Vol 16 No 12, December 2007
1009-1963/2007/16(12)/3560-06
Chinese Physics
© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

The double Lagrangian with $U(1, J)$ symmetry and its applications to the cosmology*

Wu Ya-Bo(吴亚波)†, Lü Jian-Bo(吕剑波), He Jing(贺 敬), and Fu Ming-Hui(付明慧)

Department of Physics, Liaoning Normal University, Dalian 116029, China

(Received 4 December 2006; revised manuscript received 20 March 2007)

By introducing the double complex scalar field, this paper constructs the double Lagrangian with the potential $V(\Phi(J), \Phi^*(J)) = V(|\Phi(J)|)$, which not only can describe the evolution of quintom Universe, but also can naturally give the spintessence and hessence Universe. Furthermore, the $U(1, J)$ symmetry of the double complex Lagrangian is verified, and the total conserved charge within the physical volume is derived by means of the Noëther theorem. Moreover, it can point out that the 'imaginary motion' of the angular parameter in Ref.[14], in fact, is only a real phase displacement in our model.

Keywords: double Lagrangian density, canonical scalar field, dark energy
PACC: 0220, 0420, 1130L, 9880

1. Introduction

The type Ia supernova (SNe Ia) searches[1], the cosmic microwave background results from balloon and ground experiments[2] and recent Wilkinson Microwave Anisotropy Probe (WMAP)[3] observations all suggest that the Universe is speeding up rather than slowing down. In order to explain this fact, many kinds of models have already been constructed such as the cosmological constant Λ plus cold dark matter (ΛCDM) model,[4] quintessence[5], phantom[6], generalized Chaplygin gas,[7] k-essence,[8] and so forth.

On the other hand, by fitting the SNe Ia data, marginal evidence for the equation of state (EOS) $\omega_{\rm de}(z) < -1$ at $z < 0.2$ has recently been found[9]. Moreover, the present data seem to slightly favour an evolving dark energy with $\omega_{\rm de}$ being below -1 around present epoch from $\omega_{\rm de} > -1$ in the near past[10]. Hence a class of proposals such as quintom[10,11], the geometric approach[12], holographic dark energy[13], etc., in which the EOS or effective EOS crosses over the cosmological constant boundary (or called the phantom divide) has recently presented.

Furthermore, the advice of using a single ordinary complex (or called elliptical complex) non-canonical scalar field as dark energy has already been established, for example, hessence.[14] However, there are some problems we have to face: (1) how to understand the so-called internal 'imaginary motion' mentioned in Ref.[14]; (2) whether there is a method of avoiding using the non-canonical complex scalar field in Ref.[14]; (3) whether we can obtain the EOS for dark energy from a more fundamental theory such as field theory. We will answer these questions affirmatively by introducing a double complex canonical scalar field to describe the dark energy.

It is well known that using the ordinary (elliptical) complex scalar field as dark energy has been extensively applied to cosmology, such as Refs.[14–17], etc. However, hyperbolic complex field functions are seldom involved in this field although they have been already applied to other fields in physics.[18−20] Therefore, in this paper, we will extend the Lagrangian density constructed by an ordinary complex scalar field in Ref.[15] to the double complex form, i.e., construct a double Lagrangian, which can respectively describe the evolution of the quintessence/spintessence and quintom/hessence dark energy by introducing the ordinary and hyperbolic complex scalar fields in the following two different ways. Exactly, on the one hand, by utilizing the form of the double complex scalar field $\Phi(J) = \phi_1(J) + J\phi_2(J)$, we obtain the quintessence and quintom dark energy models and it is worth noting that our method of obtaining both of them is different from that of the previous literature.[5,10,11] On the other hand, we find that the double Lagrangian

*Project supported by the National Natural Science Foundation of China (Grant No 10475036) and the Scientific Research Foundation of the Higher Education Institute of Liaoning Province under (Grant No 05L215).
†E-mail: ybwu61@163.com or ybwu@lnnu.edu.cn

constructed by us has the interesting feature of $U(1,J)$ symmetry, where we express the form of the double complex scalar field as $\Phi(J)=\phi(J)\mathrm{e}^{J\theta}$ in order to discuss the symmetry easily. The spintessence and hessence Universe models can be obtained under this description of the double complex scalar field $\Phi(J)$. Next, we can definitely point out that the 'imaginary motion' of the angular parameter in Ref.[14], in fact, is only a real phase displacement in our model. Furthermore, according to the internal $U(1,J)$ symmetry of the double Lagrangian, we obtain the total conserved charge within the physical volume by means of the Noëther theorem. Finally, some discussions about the conserved charge and conserved current will be given, too.

For the sake of convenience, it is necessary to review the related knowledge to the double complex function method (DCFM)[18] in this paper. It is well known that, besides the ordinary (elliptical) complex numbers, of which the imaginary unit is i ($\mathrm{i}^2=-1$), there exist other generalized complex numbers, namely the hyperbolic complex numbers, of which the imaginary unit is ϵ, where $\epsilon^2=+1, \epsilon\neq 1$. Corresponding to the ordinary complex number C, hyperbolic complex numbers can be described as $a+\epsilon b$ (a, b are real). Next, let J denote the double imaginary unit, i.e., J may be the ordinary unit i, it also may be the hyperbolic imaginary unit ϵ. An element $Z=a+Jb$ is called a double complex number, the conjugate element of Z is $Z^*=a-Jb$. According to the common way, in the set DC of all double complex numbers we can introduce addition, multiplication, etc.. So $\mathrm{DC}=C$ when $J=\mathrm{i}$, and $\mathrm{DC}=H$ when $J=\epsilon$. The inverse of $a+\mathrm{i}b$ exists if $|a|+|b|\neq 0$, and the inverse of $a+\epsilon b$ exists if $|a|-|b|\neq 0$. Therefore, in the following we suppose that $|a|\neq|b|$ when an abstract double complex number $Z=a-Jb$ is taken as a divisor. The Z corresponds to a pair $(Z_{\mathrm{C}}, Z_{\mathrm{H}})$, where

$$\begin{aligned} Z_{\mathrm{C}} &= Z(J=\mathrm{i})=a+\mathrm{i}b,\\ Z_{\mathrm{H}} &= Z(J=\epsilon)=a+\epsilon b. \end{aligned} \tag{1}$$

Below, we will give the definition of a double complex function and consider the case in which the function has four independent real variables ($x^\mu=t,x,y,z$, here $\mu=0,1,2,3$ denotes spacetime index).

Definition:

$$\Phi=\Phi(J)=\Phi(x^\mu;J)=\phi_1(x^\mu;J)+J\phi_2(x^\mu;J)$$

is called a double complex function defined on a region $N\subset R^4$ if the following conditions are satisfied.

(a) $\Phi_{\mathrm{C}}=\phi_{1\mathrm{C}}+\mathrm{i}\phi_{2\mathrm{C}}$ and $\Phi_{\mathrm{H}}=\phi_{1\mathrm{H}}+\epsilon\phi_{2\mathrm{H}}$ are an ordinary complex function and a hyperbolic complex function, respectively, where

$$\begin{aligned} \phi_{1\mathrm{C}} &= \phi_1(J=\mathrm{i})=\phi_{1\mathrm{C}}(x^\mu),\\ \phi_{1\mathrm{H}} &= \phi_1(J=\epsilon)=\phi_{1\mathrm{H}}(x^\mu),\\ \phi_{2\mathrm{C}} &= \phi_2(J=\mathrm{i})=\phi_{2\mathrm{C}}(x^\mu),\\ \phi_{2\mathrm{H}} &= \phi_2(J=\epsilon)=\phi_{2\mathrm{H}}(x^\mu) \end{aligned} \tag{2}$$

are all real functions defined on N.

(b) J is an 'analysis link' between Φ_{C} and Φ_{H}. This means that if we take t, x, y and z as constants, and take J as a formal real independent variable, then $\phi_1(J)$ and $\phi_2(J)$ are real analytic functions of J, and they can be written as an infinite series:

$$\begin{aligned} \phi_1(J) &= \sum_{n=0}^{\infty} a_n J^{2n},\\ \phi_2(J) &= \sum_{n=0}^{\infty} b_n J^{2n}, \end{aligned} \tag{3}$$

where $a_n(x^\mu)$ and $b_n(x^\mu)$ are real functions, and the series $\sum a_n$, $\sum(-1)^n a_n$, $\sum b_n$, and $\sum(-1)^n b_n$ are all convergent on N.

Notice that the above condition (b) of analyticity links Φ_{C} with Φ_{H}. As a result of condition (b), $\Phi(J)$ is not a mechanical combination of Φ_{C} and Φ_{H} in which J only plays a role as a discrete index ($J=\mathrm{i},\epsilon$), there is certain analytic relation between the ordinary complex function $\Phi(x^\mu;\mathrm{i})$ and the hyperbolic complex function $\Phi(x^\mu;\epsilon)$.

In this paper, we shall use the double exponential function $\mathrm{e}^{\theta J}$. It is defined as follows:

$$\mathrm{e}^{\theta J}=\sum_{n=0}^{\infty}\frac{1}{n!}\theta^n J^n=C[\theta,J]+JS[\theta,J], \tag{4}$$

where $\theta\equiv\theta(x^\mu,J)$ is a real function, i.e., $\theta_{\mathrm{C}}=\theta(J=\mathrm{i})=\theta_{\mathrm{C}}(x^\mu)$, $\theta_{\mathrm{H}}=\theta(J=\epsilon)=\theta_{\mathrm{H}}(x^\mu)$, and $C[\theta,J]$, $S[\theta,J]$ have the following definitions:

$$\begin{aligned} C[\theta,J] &\equiv \begin{cases} \cos\theta_{\mathrm{c}} & J=\mathrm{i}\\ \cosh\theta_{\mathrm{H}} & J=\epsilon, \end{cases}\\ S[\theta,J] &\equiv \begin{cases} \sin\theta_{\mathrm{c}} & J=\mathrm{i}\\ \sinh\theta_{\mathrm{H}} & J=\epsilon. \end{cases} \end{aligned} \tag{5}$$

Furthermore, for simplicity, we will displace the real function $\theta(x^\mu,J)$ with θ in the following. Thus, we can generally describe a double complex function as follows:

$$\Phi(J)=\phi(J)\mathrm{e}^{J\theta}=\phi(J)(C[\theta,J]+JS[\theta,J]),$$

$$\Phi^*(J) = \phi(J)\mathrm{e}^{-J\theta} = \phi(J)(C[\theta, J] - JS[\theta, J]), \quad (6)$$

where

$$\phi(J) \equiv |\Phi(J)| = \sqrt{\Phi(J)\Phi^*(J)} = \sqrt{\phi_1^2(J) - J^2\phi_2^2(J)},$$

i.e., $\phi_{\mathrm{C}}^2 = \phi^2(J = \mathrm{i}) = \phi_{1\mathrm{C}}^2 + \phi_{2\mathrm{C}}^2$, $\phi_{\mathrm{H}}^2 = \phi^2(J = \epsilon) = \phi_{1\mathrm{H}}^2 - \phi_{2\mathrm{H}}^2$, and θ, a double real parameter. $\phi(J)$ and θ are respectively called "modulus" and "phase angular" of the double complex function $\Phi(J)$.

2. The double Lagrangian with $U(1, J)$ symmetry and its applications to the cosmology

Consider the Robertson-Walker Universe

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}\boldsymbol{x}^2, \quad (7)$$

with the action

$$S(\Phi(J), \Phi^*(J)) = \int \mathrm{d}^4x\sqrt{-g}\Big(-\frac{1}{16\pi G}\mathcal{R} + \mathcal{L}_{\Phi(J)}\Big), \quad (8)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $\mathcal{R}$ is the Ricci scalar, and $\mathcal{L}_{\Phi(J)}$ is the double Lagrangian density with respect to a double complex scalar field $\Phi(J)$, which can be constructed as follows:

$$\mathcal{L}_{\Phi(J)} = \frac{1}{2}g^{\mu\nu}(\partial_\mu\Phi^*(J))(\partial_\nu\Phi(J)) - V(\Phi(J), \Phi^*(J)). \quad (9)$$

In order to get the double real Lagrangian and the corresponding real energy-momentum tensor of matter, we will consider a special Lagrangian density for a double complex scalar field $\Phi(J)$, in which the potential is forced to depend only on the modulus of $\Phi(J)$, i.e., $V(\Phi(J), \Phi^*(J)) = V(|\Phi(J)|)$. Thus, the Lagrangian density of the double complex scalar field in Eq.(9) can reduce into the form

$$\begin{aligned}\mathcal{L}_{\Phi(J)} &= \frac{1}{2}g^{\mu\nu}(\partial_\mu\Phi^*(J))(\partial_\nu\Phi(J)) - V(|\Phi(J)|)\\ &= \frac{1}{2}\partial_\mu\phi_1(J)\partial^\mu\phi_1(J)\\ &\quad - \frac{1}{2}J^2\partial_\mu\phi_2(J)\partial^\mu\phi_2(J) - V(\phi(J)),\end{aligned} \quad (10)$$

where $\phi_1(J)$ and $\phi_2(J)$ are two double real scalar fields of $\Phi(J)$, and the square of the modulus of $\Phi(J)$ is $\phi^2(J) = \phi_1^2(J) - J^2\phi_2^2(J)$. Obviously, when we take $J = \mathrm{i}$, Eq.(10) changes into

$$\begin{aligned}\mathcal{L}_{\Phi_{\mathrm{C}}} &= \frac{1}{2}g^{\mu\nu}(\partial_\mu\Phi_{\mathrm{C}}^*)(\partial_\nu\Phi_{\mathrm{C}}) - V(|\Phi_{\mathrm{C}}|)\\ &= \frac{1}{2}\partial_\mu\phi_{1\mathrm{C}}\partial^\mu\phi_{1\mathrm{C}} + \frac{1}{2}\partial_\mu\phi_{2\mathrm{C}}\partial^\mu\phi_{2\mathrm{C}} - V(\phi_{\mathrm{C}}),\end{aligned}$$

but when we take $J = \epsilon$, the corresponding Lagrangian density in Eq.(10) is

$$\begin{aligned}\mathcal{L}_{\Phi_{\mathrm{H}}} &= \frac{1}{2}g^{\mu\nu}(\partial_\mu\Phi_{\mathrm{H}}^*)(\partial_\nu\Phi_{\mathrm{H}}) - V(|\Phi_{\mathrm{H}}|)\\ &= \frac{1}{2}\partial_\mu\phi_{1\mathrm{H}}\partial^\mu\phi_{1\mathrm{H}} - \frac{1}{2}\partial_\mu\phi_{2\mathrm{H}}\partial^\mu\phi_{2\mathrm{H}} - V(\phi_{\mathrm{H}}).\end{aligned}$$

Thus, based on the double Lagrangian density, i.e., Eq.(10), we can obtain the evolution of quintessence and quintom Universe for the case of $J = \mathrm{i}$ and $J = \epsilon$, respectively. Concretely, we consider a spatially flat Universe, and assume that $\phi_1(J)$ and $\phi_2(J)$ are homogeneous, then the Friedmann equation can be written as

$$H^2 = \frac{4\pi G}{3}[\dot{\phi_1}^2(J) - J^2\dot{\phi_2}^2(J) + V(\phi(J))], \quad (11)$$

where $H \equiv (\dot{a}/a)$ is the Hubble parameter, 'dot' denotes the derivative with respect to the cosmic time t. The energy density and pressure for the double complex scalar field has the following form:

$$\rho_{\Phi(J)} = \frac{1}{2}\dot{\phi_1}^2(J) - \frac{1}{2}J^2\dot{\phi_2}^2(J) + V(\phi(J)), \quad (12)$$

$$p_{\Phi(J)} = \frac{1}{2}\dot{\phi_1}^2(J) - \frac{1}{2}J^2\dot{\phi_2}^2(J) - V(\phi(J)). \quad (13)$$

Moreover, the corresponding EOS parameter is given by

$$\omega_{\Phi(J)} = \frac{p_{\Phi(J)}}{\rho_{\Phi(J)}} = \frac{\frac{1}{2}\dot{\phi_1^2}(J) - \frac{1}{2}J^2\dot{\phi_2}^2(J) - V(\phi(J))}{\frac{1}{2}\dot{\phi_1^2}(J) - \frac{1}{2}J^2\dot{\phi_2}^2(J) + V(\phi(J))}. \quad (14)$$

It is easy to see that, for the case of $J = \epsilon$, the EOS parameter can cross over the cosmological constant boundary along with the evolution of the field components $\phi_{1\mathrm{H}}(t)$ and $\phi_{2\mathrm{H}}(t)$. Exactly, when $\dot{\phi}_{1\mathrm{H}}^2 < \dot{\phi}_{2\mathrm{H}}^2$, the pressure will be negative and $\omega_{\Phi_{\mathrm{H}}} < -1$, which satisfies the phantom equation of state; instead, when $\dot{\phi}_{1\mathrm{H}}^2 > \dot{\phi}_{2\mathrm{H}}^2$, the EOS parameter exceeds -1; and when $\dot{\phi}_{1\mathrm{H}}^2 = \dot{\phi}_{2\mathrm{H}}^2$, this corresponds exactly to the phantom divide line. Thus, we conclude that the quintom Universe can also be obtained by using the hyperbolic complex scalar field as dark energy. On the other hand, for $J = \mathrm{i}$, the evolution of EOS parameter is similar to the case of quintessence, namely, $\omega_{\Phi_{\mathrm{C}}} > -1$

always, because of the same positive sign of $\dot{\phi}_{1\mathrm{C}}^2$ and $\dot{\phi}_{2\mathrm{C}}^2$ in Eq.(14).

Below, we will verify that the double Lagrangian (10) constructed by us has the interesting feature of the $U(1,J)$ symmetry. In order to discuss the symmetry easily, we express the form of the double complex scalar field as $\Phi(J)=\phi(J)\mathrm{e}^{J\theta}$, where $\phi(J)$ and θ are respectively 'modulus' and 'phase angular' of the double complex function $\Phi(J)$, and both of them are real. Thus, the Lagrangian density (10) can be rewritten as

$$\mathcal{L}_{\Phi(J)}=\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi(J))(\partial_\nu\phi(J)) - \frac{1}{2}J^2\phi^2(J)g^{\mu\nu}(\partial_\mu\theta)(\partial_\nu\theta)-V(\phi(J)). \quad (15)$$

Furthermore, based on the $U(1,J)$ symmetry of the double Lagrangian (15), we will point out: (1) the uncanny 'imaginary motion' of angular parameter for the ordinary complex non-canonical scalar field in Ref.[14] is only an internal real phase displacement in our model; (2) it can be verified that there is conserved current and corresponding conserved charge in our Universe and they are the dual charges when we take $J=\mathrm{i}$ and $J=\epsilon$; (3) the cosmological evolution can be described by the double complex scalar field $\Phi(J)$, i.e., the spintessence model and hessence model of dark energy can be obtained in a natural way.

Let $M(J)$ be any element of the double unitary group $U(1,J)$, i.e. $M(J)\in U(1,J)$, and satisfy $M(J)M(J)^\dagger=I$. Note that

$$U(1,J=\mathrm{i})=U(1,C),\quad U(1,J=\epsilon)=U(1,H), \quad (16)$$

are respectively called the ordinary and hyperbolic unitary groups.

If we take $M(\alpha,J)=\mathrm{e}^{\alpha J}$, and consider the transformation

$$\Phi'(J)=M(J)\Phi(J)=\mathrm{e}^{\alpha J}\Phi(J),$$
$$\Phi'^*(J)=M^*(J)\Phi^*(J)=\mathrm{e}^{-\alpha J}\Phi^*(J), \quad (17)$$

where star and α denote respectively complex conjugate and 'angular' parameter (a real constant), we can get the following results.

(1) When $\Phi(J)$ and $\Phi^*(J)$ are expressed as the exponential forms like the Eq.(6), the transformation Eq.(17) changes into $\Phi'(J)=\phi(J)\mathrm{e}^{J(\alpha+\theta)}$, $\Phi'^*(J)=\phi(J)\mathrm{e}^{-J(\alpha+\theta)}$, i.e.

$$\phi'(J)=\phi(J),\quad \theta'=\alpha+\theta, \quad (18)$$

which indicates that under the transformation Eq.(17) there is only a phase displacement of $\Phi(J):\theta\to\alpha+\theta$. It follows that the 'imaginary motion' of angular parameter for the ordinary complex non-canonical scalar field in Ref.[14] is only an internal real phase displacement for the hyperbolic ($J=\epsilon$) complex canonical scalar field in our model. Meanwhile, we can verify that under the transformation Eq.(17) $|\Phi'(J)|^2=|\Phi(J)|^2=\Phi^*(J)\Phi(J)$, i.e., $\phi'(J)=\phi(J)$. Hence the Lagrangian Eq.(15) is also invariant, i.e. $\mathcal{L}_{\Phi'(J)}=\mathcal{L}_{\Phi(J)}$. Thus, for the case of $J=\epsilon$, here we also dub its corresponding dark energy model as hessence in order to accord with Ref.[14]. But we think that the hessence model of dark energy reconstructed by us in the more natural way than one in the Ref.[14] because of the real 'angular motion' in our method.

(2) Since the double Lagrangian Eq.(15) is invariant under the transformation Eq.(17), according to the well-known Noëther theorem, this internal symmetry implies that there may be some conserved quantity. For the double complex canonical scalar field $\Phi(J)=\phi(J)\mathrm{e}^{J\theta}$, the global symmetry of the Lagrangian under $\Phi(J)\longrightarrow\Phi(J)\mathrm{e}^{J\alpha}$ gives rise to the conserved current

$$J^\mu(J)=\frac{1}{2}J[(\partial^\mu\Phi^*(J))\Phi(J)-(\partial^\mu\Phi(J))\Phi^*(J)]. \quad (19)$$

From it we can find the corresponding conserved charge by the integral over the fourth component of the current:

$$Q(J)\equiv\int \mathrm{d}^3xJ^0(J)=-J^2a^3\phi^2(J)\dot{\theta},$$
$$\dot{\theta}=\frac{Q(J)}{-J^2a^3\phi^2(J)}. \quad (20)$$

It follows that the conserved charges $Q(J)$ has the different signs for two cases of $J=\mathrm{i}$ and $J=\epsilon$, namely, $Q(J=\mathrm{i})=Q_{\mathrm{C}}=a^3\phi_{\mathrm{C}}^2\dot{\theta}_{\mathrm{C}}$, $Q(J=\epsilon)=Q_{\mathrm{H}}=-a^3\phi_{\mathrm{H}}^2\dot{\theta}_{\mathrm{H}}$. Therefore, here we call them the dual charges.

In addition, for the double complex canonical scalar field $\Phi(J)$ the energy conservation equation $\dot{\rho}_{\Phi(J)}+3H(\rho_{\Phi(J)}+p_{\Phi(J)})=0$ can be described as

$$\ddot{\phi}(J)+3H\dot{\phi}(J)+J^2\frac{Q^2(J)}{a^6\phi^3(J)}+V'(\phi(J))=0. \quad (21)$$

The total charge conservation equation $\mathrm{d}Q(J)/\mathrm{d}t=0$ and the corresponding current conservation equation can be also derived as follows:

$$\phi^2(J)\ddot{\theta}+2\phi(J)\dot{\phi}(J)\dot{\theta}+3H\phi^2(J)\dot{\theta}=0, \quad (22)$$
$$\phi^2(J)\ddot{\theta}+2\phi(J)\dot{\phi}(J)\dot{\theta}=0. \quad (23)$$

Integrating Eq.(23), we have

$$\phi^2(t,J)\dot{\theta}(t,J)=B(t,J), \quad (24)$$

where $B(t,J)$ is an integration constant, but it depends on time t. On the other hand, by using Eq.(19), we get the charge density $\rho(t,J)$

$$\rho(t,J) \equiv J^0(t,J) = \phi^2(t,J)\dot{\theta}(t,J). \tag{25}$$

Comparing Eq.(24) with (25), we can discover that the integration constant $B(t,J)$ denotes the charge density. Besides, from Eq.(25), we have $\dot{\theta}(t,J) > 0$ due to $\rho(t,J) > 0$. Moreover, according to Eq.(20) we can give the results that $Q_{\rm C} > 0$ and $Q_{\rm H} < 0$ always come into existence.

(3) Considering a spatially flat Friedmann–Robertson–Walker (FRW) Universe and assuming that the scalar fields $\phi(J)$ and θ are homogeneous, we find that the evolving double complex scalar field contributes the energy density $\rho_{\Phi(J)}$ and pressure $p_{\Phi(J)}$ as follows:

$$p_{\Phi(J)} = \frac{1}{2}(\dot{\phi}^2(J) - J^2\phi^2(J)\dot{\theta}^2) - V(\phi(J)), \tag{26}$$

$$\rho_{\Phi(J)} = \frac{1}{2}(\dot{\phi}^2(J) - J^2\phi^2(J)\dot{\theta}^2) + V(\phi(J)). \tag{27}$$

Thus the corresponding EOS parameter is given by

$$\begin{aligned}\omega_{\Phi(J)} &= \frac{p_{\Phi(J)}}{\rho_{\Phi(J)}} \\ &= \frac{\dot{\phi}^2(J) - J^2\phi^2(J)\dot{\theta}^2 - 2V(\phi(J))}{\dot{\phi}^2(J) - J^2\phi^2(J)\dot{\theta}^2 + 2V(\phi(J))}.\end{aligned} \tag{28}$$

From the above equations it is not difficult to see that our results include some results given in the Ref.[15], i.e., spintessence, as a special case of $J = {\rm i}$. On the other hand, when $J = \epsilon$, hessence model can be obtained again, whose EOS parameter can cross over the cosmological constant boundary along with the evolution of the field components $\phi_{\rm H}(t)$ and $\theta_{\rm H}(t)$. Moreover, the corresponding Friedmann equation is

$$H^2 = \frac{4\pi G}{3}[\dot{\phi}^2(J) - J^2\phi^2(J)\dot{\theta}^2 + 2V(\phi(J))], \tag{29}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}[\dot{\phi}^2(J) - J^2\phi^2(J)\dot{\theta}^2 - V(\phi(J))]. \tag{30}$$

Furthermore, substituting Eq.(20) into Eqs. (28)–(30), we have

$$\omega_{\Phi}(J) = \frac{\dot{\phi}^2(J) - J^2\dfrac{Q^2(J)}{a^6\phi^2(J)} - 2V(\phi(J))}{\dot{\phi}^2(J) - J^2\dfrac{Q^2(J)}{a^6\phi^2(J)} + 2V(\phi(J))}, \tag{31}$$

$$H^2 = \frac{4\pi G}{3}\Big[\dot{\phi}^2(J) - J^2\frac{Q^2(J)}{a^6\phi^2(J)} + 2V(\phi(J))\Big], \tag{32}$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3}\Big[\dot{\phi}^2(J) - J^2\frac{Q^2(J)}{a^6\phi^2(J)} - V(\phi(J))\Big]. \tag{33}$$

Obviously, from Eq.(31), we see once more that, when $J = {\rm i}$ and $J = \epsilon$, the EOS parameters respectively correspond to the spintessence Universe and hessence Universe which can cross over the boundary of the cosmological constant due to the existence of the conserved charge $Q_{\rm H}$ in the Eq.(31). But when $Q(J) = 0$ the EOS parameter in Eq.(31) reduces to the quintessence model.[5] It follows that the conserved charge $Q_{\rm H}$ plays an important role in hessence Universe for crossing the boundary of $\omega = -1$.

3. Conclusions and discussions

In summary, we have constructed the double Lagrangian density with the $U(1,J)$ symmetry by using the double complex scalar field as dark energy, and discussed the evolution of the EOS parameters for the both cases of $J = {\rm i}$ and $J = \epsilon$. We find that our method includes quintessence and quintom Universe when the double complex scalar field is written as the form of $\Phi(J) = \phi_1(J) + J\phi_2(J)$. Furthermore, we not only verified the internal symmetry of $U(1,J)$ for the double Lagrangian density in Eq.(15), and presented the expressions of the conserved current and corresponding charge by means of Noëther theorem, but also pointed out that the charges $Q_{\rm C}(> 0)$ and $Q_{\rm H}(< 0)$ corresponding to $J = {\rm i}$ and $J = \epsilon$ are both dual. This seems to imply that if the ordinary complex scalar field $\Phi_{\rm C}$ is taken as a model of the dark energy (such as the spintessence model), the hyperbolic complex scalar field $\Phi_{\rm H}$ would be considered as its dual model of dark energy (such as the hessence model), which is consistent with the viewpoint mentioned in Ref.[14] that there would be certain anti-dark energy.

More importantly, based on the above discussions, the cosmological evolution has been described by using the double complex canonical scalar field $\Phi(J)$ as dark energy and spintessence model and hessence model of dark energy have been also derived in a natural way. It is worthy of noting that there is the great difference between our method and one in Ref.[14] for obtaining the hessence model. That is, in our method the double complex canonical scalar field $\Phi(J)$, rather than the ordinary complex non-canonical scalar field in Ref.[14], plays an important role in establishing hessence model of dark energy. Meanwhile, we also showed that the 'imaginary motion' of angular parameter for hessence in Ref.[14] is only a real phase displacement in our method.

References

[1] Bahall N, Ostriker J P, Perlmutter S and Steinhardt P J 1999 *Science* **284** (astro-ph/9906463)
Riess A G, Strolger L G, Tonry J, Casertano S, Ferguson H C, Mobasher B, Challis P, Filippenko A V, Saurabh J, Li W D, Chornock R, Kirshner R P, Leibundgut B, Dickinson M, Livio M, Giavalisco M, Steidel C C, Benitez N and Tsvetanov Z 2004 *Astrophys. J* **607** (astro-ph/0402512)

[2] Mason B S, Pearson T J, Readhead A C S, Shepherd M C, Sievers J L, Udomprasert P S, Cartwright J K, Farmer A J, Padin S, Myers S T, Bond J R, Contaldi C R, Pen U L, Prunet S, Pogosyan D, Carlstrom J E, Kovac J, Leitch E M, Pryke C, Halverson N W, Holzapfel W L, Altamirano P, Bronfman L, Casassus S, May J and Joy M 2003 *Astrophys. J* **591** 540
Halverson N W, Leitch E M, Pryke C, Kovac J, Carlstrom J E, Holzapfel W L, Dragovan M, Cartwright J K, Mason B S, Padin S, Pearson T J, Shepherd M C, Readhead A C S 2002 *Astrophys. J* **568** 38

[3] Spergel D N, Verde L, Peiris H V, Komatsu E, Nolta M R, Bennett C L, Halpern M, Hinshaw G, Jarosik N, Kogut A, Limon M, Meyer S S, Page L, Tucker G S, Weiland J L, Wollack E and Wright E L 2003 *Astrophys. J. Suppl.* **148** 175

[4] Weinberg S 1989 *Mod. Phys. Rev.* **61** 527

[5] Caldwell R R, Dave R and Steinhardt P J 1998 *Phys. Rev. Lett.* **80** 1582

[6] Caldwell R R, Kamionkowski M and Weinberg N N 2003 *Phys. Rev. Lett.* **91** 071301

[7] Kamenshchik A Y, Moschella U and Pasquier V 2001 *Phys. Lett.* B **511** 265 (gr-qc/ 0103004)

[8] Chiba T, Okabe T and Yamaguchi M 2000 *Phys. Rev.* D **62** 023511 (astroph/9912463)

[9] Huterer D and Cooray A 2005 *Phys. Rev.* D **71** 023506 (astro-ph/0404062)

[10] Feng B, Wang X L and Zhang X M 2005 *Phys. Lett.* B **607** 35 (astro-ph/0404224)

[11] Feng B, Li M Z, Piao Y S and Zhang X M *Preprint* (astro-ph/0407432)

[12] Cai R G, Zhang H S and Wang A 2005 *Commun. Theor. Phys.* **44** 948 (hep-th/0505186)

[13] Li M 2004 *Phys. Lett. B* **603** 1 (hep-th/0403127)

[14] Wei H, Cai R G and Zeng D F 2005 *Class. Quant. Grav.* **22** 3189

[15] Gu J A and Hwang W Y 2001 *Phys. Lett.* B **517** 1 (astro-ph/0105099); Boyle L A, Caldwell R R and Kamionkowski M 2002 *Phys. Lett.* B **545** 17 (astro-ph/0105318)

[16] Wei H and Cai R G 2005 *Phys. Rev.* D **72** 123507 (astro-ph/0509328)

[17] Zhai X H and Zhao Y B 2006 *Chin. Phys.* **10** 2465

[18] Zhong Z Z 1985 *J. Math. Phys.* **26** 2589
Wu Y B 1999 *Gen. Relat. Grav.* **31** 165

[19] Wu Y B and Li L 2002 *Chin. Phys.* **11** 323

[20] Wu Y B, Deng X M, Zhao G M, Li S, Lu J B and Yang X Y 2005 *Acta Phys. Sin.* **54** 4994 (in Chinese)