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# A novel hyperchaos evolved from three dimensional modified Lorenz chaotic system*

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This paper reports a new four-dimensional continuous autonomous hyperchaos generated from the Lorenz chaotic system by introducing a nonlinear state feedback controller. Some basic properties of the system are investigated by means of Lyapunov exponent spectrum and bifurcation diagrams. By numerical simulating, this paper verifies that the four-dimensional system can evolve into periodic, quasi-periodic, chaotic and hyperchaotic behaviours. And the new dynamical system is hyperchaotic in a large region. In comparison with other known hyperchaos, the two positive Lyapunov exponents of the new system are relatively more larger. Thus it has more complex degree.

**Keywords:** chaos, hyperchaos, four dimension chaos system, Lyapunov exponent, bifurcation diagram
**PACC:** 0545

## 1. Introduction

Over the past four decades, chaos have been studied in many fields such as mathematics and engineering communities. The first famous Lorenz system was reported by a meteorologist in 1963,[1] which is described by

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= cx - y - xz, \\ \dot{z} &= xy - bz, \end{aligned} \tag{1}$$

when $a = 10$, $b = 8/3$, $c = 28$, system (1) has a chaotic attractor. Since then, researchers endeavor to search some new chaotic systems. Later, the Rössler system,[2] Chen system,[3] Lü system,[4] Liu system[5] and other new chaotic systems[6−9] were reported one after the other. Recently Qi *et al* also proposed a new three dimensional chaotic system[10] that does not belong to the Lorenz chaotic groups, and its bifurcation characteristics were analysed by using the Central Manifolds Theorem.[11] Chaos control, synchronization and reconstruction were also studied.[12−16]

Hyperchaos, which means a chaotic system with two or more positive Lyapunov exponents, was studied with increasing interest in recent years. As we know, Lyapunov exponents are the average exponential rates of divergence or convergence of nearby orbits in phase space, so hyperchaos indicates that the dynamical system is expanded in more than one direction. The first hyperchaotic system was reported by Rössler[17] in 1979. Since then, some hyperchaos were proposed.[18−25] The researches on control and synchronization of hyperchaos were also reported.[26−28]

Hyperchaos is very useful and can be applied in the fields of Colpitts oscillators,[29] nonlinear circuits,[30] communication security[31] and fibre laser communication[32] etc.

In this paper, a new hyperchaotic system is generated from a modified Lorenz system by introducing a nonlinear state feedback controller. By computer numerical computation, we verify that the new system has two relatively large positive Lyapunov exponents. Moreover, as the parameter varies, the system is still hyperchaotic in a very wide range. The new dynamical system can also evolve into periodic, quasi-periodic, chaotic, and hyperchaotic orbits with different parameters. The new hyperchaotic system is not only demonstrated by numerical simulations but also verified with bifurcation analysis.

## 2. New hyperchaotic system

First consider Lorenz system (1). We introduce a cross-product quadratic with a constant multiplier

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into the first equation and add the control signal into the second equation. So the new four-dimensional dynamical system can be obtained, which is described as

$$\begin{aligned}\dot{x} &= a(y-x)+dyz,\\ \dot{y} &= cx-y-xz-w,\\ \dot{z} &= xy-bz,\\ \dot{w} &= kx+eyz,\end{aligned} \tag{2}$$

where $a$, $b$, $c$, $d$, $e$, $k$ are constant and $x, y, z, w$ are the state variables.

Some basic properties of system (2) are analysed in the following.

For the system, it holds that

$$\nabla V = \frac{\partial \dot{x}}{\partial x}+\frac{\partial \dot{y}}{\partial y}+\frac{\partial \dot{z}}{\partial z}+\frac{\partial \dot{w}}{\partial w} = -(a+b+1). \tag{3}$$

In order to ensure the system to be dissipative, it is required that $-(a+b+1)<0$. Therefore, system (2) converges to a subset of measure zero volume exponentially:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -(a+b+1), \tag{4}$$

which means for an initial volume $V_0$, the volume will become $V_0\mathrm{e}^{-(a+b+1)t}$ at instant $t$ through the flow generated by the system. Thus, there exists an attractor in system (2).

It is obvious that system (2) is symmetric with respect to $z$-axis for the system is invariant under the coordinate transform $(x, y, z, w) \to (-x, -y, z, -w)$.

The equilibria of system (2) can be obtained by solving the following algebraic equations:

$$\begin{aligned}&a(y-x)+dyz = 0,\\ &cx-y-xz-w = 0,\\ &xy-bz = 0, \quad kx+eyz = 0.\end{aligned} \tag{5}$$

Obviously, $S_0 = (0, 0, 0, 0)$ is the trivial equilibrium. When $-bek > 0$, the system has three equilibria. Set

$$\begin{aligned}x_* &= \frac{a\sqrt{-bek}}{e(a+k)}, \quad y_* = \frac{\sqrt{-bek}}{e},\\ z_* &= -\frac{ak}{e(a+k)},\\ w_* &= \frac{\sqrt{-bek}}{e^2(a+k)^2}(a^2ce+acek-a^2e\\ &\quad -2aek-ek^2+a^2k),\end{aligned} \tag{6}$$

we obtain the other two nonzero equilibria

$$S_1 = (x_*, y_*, z_*, w_*),$$

$$S_2 = (-x_*, -y_*, z_*, -w_*). \tag{7}$$

The Jacobian matrix of system (2) linearized about $S_0$ is given by

$$J = \begin{bmatrix} -a & a & 0 & 0\\ c & -1 & 0 & -1\\ 0 & 0 & -b & 0\\ r & 0 & 0 & 0\end{bmatrix}. \tag{8}$$

and its characteristic equation is

$$(\lambda+b)(\lambda^3+(a+1)\lambda^2+(a-ac)\lambda+ak) = 0. \tag{9}$$

From Eq.(9), we can easily know one characteristic value $\lambda_1 = -b$, and the other three characteristic values which satisfy $\lambda_2+\lambda_3+\lambda_4 = -(a+1)$. Suppose $a > 0$, according to the Routh criterion, the system is unstable when $(a+1)(1-c)-k < 0$. Furthermore, when $k > 0$, Eq.(9) has two positive real parts roots, and when $k < 0$, the number of roots with positive real parts is only one.

The stabilities of the two nonzero equilibria which can be calculated via numerical computation are not discussed here.

## 3. Bifurcation analysis

The following simulation results are obtained by using 4-order Runge–Kutta method with the step length taken as 0.0001. Fixing $a = 80$, $b = 6$, $c = 40$, $d = 30$, $e = 30$ and varying $k$, we show the three Lyapunov exponents with respect to parameter $k$ in the Fig.1. Here we just consider $k$ varying between $-79$ and 200. And the corresponding bifurcation diagram of $y$ is given in Fig.2. Obviously, the bifurcation diagram well coincides with the spectrum of Lyapunov exponents. From the two figures, we can clearly observe that system (2) is hyperchaotic when $k$ varies in a very wide range.

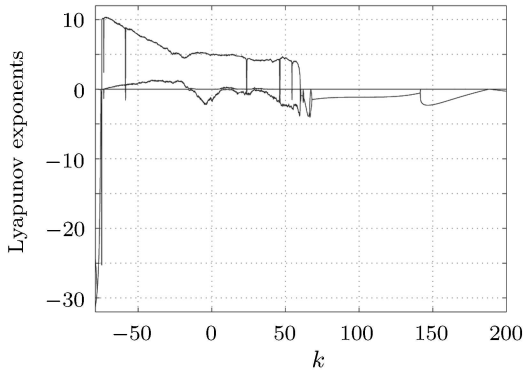



**Fig.1.** The Lyapunov exponent spectrum of system (2).

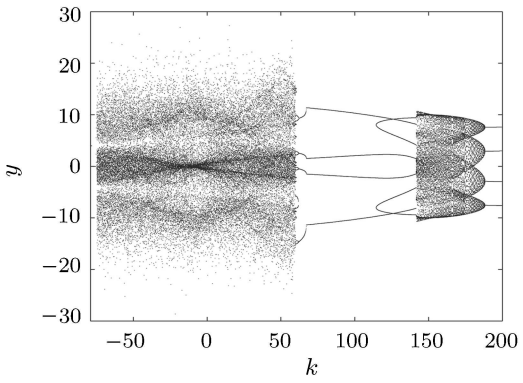



**Fig.2.** Bifurcation diagram of $y$ with increasing $k$.

We always assume $a = 80$, $b = 6$, $c = 40$, $d = 30$, $e = 30$ in the following analysis. By computer simulation, we can obtain more detailed information about the new system (2). When $k \in (60.1, 141.7)$, $(-188.8, 200)$, the largest Lyapunov exponent almost equals zero, and the others are negative, that means system (2) is periodic. There are some different periodic solutions under the different parameter conditions of $k$, as shown in Fig.3. Through observing the bifurcation diagram, we also find that there exists a reverse period-doubling bifurcation process in system (2).

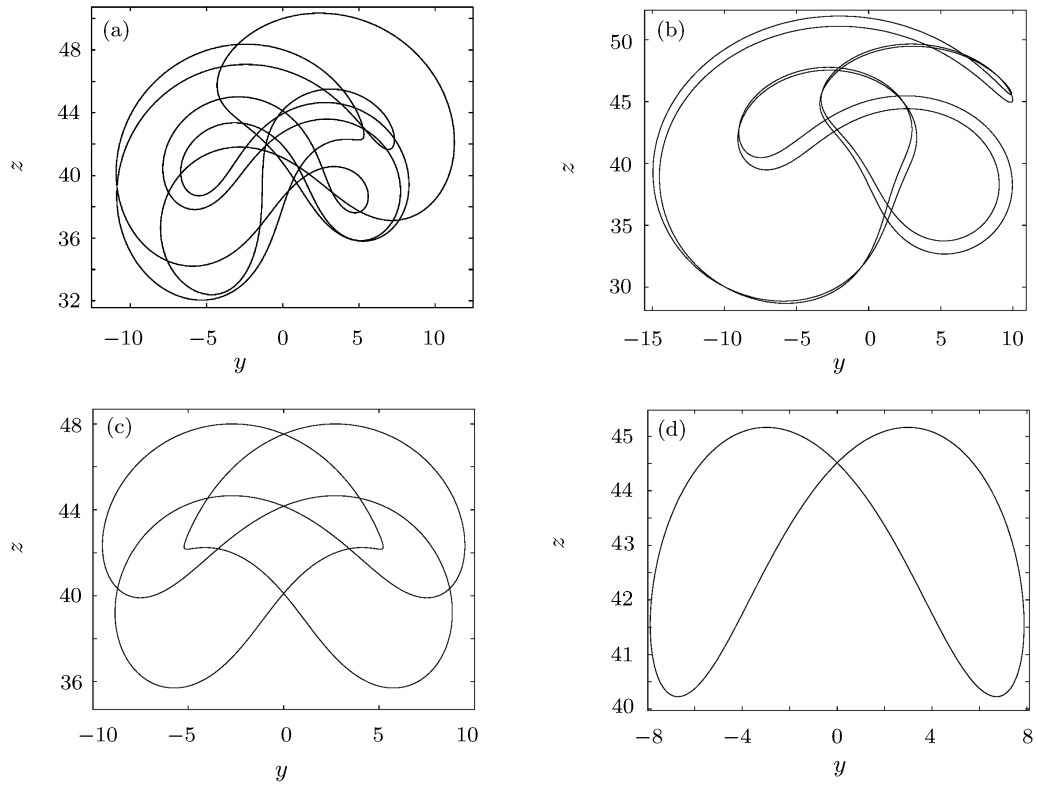



**Fig.3.** Periodic phase portraits of system (2) on the $y - z$ plane: (a) $k = 23.7$; (b) $k = 61$; (c) $k = 140$; (d) $k = 200$.

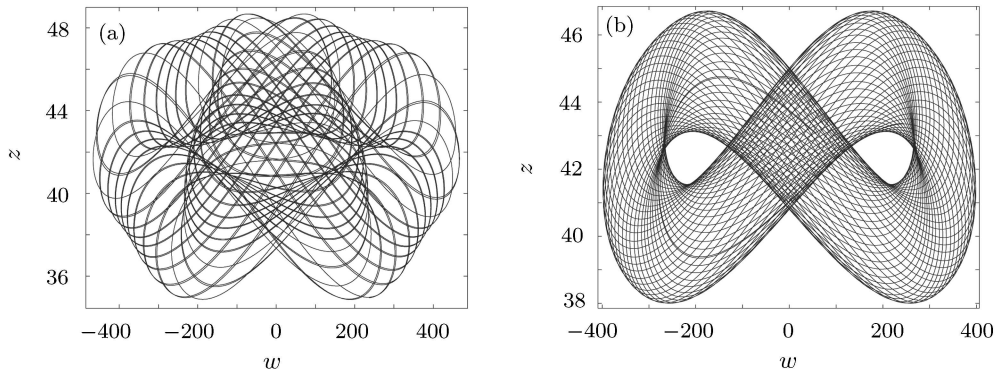



**Fig.4.** Quasi-periodic phase portraits of system (2) on the $w - z$ plane: (a) $k = 150$; (b) $k = 180$.

When $k \in (141.8, 188.7)$, both the first and the second largest Lyapunov exponents almost equal to zero, and the other two are negative, so system (2) is quasi-periodic. The quasi-periodic phase portraits when parameter $k$ equals to 150 and 180 are depicted in Fig.4, which exhibit the beautiful gridding structures.

When $k \in (-74.6, -73.2)$, $(-15.7, 7.4)$, $(14, 23.5)$, $(38.3, 46)$, system (2) has only one positive Lyapunov exponent along with one zero and two negative Lyapunov exponents, which means that the system is chaotic, some phase portraits are shown in Fig.5. When $k = -74$, the projections of the chaotic attractor on $x-y$ and $x-z$ plane are shown in Fig.5(a). And the same projections are depicted in Fig.5(b) when $k = 1$.

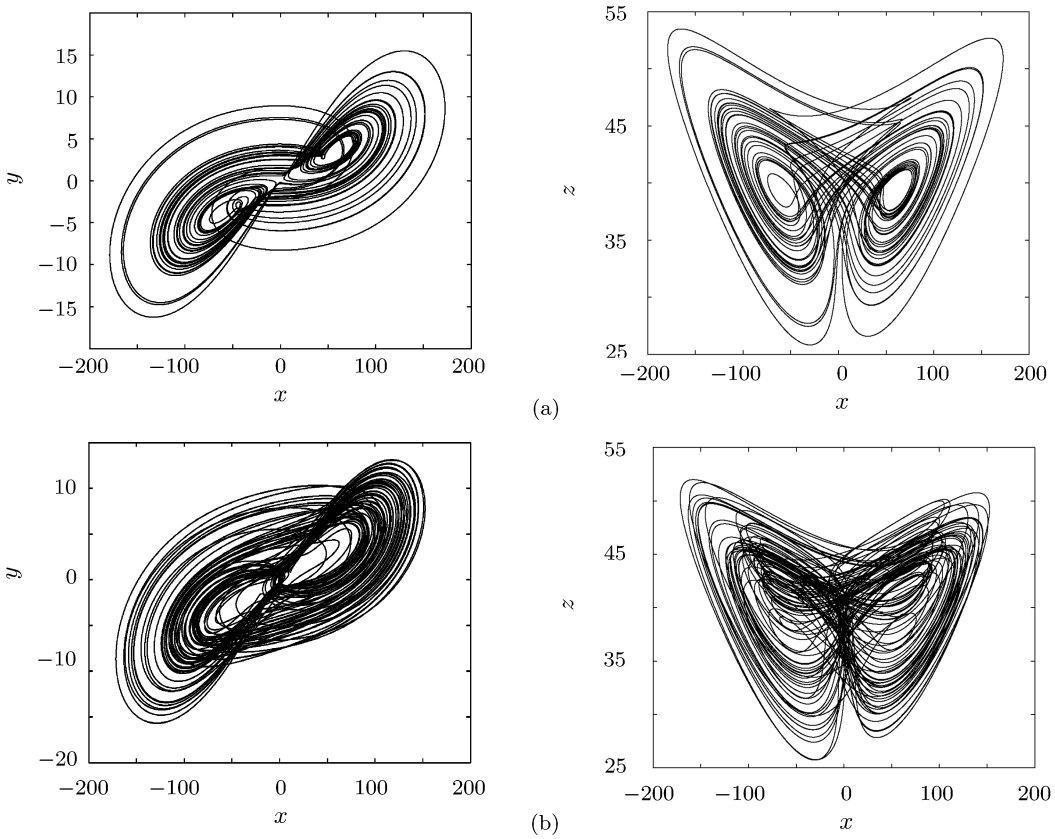



**Fig.5.** Projections of the chaotic attractors of system (2) on $x - y$ and $x - z$ plane: (a) $k = -74$; (b) $k = 1$.

When $k \in (-73.3, -15.7)$, $(7.5, 14.7)$, system (2) has two positive Lyapunov exponents, one zero and one negative Lyapunov exponents, that is to say system (2) is hyperchaotic. When $k = -69$, the projections of the hyperchaotic attractor are shown in Fig.6(a). In comparison with the above chaotic attractors, the hyperchaotic attractors are more disordered because there are simultaneously two different expanding directions. Figure 6(b) shows some other projections of the hyperchaotic attractor when $k$ equals to $-38.5$.

When $a = 80$, $b = 6$, $c = 40$, $d = 30$, $e = 30$, some typical hyperchaotic systems with different $k$ and their Lyapunov exponents are listed in Table 1. It is clear that the two positive Lyapunov exponents are relatively big.

**Table 1.** Some typical parameter values of $k$ to generate hyperchaos and their Lyapunov exponents.

| $k$ | $LE_1$ | $LE_2$ | $LE_3$ | $LE_4$ |
|---|---|---|---|---|
| $-69$ | 10.156 | 0.224 | 0 | $-97.380$ |
| $-61.5$ | 9.094 | 0.604 | 0 | $-96.699$ |
| $-38.5$ | 6.765 | 1.312 | 0 | $-95.075$ |
| $-20.8$ | 4.797 | 1.198 | 0 | $-92.996$ |

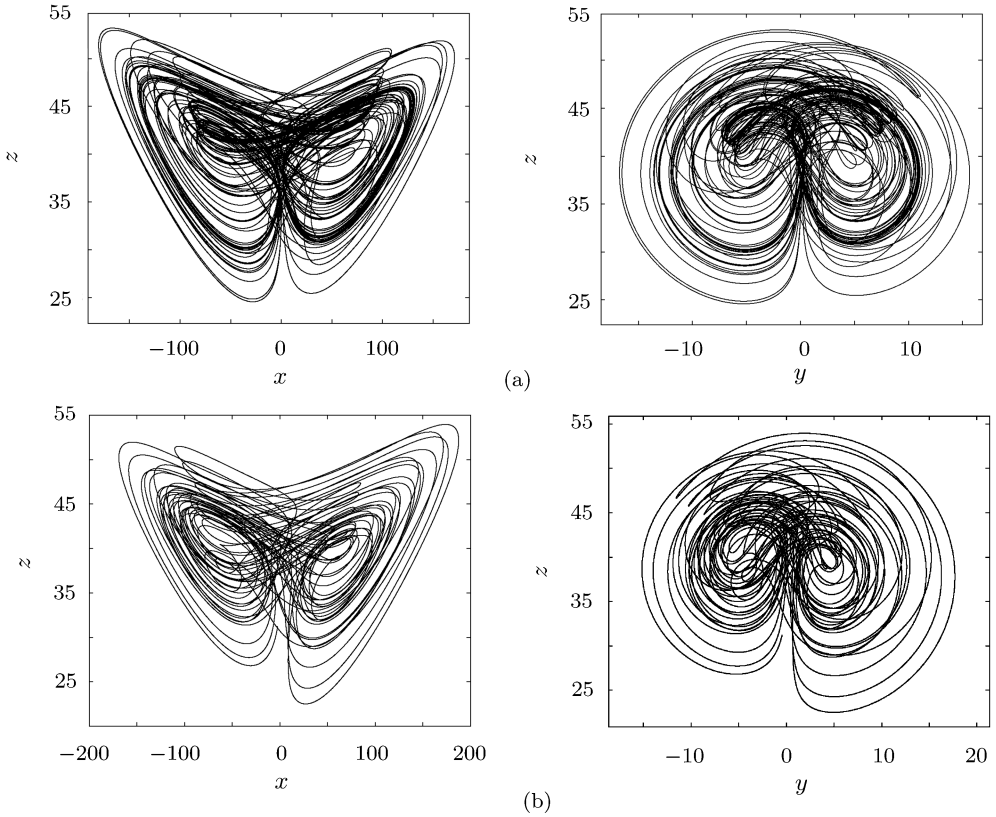



**Fig.6.** Projections of the hyperchaotic attractors on $x-z$ and $y-z$ plane: (a) $k=-69$; (b) $k=-38.5$.

# 4. Conclusions

In this paper, we have proposed a new hyperchaotic system generated from a modified Lorenz system. By introducing a nonlinear state feedback controller into the second equation and adding a cross-product quadratic item into the first equation, we obtained the four-dimensional dynamical system. As parameters vary, the new system can be periodic, quasi-periodic, chaotic and hyperchaotic respectively. The proposed system is hyperchaotic in large regions and possesses two relatively big positive Lyapunov exponents. The hyperchaos has been verified by computer numerical computation. Some complex characteristics of the system and other methods for generating hyperchaos are left for further researches.

# References


[1] Lorenz E N 1963 *J. Atmos. Sci.* **20** 130
[2] Rossler O E 1976 *Phys. Lett.* A **57** 397
[3] Chen G R and Ueta T 1999 *Int. J. Bifur. Chaos* **9** 1465
[4] Lü J H and Chen G R 2002 *Int. J. Bifur. Chaos* **12** 659
[5] Wang F Q and Liu C X 2006 *Acta Phys. Sin.* **55** 5061 (in Chinese)
[6] Yu S M, Lin Q H and Qiu S S 2004 *Acta Phys. Sin.* **53** 2084 (in Chinese)
[7] Wang G Y, Qiu S S and Xu Z Y 2006 *Acta Phys. Sin.* **55** 3295 (in Chinese)
[8] Wang J Z, Chen Z Q and Yuan Z Z 2006 *Acta Phys. Sin.* **55** 3956 (in Chinese)
[9] Zhang Y H, Qi G Y, Liu W L and Yan Y 2006 *Acta Phys. Sin.* **55** 3307 (in Chinese)
[10] Qi G Y, Chen G R, Du S Z, Chen Z Q and Yuan Z Z 2005 *Physica* A **352** 295
[11] Wang F Z, Qi G Y, Chen Z Q, Zhang Y H and Yuan Z Z 2006 *Acta Phys. Sin.* **55** 4005 (in Chinese)
[12] Li L X, Peng H P, Lu H B and Guan X P 2001 *Chin. Phys.* **10** 796
[13] Hua C C and Guan X P 2004 *Chin. Phys.* **13** 1391
[14] Zhang X Y, Guan X P and Li H G 2005 *Chin. Phys.* **14** 279
[15] Ma J H, Chen Y S and Liu Z Y 1999 *Appl. Math. Mech.* **20** 1214
[16] Ma J H and Chen Y S 2000 *Appl. Math. Mech.* 2000 **21** 1237
[17] Rossler O E 1979 *Phys. Lett.* A **71** 155
[18] Li Y X, Chen G R and Wallace K S T 2005 *IEEE Trans.* CAS-II **52** 204

[19] Li Y X, Wallace K S T and Chen G R 2005 *Int. J. Circ. Theor. Appl.* **33** 235
[20] Kapitaniak T and Chua L O 1994 *Int. J. Bifur. Chaos* **4** 477
[21] Cafagna D and Grassi G 2003 *Int. J. Bifur. Chaos* **13** 2889
[22] Yu S M, Lin Q H and Qiu S S 2003 *Acta Phys. Sin.* **52** 25 (in Chinese)
[23] Wang F Q and Liu C X 2006 *Chin. Phys.* **15** 963
[24] Wang J Z, Chen Z Q and Yuan Z Z 2006 *Chin. Phys.* **15** 1216
[25] Gao T G, Chen Z Q, Yuan Z Z and Chen G R 2006 *Int. J. Mod. Phys.* C **17** 471
[26] Wang F Q and Liu C X 2006 *Chin. Phys.* **15** 1971
[27] Song Y Z, Zhao G Z and Qi D L 2006 *Chin. Phys.* **15** 2266
[28] Yu S M, Ma Z G, Qiu S S, Peng S G and Lin Q H 2004 *Chin. Phys.* **13** 317
[29] Cenys A, Tamaservicius A, Baziliauskas A, Krivickas R and Lindberg E 2003 *Chaos, Solitons and Fractals* **17** 349
[30] Barbara C and Silvano C 2002 *Int. J. Circuit Theory and Applications* **30** 625
[31] Udaltsov V S, Goedgebuer J P, Larger L, Cuenot J B, Levy P and Rhodes W T 2003 *Optics and Spectroscopy* **95** 11
[32] Zhang S H and Shen K 2003 *Chin. Phys.* **12** 149