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New scheme of anticipating synchronization for arbitrary anticipation time and its application to long-term prediction of chaotic states*

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How to predict the dynamics of nonlinear chaotic systems is still a challenging subject with important real-life applications. The present paper deals with this important yet difficult problem via a new scheme of anticipating synchronization. A global, robust, analytical and delay-independent sufficient condition is obtained to guarantee the existence of anticipating synchronization manifold theoretically in the framework of the Krasovskii–Lyapunov theory. Different from 'traditional techniques (or regimes)' proposed in the previous literature, the present scheme guarantees that the receiver system can synchronize with the future state of a transmitter system for an arbitrarily long anticipation time, which allows one to predict the dynamics of chaotic transmitter at any point of time if necessary. Also it is simple to implement in practice. A classical chaotic system is employed to demonstrate the application of the proposed scheme to the long-term prediction of chaotic states.

Keywords: anticipating synchronization, long-term predictability, chaotic systems
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1. Introduction

It is well known that long-term unpredictability is a generic feature of chaotic systems, so modelling and predicting the dynamics of chaotic systems are an important yet difficult problem in reality. Several techniques have been used to deal with this challenging subject, for instance, neural network, etc. Recently, a novel approach, named anticipating synchronization discovered by Voss,[1] was suggested as a good candidate for predicting the dynamics of chaotic systems.

In Voss's work,[1] a novel regime of anticipating synchronization was first discovered, and in this regime a receiver system may synchronize with the future state of a transmitter system, i.e. the receiver dynamics may lead the transmitter to use appropriate delayed lines. Two coupling schemes, i.e. complete replacement and delayed coupling, were considered in his paper. In the scheme of complete replacement, Voss showed theoretically that the anticipation time might be arbitrarily long. However, one will see that at least one time delay needs to be contained in the system under consideration by using this scheme and the anticipation time equal to this time delay. So if there is no time delay contained in the system under consideration, for example, continuous chaotic systems without a time delay, the regime of anticipating synchronization will fail. In the scheme of delayed coupling, Voss judged numerically the existence of the invariant anticipating manifold to guarantee anticipating synchronization between the transmitter system and receiver system. Some constraints were required of the anticipation time and coupling strength K, which implies that to guarantee to achieve anticipating synchronization one cannot adjust the time delay in the delayed coupling term arbitrarily, namely, the anticipation time (or equivalently forecast horizon) is not adjustable or controllable in his regime. Despite the existence of the above limitations, the scheme of delayed coupling received abroad welcome in the study of the task of anticipating synchronization, for example, in numerical studies[2−5] and experimental studies.[6−10] Theoretically, an arbitrary long anticipation time can be obtained via a chain of anticipating synchronized exact replicas of the original

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system.[11,12] However, it has been recently shown in Ref.[13] that this scheme is unstable to propagate perturbations since the spatiotemporal character of the coupled chain introduces a convective-like instability into the synchronization manifold.

Motivated by the above discussions, we study this challenging problem again using a new scheme of bidirectional delayed coupling in this paper. A global, robust, analytical and delay-independent sufficient condition is obtained in the framework of the Krasovskii–Lyapunov theory.[14] Different from 'traditional techniques (or regimes)' proposed in the previous literature, by utilizing the present regime the receiver system can synchronize with the future state of a transmitter system for an arbitrarily long anticipation time, i.e. the forecast horizon can be arbitrarily long. Moreover, the anticipation time (or equivalently forecast horizon) is adjustable and controllable arbitrarily in this regime, that is to say, one can predict the dynamics of chaotic transmitter at any point of time if necessary. Note that the system under consideration is very general and almost no restriction is added to it, also the proposed procedure is simple to implement in practical.

This paper is structured as follows. Theoretical analysis of anticipating synchronization is made in Section 2. We restrict ourselves in this section to proving theoretically the existence of an invariant anticipating manifold, and the main result is described in theorem 1. A benchmark chaotic system, the Rössler model, is employed to demonstrate the application of the proposed scheme to the long-term predictability of chaotic states in Section 3, and good effectiveness is observed. Finally, concluding remarks are given in Section 4.

2. Theoretical analysis of anticipating synchronization

The chaotic system under consideration is

$$\dot{x} = Ax + f(x), \tag{1}$$

where $x \in (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in R^n$ is the state vector, and $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))^{\mathrm{T}} : \quad R^n \to R^n$ is the nonlinear vector function. Let $\Omega \in R^n$ be a chaotic bounded set of Eq.(1) which is globally attractive.

For the nonlinear vector function $f(x)$, we make a general assumption as follows: for any $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ and $y = (y_1, y_2, \cdots, y_n)^{\mathrm{T}} \in \Omega$, there exist constants $l_i > 0$ $(i = 1, 2, \cdots, n)$ such that

$$|f_i(x) - f_i(y)| \le l_i \max_{1 \le j \le n} |x_j - y_j|, \quad i = 1, 2, \cdots, n. \tag{2}$$

Actually, the above condition is the so-called the uniform Lipschitz condition and l_i is referred to as the uniform Lipschitz constant. One can easily check that a wide variety of delayed dynamical systems in the form of Eq.(1) satisfy the above condition. For example, condition (2) will hold as long as the partial differential $\partial f_i / \partial x_j$ $(i, j = 1, 2, \cdots, n)$ are bounded in Ω. Therefore, the category of systems in the form of Eq.(1) includes almost all well-known chaotic systems.

We start our analysis with the following two mutually coupled systems:

$$\dot{x} = Ax + f(x) + K_1(x - y_\tau), \tag{3}$$

$$\dot{y} = Ay + f(y) + K_2(y_\tau - x), \tag{4}$$

where $y_\tau = y(t - \tau)$. If $y(t - \tau)$ can track the target $x(t)$, that is,

$$\lim_{t \to \infty} ||x(t) - y(t - \tau)|| = 0, \tag{5}$$

receiver (4) synchronizes with the future state of the transmitter (3) for an anticipation time τ, namely, the anticipating synchronization has been achieved.

Throughout the present paper, $||\bullet||$ denotes the Euclidean norm, which is a function of time. The anticipating synchronization problem is then converted into the determination of the control gain matrices K_1 and K_2 to achieve the goal (5).

Now defining an error system on an anticipating manifold, that is, $e(t) = x(t) - y(t - \tau)$, it becomes

$$\dot{e}(t) = Ae(t) + F(e(t)) + K_1 e(t) + K_2 e(t - \tau), \tag{6}$$

where $F(e) = f(e) - f(x - e) = (f_1(e) - f_1(x - e), f_2(e) - f_2(x - e), \cdots, f_n(e) - f_n(x - e))^{\mathrm{T}}$. Based on the above analysis, goal (5) can be achieved if the origin of error system (6) is asymptotically stable. The following theorem gives the sufficient condition which guarantees that the origin of error system (6) is asymptotically stable based on Lyapunov stabilization arguments in the framework of the Krasovskii–Lyapunov theory.[14]

Theorem 1 The origin of error system (6) is asymptotically stable if there exist constant matrices

K_1 and K_2 such that the following matrix:

$$\begin{bmatrix} A + K_1 + A^{\mathrm{T}} + K_1^{\mathrm{T}} + (2ml+1)I & K_2 \\ K_2^{\mathrm{T}} & -I \end{bmatrix}$$

is negative definite, where $l = \max\{l_1, l_2, \cdots, l_n\}$, m denotes the number of nonzero l_i and I indicates the identity matrix.

Proof Construct a Lyapunov function in the following form:

$$V(t) = e(t)e(t) + \int_{t-\tau}^{t} e(s)e(s)\mathrm{d}s.$$

The differential of the Lyapunov function along the trajectory of error system is

$$\begin{aligned} \dot{V}(t) &= e^{\mathrm{T}}(t)\dot{e}(t) + \dot{e}^{\mathrm{T}}(t)e(t) + e^{\mathrm{T}}(t)e(t) - e^{\mathrm{T}}(t-\tau)e(t-\tau) \\ &= e(t)(A + K_1 + A^{\mathrm{T}} + K_1^{\mathrm{T}} + I)e(t) + e^{\mathrm{T}}(t)K_2e(t-\tau) + e^{\mathrm{T}}(t)F(e) + F^{\mathrm{T}}(e)e(t) \\ &\quad + e^{\mathrm{T}}(t)K_2e(t-\tau) + e^{\mathrm{T}}(t-\tau)K_2^{\mathrm{T}} - e^{\mathrm{T}}(t-\tau)e(t-\tau). \end{aligned}$$

Applying inequality (2), we have

$$\begin{aligned} \dot{V}(t) &\leq e(t)(A + K_1 + A^{\mathrm{T}} + K_1^{\mathrm{T}} + (2ml+1)I)e(t) + e^{\mathrm{T}}(t)K_2e(t-\tau) + e^{\mathrm{T}}(t)K_2e(t-\tau) \\ &\quad + e^{\mathrm{T}}(t-\tau)K_2^{\mathrm{T}} - e^{\mathrm{T}}(t-\tau)e(t-\tau) \\ &= (e^{\mathrm{T}}(t), e^{\mathrm{T}}(t-\tau)) \begin{bmatrix} A + K_1 + A^{\mathrm{T}} + K_1^{\mathrm{T}} + (2ml+1) & K_2 \\ K_2^{\mathrm{T}} & -I \end{bmatrix} \begin{pmatrix} e(t) \\ e(t-\tau) \end{pmatrix}. \end{aligned}$$

Therefore, if the matrix $\begin{bmatrix} A + K_1 + A^{\mathrm{T}} + K_1^{\mathrm{T}} + (2ml+1)I & K_2 \\ K_2^{\mathrm{T}} & -I \end{bmatrix}$ is negative definite, it can be easily obtained as $\dot{V}(t) \leq 0$.

Clearly, $\dot{V}(t) = 0$ if and only if $e(t) = 0$. *QED*.

The proposed scheme is simple to implement in practice for it is similar to the well-known DFC (delayed feedback control method). Furthermore, no specific restriction is added to chaotic system (1) except that the nonlinear vector function $f(x)$ is required to satisfy uniform Lipschitz condition (2) which is very loose in nature, so the proposed scheme can be broadly applied to achieving the anticipating synchronization.

Most importantly, a delay-independent sufficient condition for anticipating synchronization between two chaotic systems is given in theorem 1 based on Lyapunov functions, which guarantees the existence of anticipating chaotic synchronization for an arbitrary anticipation time τ. Obviously, such a synchronization motion is strictly (i.e. high-qualitative), globally (as long as the chaotic attractor is globally attractive), and nonlinearly stable, so the present scheme is strongly against turbulence, i.e. robust against the effect of noise. Therefore, it is possible for anticipating synchronization to be utilized to predict the long-term chaotic states.

3. Forecasting chaotic systems via anticipating synchronization

A generic feature of chaotic system is long-term unpredictability, that is, a tiny perturbation to initial values may result in tremendous differences among trajectories. From a theoretical point of view, the Lyapunov theory provides a sound framework for this problem, and the inverse of the largest Lyapunov exponent gives a theoretical limit to the predictive horizon attainable. In this section, we discuss this important yet difficult problem and the long-term predictability of chaotic behaviour again via anticipating synchronization inspired by the previous work of Voss.[1] A benchmark chaotic system, the Rössler model, is used to illustrate the present scheme, which

will indicate that by the proposed strategy a long-term prediction of chaotic state is achievable.

Firstly, we construct a transmitter and a receiver as follows:

$$\begin{aligned} \dot{x}_1 &= -y_1 - z_1 + k_{11}(x_1 - x_{2,\tau}), \\ \dot{y}_1 &= x_1 + ay_1 + k_{12}(y_1 - y_{2,\tau}), \\ \dot{z}_1 &= b + z_1(x_1 - c) + k_{13}(z_1 - z_{2,\tau}), \end{aligned} \tag{7}$$

and

$$\begin{aligned} \dot{x}_2 &= -y_2 - z_2 + k_{21}(x_{2,\tau} - x_1), \\ \dot{y}_2 &= x_2 + ay_2 + k_{22}(y_{2,\tau} - y_1), \\ \dot{z}_2 &= b + z_2(x_2 - c) + k_{23}(z_{2,\tau} - z_1). \end{aligned} \tag{8}$$

It is known that when $a = 0.15,\ b = 0.2$ and $c = 10$, the Rössler system is chaotic with a corresponding largest Lyapunov exponent $\lambda = 0.09$, giving a theoretical predictive horizon $t = 1/\lambda = 11.1$.[15]

It is easy to test whether the Rössler system satisfy uniform Lipschitz condition (2). According to theorem 1, we choose $k_{11} = -31,\ k_{12} = -31,\ k_{13} = -22$ and $k_{21} = k_{22} = k_{23} = 0$. The results are shown in Figs.1–4 with $\tau = 10$ and $\tau = 100$.

It can be seen from the above figures that the receiver variable $y(t)$ synchronizes with the future state of the transmitter variable $x(t)$ with an anticipating time τ after a short transient process. The anticipating times we choose here for illustration are $\tau = 10$ and $\tau = 100$, respectively. Obviously, the second anticipating time $\tau = 100$ much exceeds the theoretical prediction horizon $t = 1/\lambda = 11.1$. Actually, the anticipating time can be an arbitrary value, since the sufficient condition of theorem 1 is delay-independent.

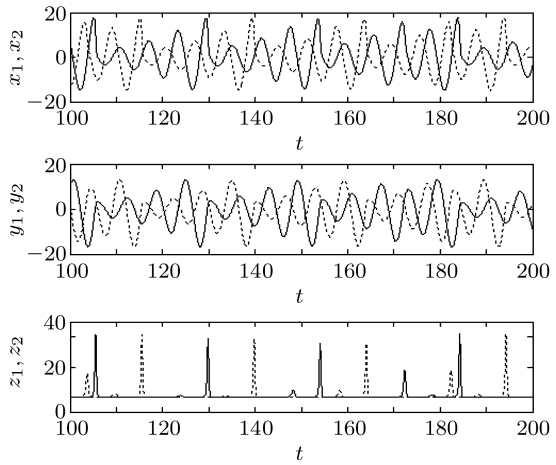


Fig.1. Numerically simulation results for receiver variable $y(t)$ (solid line) anticipating the transmitter variable $x(t)$ (dotted line) with an anticipating time $\tau = 10$.

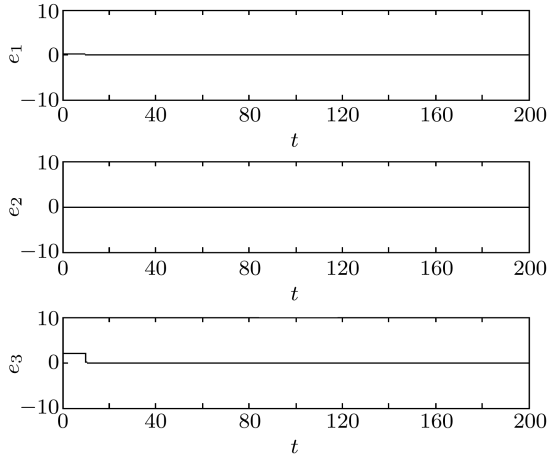


Fig.2. Numerically simulated error $e(t)$ between $x(t)$ and $y(t-\tau)$ with $\tau = 10$.

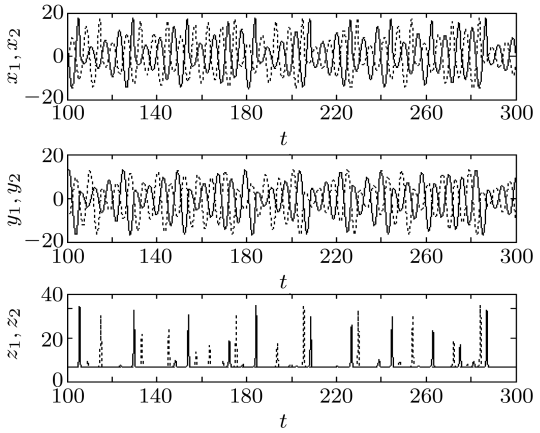


Fig.3. Numerically simulation results for receiver variable $y(t)$ (solid line) anticipating the transmitter variable $x(t)$ (dotted line) with an anticipating time $\tau = 100$.

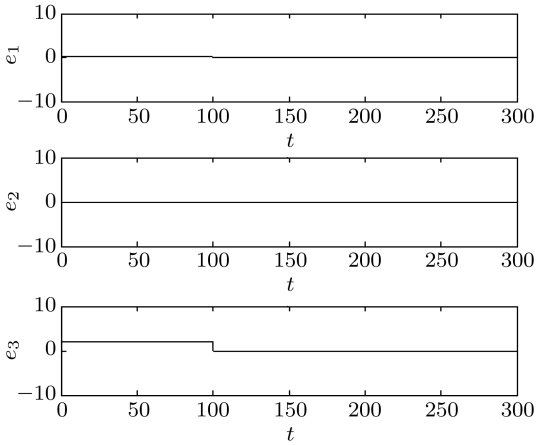


Fig.4. Numerically simulated error $e(t)$ between $x(t)$ and $y(t-\tau)$ with $\tau = 100$.

Note that a significant difference may be observed between the theoretical thresholds of feedback gains

$K_i\,(i = 1, 2)$ and the numerical ones. This is reasonable, in that theorem 1 gives a sufficient, but not necessary condition of stability. The method assures anticipating synchronization when the conditions of theorem 1 are satisfied, but does not forbid a possibility of anticipating synchronization when these conditions are not satisfied.

4. Concluding remarks

The anticipating synchronization and its possible application to the long-term prediction of chaotic states are studied in this paper. Firstly, the theoretical analysis of anticipating synchronization is performed in two bidirectional delayed coupled chaotic systems and a global, robust, analytical and delay-independent sufficient condition is obtained to guarantee the existence of anticipating synchronization manifold theoretically in the framework of Krasovskii–Lyapunov theory.[14] By using the new scheme proposed in this paper, the receiver system can synchronize with the future state of a transmitter system for an arbitrarily long anticipation time. Therefore, the present scheme can be applied to the fast and robust prediction of chaotic systems. Particularly, the anticipation time is totally adjustable and controllable in this regime; this makes it possible to forecast the future dynamics at any point of time if necessary. Then, a classical chaotic system, the Rössler model, is take for example to demonstrate the application of the proposed scheme to the long-term predictability of chaotic states.

Note that the model considered in our present work is a very general one which can include almost all well-known nonlinear dynamical systems. On the other hand, the parameters and the structure of the model are available, that is to say, no uncertainty is involved in model (1). It should be pointed out that the uncertainty does exist in some practical situations, for example, the parameters or the functional form of the experimental systems might not be known beforehand and what is known is only a time series of the system dynamics (see Refs.[16–21] and the references cited therein). Thus, it is desirable to carry out further investigation on the long-term prediction of chaotic dynamics when uncertainty is present in the matter. In fact, this is an open problem and further effort is required in future.

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