lagrangian is 4 and it is so low that the equations can be solved simply by hand-calculation, but the process is still very complicated. When the dimension is high, it is so complicated that it cannot be solved by hand-calculation, but using algorithm 3, it is solved very easily with the help of computers.

5. A new algorithm based on Faddeev–Jackiw approach

Although the Faddeev–Jackiw approach is improved by the Wu elimination method or other methods, it is still very complicated to study constrained dynamics. On the one hand, with the new constrains appearing and the multipliers being introduced, the scales of the Euler-Lagrange equations become larger and larger, so that they are more and more difficult to be solved. On the other hand, once the new constrains appear, the one-form Lagrangian and the symplectic matrix are required to be calculated again, and thus the computation process is complicated. In addition, when the auxiliary variables are introduced, some constraints may appear, but they are not included in the Faddeev–Jackiw approach. For example, the primary constraints as the auxiliary variables are momenta.

After introducing auxiliary variables p, the auxiliary variables p and generalized coordinates q combine with the symplectic variables ξ. So the variables in the formulas including auxiliary variables are symplectic variables. Then the formulas are regarded as equations in unknown symplectic variables $\dot{\xi}$, and denoted by Ps. Ps are rewritten as linear equations with respect to $\dot{\xi}$, such as $\dot{\xi}^2 = \xi$ is rewritten as $\dot{\xi} = \sqrt{\xi}$. The rewriting is to use the theory of linear equation systems. As the auxiliary variables are introduced, the Euler-Lagrange equations of motion are

$$f_{ij}\dot{\xi}_j = \frac{\partial V^{(0)}(\xi)}{\partial \xi_i}, \tag{23}$$

which is a linear equation system in unknown $\dot{\xi}$, and denoted as Ls. Because Ls and Ps are all linear equations in unknown $\dot{\xi}$, we let $Ls = Ls \cup Ps$. Then the constrains, as the auxiliary variables are introduced, are included. Supposing that the rank of the coefficient of Ls is R and $R < n$, and using the primary transformation of rows, Ls becomes

$$\begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & \vdots & \vdots \\ g_{R1} & \cdots & g_{Rn} \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \vdots \\ \dot{\xi}_R \\ \dot{\xi}_{R+1} \\ \vdots \\ \dot{\xi}_n \end{pmatrix} = \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_R \\ \Omega_{R+1} \\ \vdots \\ \Omega_n \end{pmatrix}. \tag{24}$$

According to the theory of linear equation systems, Eq.(24) has a solution iff the coefficient matrix and the augmented matrix have the same ranks. The inhomogeneous entries for Eq.(24) are functions, so there are three possibilities for $\Omega_k, k = R+1, \ldots, n$. (1) $\Omega_k, k = R+1, \ldots, n$ are all zero, then Eq.(24) has an infinite solution; in Ref.[10], Eq.(24) becomes only one solution by the gauge fixing term introduced into symplectic potential; (2) nonzero constants appear in $\Omega_k, k = R+1, \ldots, n$, then Eq.(24) has no solution, so the system does not have a stationary-point; (3) $\Omega_k, k = R+1, \ldots, n$ are all not zero, i.e. there exist nonzero constants, so some are functions, and it might as well let $\Omega_k, k = R+1, \ldots, n$ be all functions. Here, only the third case is studied. Eq.(24) has solutions iff

$$0 = \Omega_k, \quad k = R+1, \ldots, n. \tag{25}$$

Equations (25) are new equations, and called constrains. Because Eqs.(25) are equations, the derivative of their both sides with respect to t are also equations,

$$0 = \dot{\Omega}_k = \frac{\partial \Omega}{\partial \xi_i}\dot{\xi}_i, \quad k = R+1, \ldots, n. \tag{26}$$

Equations (26) are consistent conditions and they are linear equations in unknown $\dot{\xi}$. Eqs.(23) and (26) constitute a new system of linear equations. Solving the new linear equation system, new constrains can be obtained, and then the consistent conditions can be worked out. The steps continue until no new constraints emerge. In addition, some constrains can appear when the auxiliary variables are introduced. Combing the above discussion yields the following algorithm.

Algorithm 4

Input: Lagrange function L and Ps

Output: Constrained Euler-Lagrange equations

Step1 Calculate the one-form Lagrangian and the Euler-Lagrange equations of motion, denoted by Ls;

Step2 $Ls = Ls \cup Ps$. Let $Cs = \{\}$;

Step3 Solve equations Ls, and choose the constraints and denoted by Ss;

Step4 If Ss are the subset of Cs, jump to Step5; otherwise let $Cs = Cs \cup Ss$; calculate the consistent conditions of Ss, and denoted by Ds; let $Ls = Ls \cup Ds$ and return to Step3;

Step5 Output Ls and Cs.

In algorithm 4, Constraints appear from the process of solving equations, and the consistent conditions are obtained from the law of motion and the same as those obtained from the Faddeev–Jackiw approach. Therefore algorithm 4 is correct. For a algorithm, besides the correctness, it is also very important that the algorithm can terminate in finite steps. Next we prove that the new algorithm can terminate in finite steps for the constraints are finite in the form of theorem.

Theorem 4 Algorithm 4 can terminate in finite steps.

Proof Let the initial domain of the one-form Lagrangian be taken as the whole $(2n+1)$-dimensional $(t, \xi, \dot{\xi})$-space M. If constrains (25) appear, then the variables $t, \xi, \dot{\xi}$ of the dynamical system satisfy Eqs.(25) and (26). So the Lagrangian is now defined only in the $(2n+1-R)$-dimensional surface S' which is embedded in M. If the new constrains appear, then the Lagrangian is now defined only in the $(2n+1-R-R')$-dimensional surface S'' which is embedded in S'. Continuously, the dimension of the surface which the Lagrangian is defined in is reduced with the new constrains appearing, but a minimum value of the dimension must be bigger than zero, so the dimension of embedded surface must be equal to a minimum value in finite steps. From the discussion above, it is obvious that the achievable minimum dimension of embedded surface is equivalent to that in the case without new constrains appearing. So this algorithm can terminate in finite steps.

The new algorithm is simpler than the Faddeev–Jackiw approach, because it need not calculate the one-form Lagrangian and symplectic matrix, and introduce the multipliers repeatedly with the constraints appearing. For the polynomial type, in step 3, the Wu elimination method is still used to solve the equations.

The problem in example 1 can be solved by Algorithm 4. Here, the second Cawley's counterexample is solved by Algorithm 4. The counterexample is presented to Dirac's conjecture. For the counterexample, the Dirac method and Faddeev–Jackiw approach lead to wrong outcomes. The example is not of a polynomial type, so the simple form can be calculated by hand-calculation.

Example 2 The Lagrange function is

$$L = \dot{x}\dot{z}^2 + yz. \tag{27}$$

The auxiliary variables are momenta p_x, p_y, p_z

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = \dot{z}^2, \quad p_y = \frac{\partial L}{\partial \dot{y}} = 0, \\ p_z &= \frac{\partial L}{\partial \dot{z}} = 2\dot{x}\dot{z}. \end{aligned} \tag{28}$$

Equations (28) are rewritten as

$$\dot{z} - p_x^{1/2} = 0, \quad p_y = 0, \quad 2p_x\dot{x} - p_z = 0. \tag{29}$$

The symplectic variables are $\xi = \{x, y, x, p_x, p_y, p_z\}$. The one-form lagrangian is

$$L = p_x\dot{x} + p_z\dot{z} - p_x^{1/2}p_z - yz. \tag{30}$$

The Euler-Lagrange equations of motion are

$$\begin{aligned} &\dot{x} - \frac{1}{2}p_x^{-1/2}p_z = 0, \quad \dot{z} - p_x^{1/2} = 0, \quad \dot{p}_x = 0, \\ &\dot{p}_z - y = 0, \quad z = 0. \end{aligned} \tag{31}$$

Equations (29) and (31) constitute new linear equations. The first equation in Eqs.(31) is the same as the third equation in Eqs.(29). Using Algorithm 4, the results are as follows:

$$\dot{x} - \frac{1}{2}p_x^{-1/2}p_z = 0, \quad \dot{y} = 0, \quad \dot{z} - p_x^{1/2} = 0; \tag{32}$$

$$\dot{p}_x = 0, \quad \dot{p}_y = 0, \quad \dot{p}_z = 0; \tag{33}$$

$$\begin{aligned} &y = 0, \quad z = 0, \quad p_x = 0, \\ &p_y = 0, \quad p_z = 0. \end{aligned} \tag{34}$$

Expressions (32) and (33) are equations of motion, and expressions (34) are constraints. A lot of mathematicians and physicians have studied the Cawley's counterexamples now. They try to obtain a general theory containing the Dirac's conjecture and the counterexamples, however at present, the correct result has not been obtained yet.

6. Conclusions

In this paper, firstly, for the polynomial-type constrained dynamics, the Faddeev–Jackiw approach is improved by the Wu elimination method. For the Faddeev–Jackiw approach, it is very complicated to treat the process of solving constraints. The burden of computation is so heavy that it is very difficult to do and prone to error by hand-calculation. But for the

improved algorithm, on the one hand, the complicated and a great many computations can be executed by computers, so the constrained dynamics can be studied with the help of computers. On the other hand, the result form of the improved algorithm is very good and redounds to further study. Based on the Faddeev–Jackiw approach, a new algorithm is presented. Besides its being simpler than the Faddeev–Jackiw approach in application, the new algorithm is stricter than the Faddeev–Jackiw approach in deduction. For the polynomial type, the Wu elimination can also be used to solve the constraints. So the new algorithm can be implemented by computers more easily than the improved algorithm.

The improved algorithm and the new algorithm can be executed by computers, but the complexities of time and space need further studying. And although the constraints have been solved, it is required to further study the quantization, gauge transformation and other qualities of dynamics that are computed by computers. The programs of the improved algorithm and the new algorithm are compiled with Mathematica symbolic software. Besides the above examples, other examples are studied by the programs, such as the $SU(2)$ Yang–Mills equation in 3+1 dimensional space, the $SU(2)$ Yang–Mills equation in 3+1 dimensional space, the $SU(2)$ Yang–Mills equation in 0+1 dimensional space and the system Lagrangian $L = q_1(\dot{q}_2 - q_3) - \dot{q}_1 q_2$, etc. In the Intel CPU 2.0G, the results are obtained in 5 seconds.

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