Vol 16 No 11, November 2007
1009-1963/2007/16(11)/3231-07

Chinese Physics

© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

Adaptive projective synchronization of unified chaotic systems and its application to secure communication*

Hu Man-Feng(胡满峰)[a)b)†] and Xu Zhen-Yuan(徐振源)[a)‡]

[a)] *School of Science, Southern Yangtze University, Wuxi* 214122, *China*

[b)] *School of Information Technology, Southern Yangtze University, Wuxi* 214122, *China*

(Received 3 January 2007; revised manuscript received 14 May 2007)

In this paper, a simple adaptive linear feedback control method is proposed for controlling the scaling factor between two coupled unified chaotic systems to a desired value, based on the invariance principle of differential equations. Under this control strategy, one can arbitrarily select the scaling factor. Numerical simulations are given to support the effectiveness of the proposed method and show the robustness against noise. Furthermore, a secure communication scheme based on the adaptive projective synchronization of unified chaotic systems is presented and numerical simulation shows its feasibility.

Keywords: projective synchronization, adaptive control, unified chaotic system, secure communication

PACC: 0545

1. Introduction

As a key technique of secure communications, chaos synchronization has attracted increasing interest from scientists and engineers since the seminal paper by Pecora and Carroll.[1] Over the last decades, a great variety of methods and techniques have been proposed for chaos synchronization such as backstepping design technique,[2] adaptive control approach,[3−5] time-delay feedback approach,[6,7] observer method,[8,9] passive control approach,[10,11] impulsive control[12] etc. In addition, different types of synchronization phenomena have been observed and investigated in a variety of chaotic or/and hyper-chaotic systems, such as identical (complete) synchronization, phase synchronization,[13] lag synchronization,[14] generalized synchronization,[15,16] robust synchronization[17] and so on (for more references, one can see Refs.[18,19]). Amongst all kinds of chaos synchronization, projective synchronization(PS) is one of the most interesting problems.

In a partially linear chaotic system, PS may occur with the characteristic that the states of two coupled systems synchronize up to a constant ratio called scaling factor. In 1999, Mainieri and Rehacek[20] first reported this phenomenon in both Lorenz and disc dynamo systems, and explained the mechanism of the formation of PS in three-dimensional systems, and further attempted to predict the scaling factor by introducing a vector field. However, they only provided a guideline of predicting the scaling factor rather than a concrete theoretical solution. In fact, Xu's study[21] showed that it is difficult to anticipate the scaling factor because it is dependent on the initial condition and the underlying chaotic dynamics. From the application point of view, it is necessary to control the system in order to select and direct the solution of synchronization in a defined way. Motivated by this idea, Xu[21−23] introduced three types of feedback controllers to the drive system to lead the scaling factor to a desired value. Since then, the problem of controlling the scaling factor of PS becomes an interesting topic in chaos synchronization. Liu *et al*[24] also introduced a feedback controller to the drive system for realizing PS with a desired scaling factor. For other interesting works regarding PS, one can refer to Refs.[25–30,46]. In most of the previous works the proposed schemes are often very specific, the added controller may sometimes be too big to implement physically. Thus it is desirable to look for a simple and physically available controller for realizing the PS between two coupled

*Project supported in part by the National Natural Science Foundation of China (Grant Nos 10372054 and 60575038) and the Science Foundation of Southern Yangtze University of China (Grant No 000408).

†Corresponding author. E-mail: humanfeng@vip.sytu.edu.cn

‡E-mail:Xu_zhenyuan@yahoo.com.cn

chaotic systems.

In 1963, Lorenz[31] presented the first chaotic attractor model (Lorenz system) in a three-dimensional autonomous system. In 1999, Chen and Ueta[32] found another similar but not topologically equivalent chaotic attractor(Chen system). In 2002, Lü and Chen[33] found the critical chaotic attractor (Lü system) lying between Lorenz and Chen attractors. Recently, Lü *et al*[34,35] presented a new system (unified chaotic system) with a parameter which can be changed to observe the above three attractors. Unified chaotic system plays a key role in the study of the generalized Lorenz family.[34]

Very recently, Huang proposed a simple adaptive feedback control method to effectively synchronize two almost arbitrary identical chaotic systems in his series paper.[4,5,36] The quality of this method is simple to implement in practice and quite robust against the effect of noise. This technique has been adopted by some authors to realize the identical synchronization of almost all kinds of coupled identical neural networks with time-varying delay[37] and the complete synchronization in uncertain complex networks.[38]

Motivated by the above discussion, we will study the PS between unified chaotic systems by adopting the simple adaptive feedback idea proposed by Huang.[4,5,36] To the best of our knowledge, this adaptive feedback control method is applied to the nonidentical synchronization for the first time in this paper. On the other hand, less attention was paid to the control problem of PS by using adaptive linear feedback control technique.

The remaining part is organized as follows. In Section 2, a brief description of the unified chaotic system is introduced and the definition of PS is given. In Section 3, we investigate the design of adaptive feedback controller for the PS of two coupled unified chaotic systems. In Section 4, a scheme of secure communication based on the adaptive PS of unified chaotic systems is presented. We conclude the paper in Section 5.

2. The definition of PS and the unified chaotic system

First, we give the following definition of partially linear system and PS in coupled partially linear systems.

Definition 1 Partially linear systems are given by

$$\boldsymbol{u} = M(z)\boldsymbol{u}, \quad \dot{z} = f(\boldsymbol{u}, z), \tag{1}$$

where $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)^{\mathrm{T}} \in R^n$ is the state vector, z is the one-dimensional variable; $M(z)$ is a linear matrix function of z and $f : R^n \times R^1 \longrightarrow R^1$ is a nonlinear function.

Definition 2 For two coupled partially linear systems, we have

$$\dot{\boldsymbol{u}}_{\mathrm{d}} = M(z)\boldsymbol{u}_{\mathrm{d}}, \quad \dot{z} = f(\boldsymbol{u}_{\mathrm{d}}, z), \quad \dot{\boldsymbol{u}}_{\mathrm{r}} = M(z)\boldsymbol{u}_{\mathrm{r}}, \tag{2}$$

where the subscripts d and r stand for the drive and response systems respectively. System (2) is said to realize PS, if there exists a constant $\alpha(\neq 0)$ such that $\lim\limits_{t\longrightarrow\infty} \|\boldsymbol{u}_{\mathrm{r}} - \alpha\boldsymbol{u}_{\mathrm{d}}\| = 0$. Here $\| \bullet \|$ denotes a norm of a vector.

The unified chaotic system is given by

$$\begin{aligned} \dot{x} &= (25\theta + 10)(y - x), \\ \dot{y} &= (28 - 35\theta)x - xz + (29\theta - 1)y, \\ \dot{z} &= xy - \frac{\theta + 8}{3}z, \end{aligned} \tag{3}$$

where $\theta \in [-0.016, 1.15]$. System (3) belongs to the generalized Lorenz system when $-0.016 \leq \theta < 0.8$ and the generalized Chen system when $0.8 < \theta \leq 1.15$. In general, we know that system (3) is the original Lorenz system for $\theta = 0$ and stands for the original Chen system for $\theta = 1$; when $\alpha = 0.8$, system (3) expresses a Lü system. Figure 1 displays these chaotic attractors of the unified chaotic system for $\theta = -0.016, 0.3, 0.8$ and 1.15.

Obviously, the state vector $(x, y, z)^{\mathrm{T}}$ of unified chaotic system can be broken into two parts $(\boldsymbol{u}, z)$ and it can be written as

$$\dot{\boldsymbol{u}} = M(z)\boldsymbol{u}, \quad \dot{z} = xy - \frac{\theta + 8}{3}z, \tag{4}$$

where $\boldsymbol{u} = (x, y)^{\mathrm{T}}$ and

$$M(z) = \begin{pmatrix} -25\theta - 10 & 25\theta + 10 \\ -z + 28 - 35\theta & 29\theta - 1 \end{pmatrix}.$$

Recently, many studies on the unified chaotic system were published.[24,39–42] The aim of this paper is to investigate the PS of the unified chaotic system (3) with a desired scaling factor using the adaptive feedback control strategy. The intention of choosing the unified chaotic system as an example is to clarify the proposed methodology. However, it is worth mentioning that our adaptive scheme not only can be applied to the unified chaotic system, but also applied to a class of chaotic systems which have a partially linear structure. Some other examples are hyperchaotic Lü system (four-dimensional system), seven-dimensional and ten-dimensional chaotic systems used in Ref.[23].

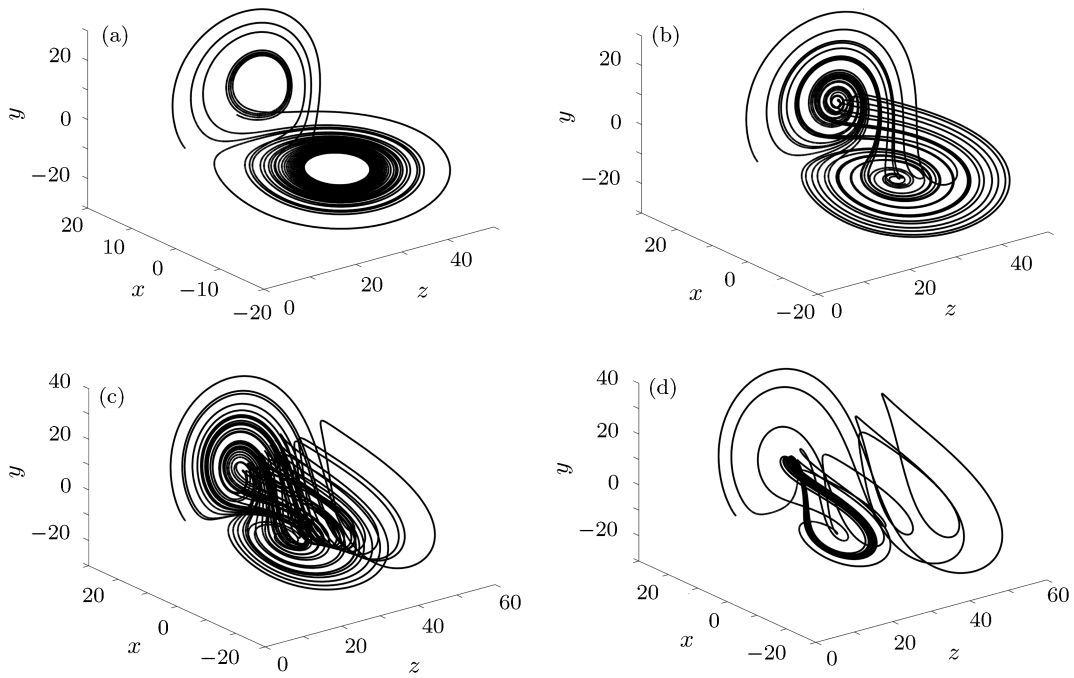


Fig.1. Generalized chaotic attractors with initial values $(x(0), y(0), z(0)) = (1, 2, 3)$; (a) $\theta = -0.016$; (b) $\theta = 0.2$; (c) $\theta = 0.8$ and (d) $\theta = 1.15$.

3. Adaptive PS between unified chaotic systems

In this section, we will investigate the PS problem of the following coupled system:

$$\begin{aligned}
\dot{x} &= (25\theta + 10)(y - x),\\
\dot{y} &= (28 - 35\theta)x - xz + (29\theta - 1)y,\\
\dot{z} &= xy - \frac{\theta + 8}{3}z,\\
\dot{x}_1 &= (25\theta + 10)(y_1 - x_1) - k_1(x_1 - \alpha x),\\
\dot{y}_1 &= (28 - 35\theta)x_1 - x_1 z + (29\theta - 1)y_1 - k_2(y_1 - \alpha y). \qquad (5)
\end{aligned}$$

When the controller is cancelled, namely, $k_1 = k_2 = 0$, the PS of system (5) may occur for arbitrary initial values and any $\theta \in [-0.016, 1.15]$, according to the criteria reported in Refs.[43,44]. Whereas the scaling factor α is unpredictable. For leading the scaling factor to the specified value α, a controller is necessary. We have the following theorem:

Theorem 1 The two coupled identical unified chaotic systems (5) can achieve PS with arbitrarily selected scaling factor α when the coupling strengths k_1, k_2 are updated by the following law:

$$\begin{aligned}
\dot{k}_1 &= \gamma_1(x_1 - \alpha x)^2 := \gamma_1 e_1^2,\\
\dot{k}_2 &= \gamma_2(y_1 - \alpha y)^2 := \gamma_2 e_2^2, \qquad (6)
\end{aligned}$$

where $\gamma_i > 0$, $i = 1, 2$ are arbitrary positive constants, $e_i (i = 1, 2)$ are PS errors.

Proof Construct the following scalar function

$$V = \frac{1}{2}(e_1^2 + e_2^2) + \frac{1}{2}\Big[\frac{1}{\gamma_1}(k_1 - L)^2 + \frac{1}{\gamma_2}(k_2 - L)^2\Big], \quad (7)$$

where L is a sufficiently big constant. By differentiating function V along the trajectories of system (5), we obtain

$$\begin{aligned}
\dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + (k_1 - L)e_1^2 + (k_2 - L)e_2^2\\
&= \boldsymbol{e}^{\mathrm{T}}\tilde{M}(z, \theta, L)\boldsymbol{e}, \qquad (8)
\end{aligned}$$

where

$$\boldsymbol{e} = (e_1, e_2)^{\mathrm{T}},$$

$$\tilde{M}(z, \theta, L) = \begin{pmatrix} 25\theta + 10) - L & 25\theta + 10 \\ -z + 28 - 35\theta & 29\theta - 1 - L \end{pmatrix}.$$

Since chaotic signals are bounded, we assume that $|z| \le B$ (B is positive constant), then according to the well known Gershgorin disc theorem, when L is sufficiently big such that all the eigenvalues of matrix $\tilde{M}(z, \theta, L)$ are located in the open left-half complex plane, we have

$$\dot{V} \le 0. \qquad (9)$$

It is obvious that $\dot{V} = 0$ if and only if $\boldsymbol{e}(t) = 0$, that is, the set $E = \{(\boldsymbol{e}, k) \in R^4 : \boldsymbol{e}(t) = 0, k = (k_1, k_2)^{\mathrm{T}} = k_0 = (k_{01}, k_{02})^{\mathrm{T}} \in R^2\}$ (k_{0i} are

two constants) is the largest invariant set contained in $M = \{\dot{V} = 0\}$ for systems (5). According to the well-known Lasalle invariance principle, the orbit of system (5) converges asymptotically to the set E, i.e., $\boldsymbol{e}(t) \longrightarrow 0, k \longrightarrow k_0$ as $t \longrightarrow \infty$. Then the PS with the desired scaling factor α is achieved under the updated strength (6). This completes the proof.

Remark 1 The above adaptive method can be extended straightforwardly to any arbitrary system, as long as its structure is partially linear.

Remark 2 From the proof of Theorem 1, we can find that the condition satisfying the theorem is very loose. In addition, it is not necessary to know the exact value of the auxiliary constant L. These, in practice, make it possible to design a simple and physically available adaptive linear feedback controller. It is noted that the value of positive constants γ_i has an effect on the synchronization quality from the numerical simulations. A sufficiently large adaptive gain γ_i would lead to fast and high-quality PS, while for sufficiently small adaptive gain γ_i, the PS can be realized only after quite a long time or even cannot be realized.

Remark 3 In Ref.[24], the controller was set to the drive system. Here we set the controller to the response system. This is because the demand of the originality of the chaotic signals from the drive should not be altered by any artificial interruption (i.e. control signals), and also due to the consideration of the potential application in secure communications in which information is masked by a driving dynamics. In addition, the control scheme[24] needs the feedback gain and the desired scaling factor keeping the same sign for the purpose of controlling. Our adaptive control scheme can overcome this restriction.

Now, we give some numerical simulations to support the effectiveness of the above results. In all simulations, we take all the initial conditions as $(x(0), y(0), z(0)) = (1, 2, 3)$, $(x_1(0), y_1(0)) = (4, 5)$, $(k_1(0), k_2(0)) = (0, 0)$. We set $\gamma_1 = \gamma_2 = 1$. The corresponding simulation results with or without the effect of noise are shown in Figs.2 and 3. Figure 2 shows the temporal evolution of the PS errors and the dynamical coupling strength (inset). Figure 3 shows the slight effect of noise added to the signal x with the strengths 1 and 2. The simulation results are sufficient to show that the adaptive linear feedback method for the PS of coupled unified chaotic systems is very effective and quite robust against noise.

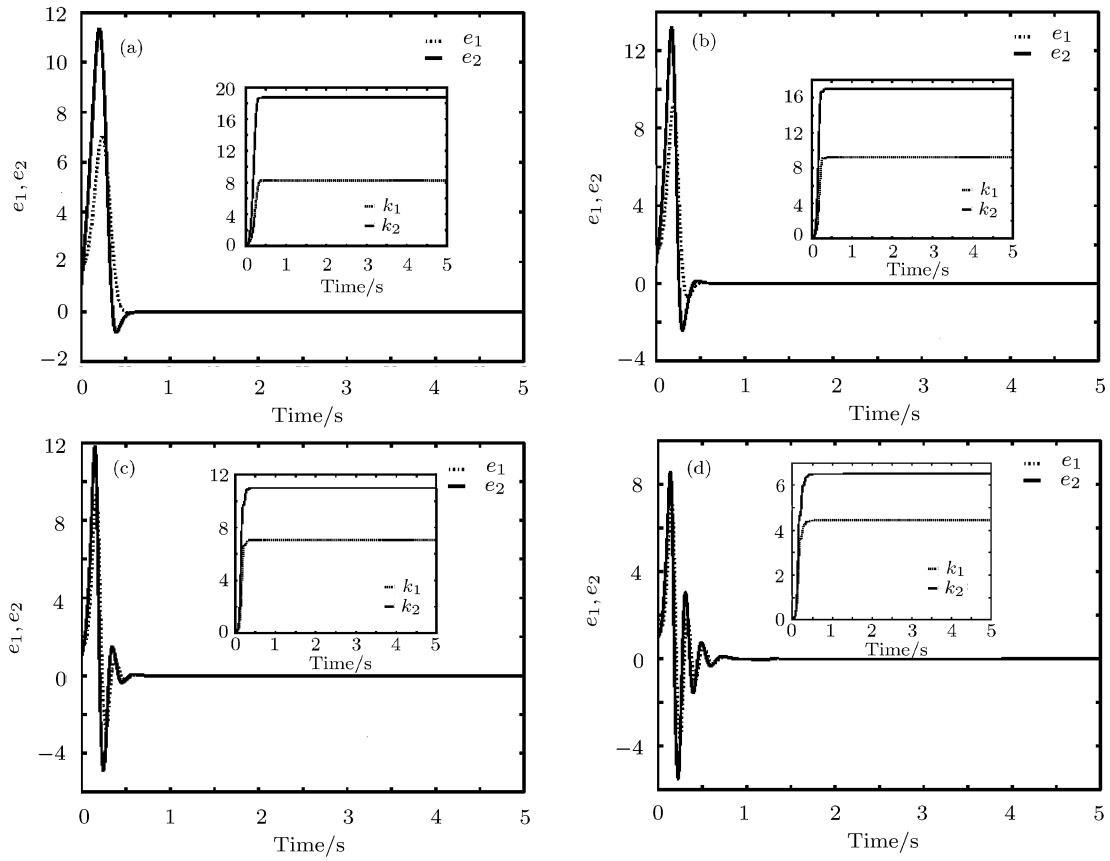


Fig.2. Variation of errors and coupling strengths(inset) for the adaptive projective synchronization of two coupled unified chaotic systems with desired scaling factor $\alpha = 2$; (a) $\theta = -0.016$, (b) $\theta = 0.3$, (c) $\theta = 0.8$ and (d) $\theta = 1.15$.

From further numerical experiments on unified chaotic systems, we find that the dynamical coupling of the state variables may be of no use for the PS, namely certain γ_i can be assumed to be 0. For instance, let $\gamma_1 = 0, \gamma_2 = 1$, i.e. only y and y_1 are chosen to be the coupling terms, the PS can also achieve (see Fig.4.(a)), while $\gamma_1 = 1, \gamma_2 = 0$, that is, the coupling of only variables x and x_1 are chosen, the PS can not be realized. But, when the coupling term of only variables x and x_1 is added to the second component of the response system(variable y_1), the PS can also achieve (Fig.4.(b)). This synchronization phenomenon has been reported in Ref.[45], in which they called it dislocated feedback synchronization.

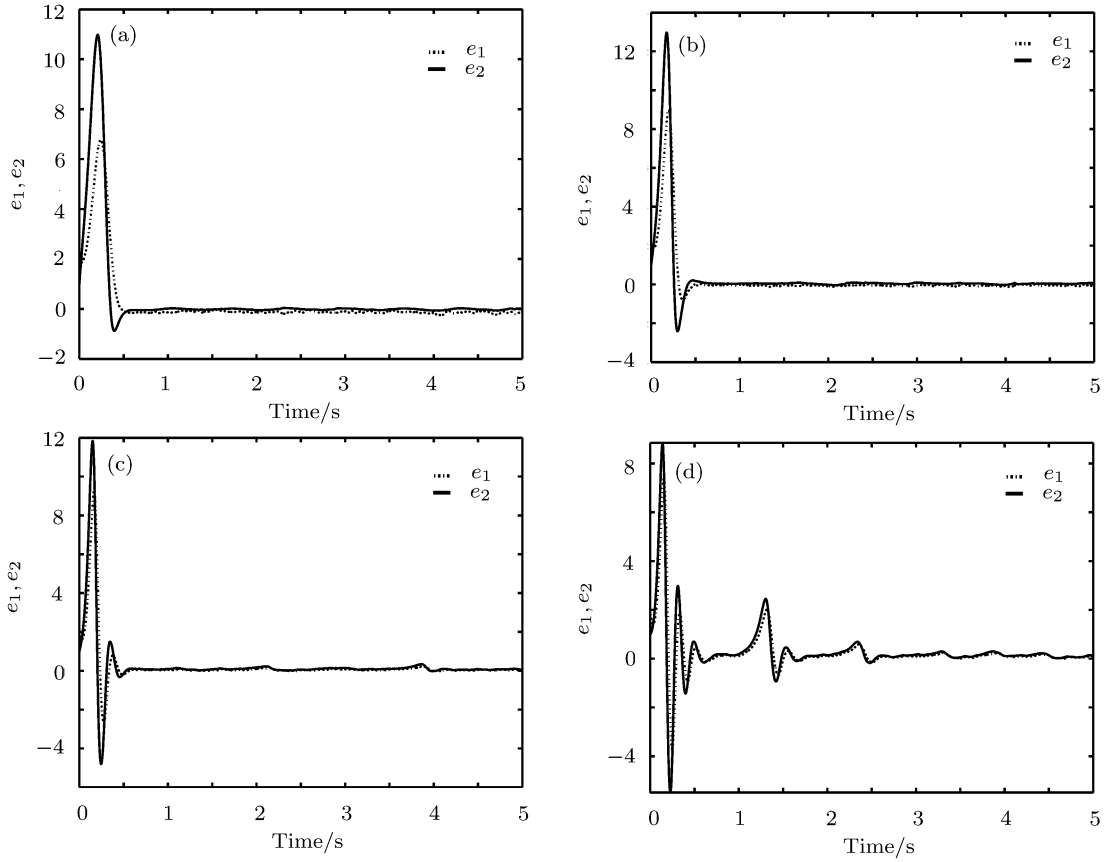


Fig.3. Variation of errors for the adaptive PS of two coupled unified chaotic systems with desired scaling factor $\alpha = 2$ when noise is added to x, for (a) $\theta = -0.016$ and (b) $\theta = 0.3$ with noise strength 2, (c)$\theta = 0.8$ and (d) $\theta = 1.15$ with noise strength 1.

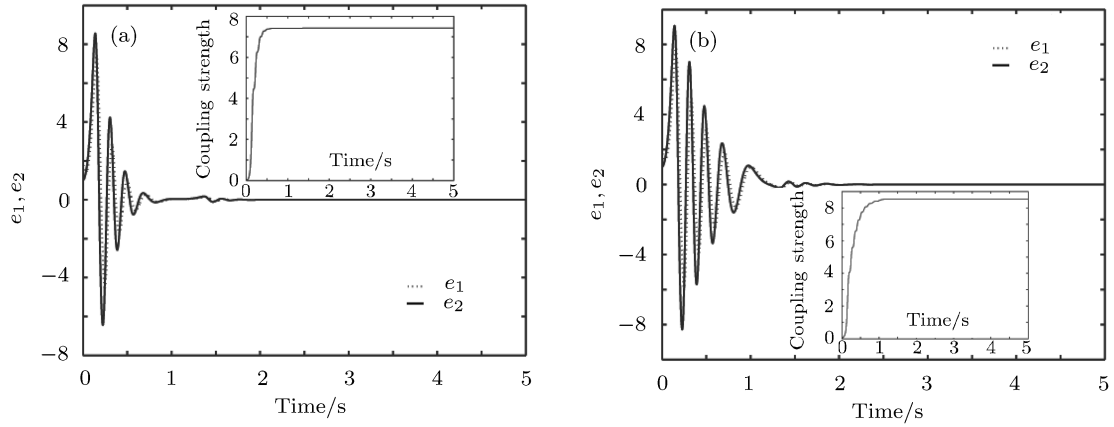


Fig.4. The temporal evolution of errors and the dynamical coupling strength (inset) with scaling factor $\alpha = 2$ and system parameter $\theta = 1.15$, (a) only y and y_1 are chosen as coupling term; (b) coupling term formed by only x and x_1 added to the variable y_1.

4. Application in secure communication

In this section, we will apply the adaptive PS scheme derived above to secure communication. Assuming that $m(t)$ is the message signal, and adding it to the right-hand side of the first equation for the transmitter (drive system) Eq.(5), then we have

$$\begin{aligned}\dot{x} &= (25\theta + 10)(y - x) + m(t),\\ \dot{y} &= (28 - 35\theta)x - xz + (29\theta - 1)y,\\ \dot{z} &= xy - \frac{\theta + 8}{3}z. \end{aligned} \tag{10}$$

Select the output $x(t)$ of system (10) as the transmitted signal, then construct the system for the receiver as follows:

$$\begin{aligned}&\dot{x}_1 = (25\theta + 10)(y_1 - x_1) + p(t) - k_1(x_1 - \alpha x),\\ &\dot{y}_1 = (28 - 35\theta)x_1 - x_1 z + (29\theta - 1)y_1 - k_2(y_1 - \alpha y),\\ &\dot{p} = q(x_1 - \alpha x), \quad \dot{k}_1 = (x_1 - \alpha x)^2,\\ &\dot{k}_2 = (y_1 - \alpha y)^2, \end{aligned} \tag{11}$$

where q is a positive parameter.

Let $e_1 = x_1 - \alpha x$, $e_2 = y_1 - \alpha y$, $e_3 = p - \alpha m$ and construct the following Lyapunov function

$$W = \frac{1}{2}(qe_1^2 + e_2^2 + e_3^2) + \frac{1}{2}(k_1 - L)^2 + \frac{1}{2}(k_2 - L)^2, \tag{12}$$

where L is a constant. Since the eigenfrequency of the message signal $m(t)$ is much less than the oscillating frequency of the chaotic system in practice, when scaling factor α is not much bigger, we have $\alpha(\mathrm{d}m/\mathrm{d}t) \approx 0$. It is easy to obtain $(\mathrm{d}W/\mathrm{d}t) < 0$. From this, $e_3 = p - \alpha m \longrightarrow 0$ can be derived as $t \longrightarrow \infty$, i.e. $p(t)/\alpha$ can recover the message signal $m(t)$.

Taking $m(t) = 0.1\sin(0.1\pi t)$, $q = 50$, the initial values are $(x(0),\ y(0),\ z(0),\ x_1(0),\ y_1(0),\ p(0),\ k_1(0), k_2(0)) = (1, 2, 3, 4, 5, 6, 0, 0)$. We fix the scaling factor at $\alpha = -0.1$, then Fig.5 shows the temporal evolutions of $p(t) - \alpha m(t)$ with parameter (a) $\theta = -0.016$; (b) $\theta = 0.3$; (c) $\theta = 0.8$ and (d) $\theta = 1.15$. From the numerical experiments, we can find that the larger the parameter θ (varying in $[-0.016, 1.12]$), the longer the time for reaching PS.

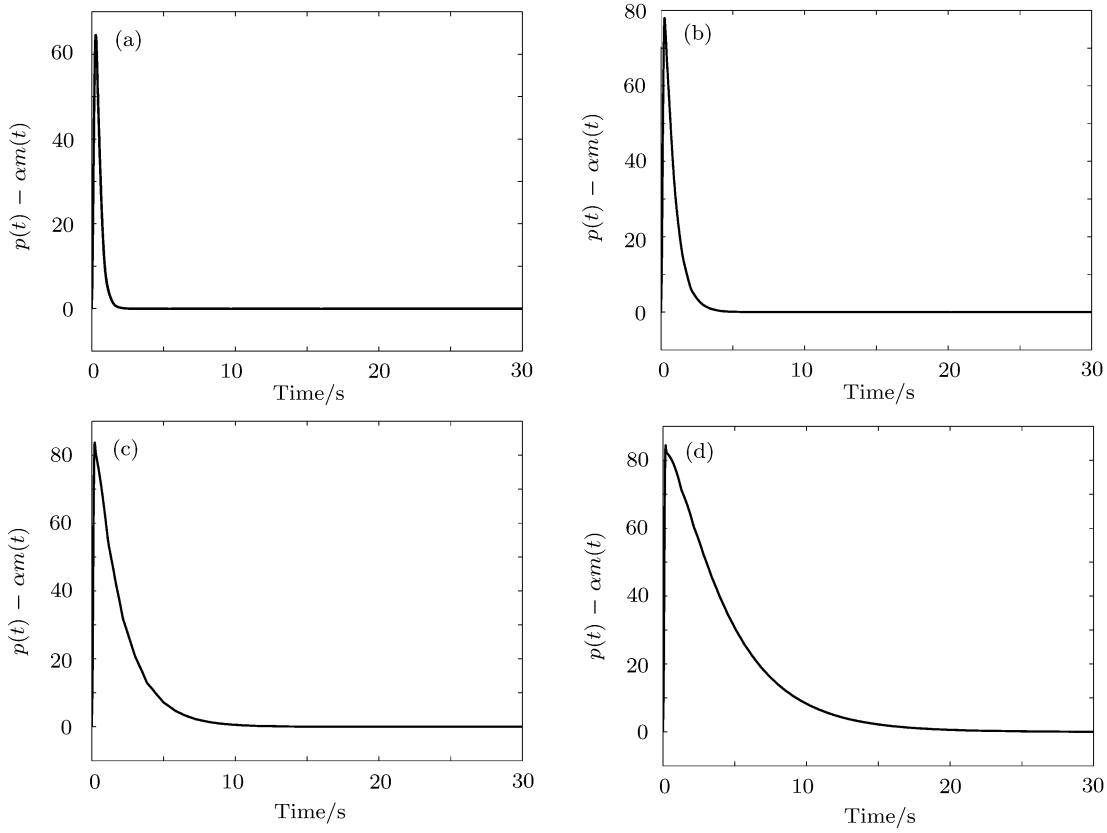


Fig.5. Variation of errors $p(t) - \alpha m(t)$ with desired scaling factor $\alpha^* = -0.1$; (a) $\theta = -0.016$, (b) $\theta = 0.3$ (c)$\theta = 0.8$ and (d) $\theta = 1.15$.

Remark 4 A similar secure communication scheme based on the identical synchronization of unified chaotic systems was proposed in Ref.[39], in which chaos synchronization depends sensitively on the parameter interval $\left[0, \frac{3}{116}\right]$. While in our secure communication scheme, parameter θ is continuously adjustable in interval $[-0.016, 1.15]$ and the scaling factor α can be arbitrary chosen, thus they can be used as the cipher key space. We believe that these features can improve the security than that of Ref.[39].

5. Conclusions

In this paper, we propose a simple adaptive linear feedback controller for the PS of unified chaotic systems coupled through one variable z based on the invariance principle of functional differential equations. Numerical simulations are given to illustrate the effectiveness of the controller and show the robustness against noise. Furthermore, a scheme for secure communication, based on PS of unified chaotic systems, is presented in theory, and its feasibility is supported by numerical simulations. It is worth pointing out that our scheme can easily and straightforwardly be extended to the partially linear system of arbitrary dimensions.

References

[1] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821
[2] Zhang H, Ma X K, Yang Y and Xu C D 2005 *Chin. Phys.* **14** 86
[3] Li Z and Han C Z 2001 *Chin. Phys.* **10** 5
[4] Huang D B 2005 *Phys. Rev.* E **71** 037203
[5] Huang D B 2006 *Phys. Rev.* E **73** 066204
[6] He G G, Zhu P, Chen H P and Cao Z T 2006 *Acta Phys. Sin.* **55** 1040 (in Chinese)
[7] Ahlborn A and Parlitz U 2004 *Phys. Rev. Lett.* **93** 264101
[8] Yao L N, Gao J F and Liao N H 2006 *Acta Phys. Sin.* **55** 35 (in Chinese)
[9] Gao T G, Chen Z Q, Yuan Z Z and Gu Q L 2004 *Acta Phys. Sin.* **53** 1305 (in Chinese)
[10] Wang F Q and Liu C X 2006 *Chin. Phys.* **15** 1971
[11] Zhao Y Z, Zhao G Z and Qi D L 2006 *Chin. Phys.* **15** 2266
[12] Zhang R and Xu Z Y 2006 *Acta Phys. Sin.* **55** 5070 (in Chinese)
[13] Rosenblum M G, Pikovsky A S and Kurths J 1996 *Phys. Rev. Lett.* **76** 1804
[14] Taherion S and Lai Y C 1999 *Phys. Rev.* E **59** 6247
[15] Rulkov N F, Sushchik M M and Tsimring L S 1995 *Phys. Rev.* E **51** 980
[16] Li F, Hu A H and Xu Z Y 2006 *Acta Phys. Sin.* **55** 590 (in Chinese)
[17] Li F, Hu A H and Xu Z Y 2006 *Chin. Phys.* **15** 507
[18] Chen G and Dong X 1998 *From Chaos to Order: Perspective, Methodologies and Applications* (Singapore: World Scientific)
[19] Boccaletti S, Kurths J, Osipov G, Valladares D L and Zhou C S 2002 *Phys. Rep.* **366** 1
[20] Mainieri R and Rehacek J 1999 *Phys. Rev. Lett.* **82** 3042
[21] Xu D 2001 *Phys. Rev.* E **63** 27201
[22] Xu D, Li Z and Bishop R 2001 *Chaos* **11** 439
[23] Xu D and Chee C Y 2002 *Phys. Rev.* E **66** 046218
[24] Liu J, Chen S H and Lu J A 2003 *Acta Phys. Sin.* **52** 1595 (in Chinese)
[25] Wang B and Bu S 2004 *Int. J. Mod. Phys.* B **18** 2415
[26] Zou Y L and Zhu J 2006 *Chin. Phys.* **15** 1965
[27] Feng C, Zhang Y and Wang Y 2006 *Chin. Phys. Lett.* **23** 1418
[28] Li C G 2006 *Progr. Theor. Phys.* **115** 661
[29] Hu M F, Xu Z Y, Zhang R and Hu A H 2007 *Phys. Lett.* A **361** 231
[30] Li G H, Zhou S P and Yang K 2006 *Phys. Lett.* A **355** 326
[31] Lorenz E N 1963 *J. Atmos. Sci.* **20** 130
[32] Chen G and Ueta T 1999 *Int. J. Bifur. Chaos* **9** 1465
[33] Lü J H and Chen G 2002 *Int. J. Bifur. Chaos* **12** 659
[34] Lü J H, Chen G, Zhang S C and Celikovsky S 2002 *Int. J. Bifur. Chaos* **12** 2917
[35] Lu J A, Tao C H, Lü J H and Liu M 2002 *Chin. Phys. Lett.* **19** 632
[36] Huang D B 2004 *Phys. Rev. Lett.* **93** 214101
[37] Cao J D and Lu J Q 2006 *Chaos* **16** 013133
[38] Chen M Y and Zhou D H 2006 *Chaos* **16** 013101
[39] Lu J A, Wu X Q and Lü J H 2002 *Phys. Lett.* A **305** 365
[40] Ucar A, Lonngren K E and Bai E W 2006 *Chaos, Solitons and Fractals* **27** 1292
[41] Yan Z Y 2005 *Phys. Lett.* A **334** 406
[42] Yu D C, Wu A G and Wang D Q 2006 *Chin. Phys.* **15** 306
[43] Li Z and Xu D 2001 *Phys. Lett.* A **282** 175
[44] Xu D, Ong W L and Li Z 2002 *Phys. Lett.* A **305** 167
[45] Tao C H 2006 *Phys. Lett.* A **348** 201
[46] Shao S Q 2007 *Chin. Phys.* **16** 2612