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Remote interactions between two d-dimensional distributed quantum systems: nonlocal generalized quantum control-NOT gate and entanglement swapping*

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We present a systematic simple method to implement a generalized quantum control-NOT (CNOT) gate on two d-dimensional distributed systems. First, we show how the nonlocal generalized quantum CNOT gate can be implemented with unity fidelity and unity probability by using a maximally entangled pair of qudits as a quantum channel. We also put forward a scheme for probabilistically implementing the nonlocal operation with unity fidelity by employing a partially entangled qudit pair as a quantum channel. Analysis of the scheme indicates that the use of partially entangled quantum channel for implementing the nonlocal generalized quantum CNOT gate leads to the problem of 'the general optimal information extraction'. We also point out that the nonlocal generalized quantum CNOT gate can be used in the entanglement swapping between particles belonging to distant users in a communication network and distributed quantum computer.

Keywords: nonlocal generalized quantum CNOT gate, qudit, partially entangled, entanglement swapping

PACC: 0365, 4250

1. Introduction

Although quantum entanglement has been a major research topic in the last decades, the nature of the relation between entanglement nonlocality and the property of physical interaction is a fairly new subject. In the recent years it has been realized that the entanglement can be used as a resource for implementing various types of remote interactions.[1−7] Optimal ways of generating entanglement using a given interaction have been investigated.[8,9]

Teleportation of quantum gate,[2] which is a particular case of remote interaction, is a critical step for the implementation of quantum communication network and distributed quantum computation. To realize the distributed quantum computation, not only quantum state but also quantum gate must be transferred between the nodes of the network efficient. The best way to achieve that is through the direct teleportation of quantum gate, i.e. one can implement a collection quantum gate from acting on local qubits to acting on remote qubits by using local operations, classical communication and shared entanglement (LOCCSE). Teleportation of quantum gates is more challenging than teleportation of quantum states[10−13] and has attracted much attention.

It is well-known that control-NOT-(CNOT) gates together with a set of general one-qubit gates are sufficient to implement any multiqubit unitary transformation.[14] These two kinds of operations have been realized locally in laboratory on physical systems such as trapped ions, neutral atoms, and so on.[14] If the qubits involved in the process are separated in distance, unitary transformations must be implemented nonlocally via LOCCSE. Eisert *et al*[3] and Collins *et al*[4] studied nonlocal CNOT gate first in 2000 and found that a nonlocal CNOT gate can be implemented by LOCCSE and consumes only one ebit (maximally entangled pair of qubits) of entanglement and two cbits (bits of classical communication). In 2001, Huelga *et al*[7] studied the remote implementation of a unitary transformation on a qubit and showed

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the existence of nontrivial protocols (i.e. using less resources than bidirectional state teleportation) which allow the perfect remote implementation of certain continuous sets of quantum operation. Thereafter, a series of researches of some special operations and arbitrary operations have been performed.[1,15−17] Recently, the teleportation of a quantum CNOT gate was implemented experimentally in a linear optics setup.[18] An experiment along this line, where a nonlocal rotation was implemented using linear optics, was reported in Ref.[19]. However, all the previous studies have dealt with the problem by using two-dimensional quantum systems, namely, qubit.

Quantum information theory has been extensively developed for a two-level spin-1/2 system. It is only recently that a considerable effort has been made to study the d-dimensional quantum systems (qudit, $d \geq 3$). Their entanglement and separability have been investigated.[20] Recent studies show that qudits are better suited for a certain purpose. For example, in the presence of noise, qudit scheme for quantum cryptograph is more secure than protocol using two-dimensional state;[21] this fact makes long-distance quantum communication, which usually suffers from the noisy channels, more feasible when using qudits instead of qubits. A pair of entangled qudits violates local realm more strongly than qubits.[22] It may be more significant that quantum computation with qudits instead of qubits is the exponential increase of the available Hilbert space with the same number of physical resources.[23] A scheme based on a parametric downconversion is proposed to generate high-dimensional entanglement between two photons.[24] Two experiments were reported,[25,26] in which one used the spatial modes of the electromagnetic field carrying orbital angular momentum to create high-dimensional entanglement.

The purpose of the present work is to present a systematic approach to constructing a nonlocal generalized CNOT quantum gate[27] on two d-dimensional distributed systems. The generalized CNOT gate is defined as

$$U_{\mathrm{GCNOT}}(c;t)\left|i\right\rangle_{c}\left|j\right\rangle_{t}=\left|i\right\rangle_{c}\left|(i-j) \bmod d\right\rangle_{\mathrm{t}}, \quad (1)$$

where c and t denote the control and the target qudits, respectively. Firstly, we present a perfect scheme by which the nonlocal generalized quantum CNOT gate can be implemented by using a maximally entangled two-qudit state as a quantum channel. Secondly, we investigate the problem in a slightly practical way. We propose an other scheme of probabilistically implementing the nonlocal operation via a partially entangled channel, with a given probability of successful implementation. Analysis of the scheme indicates that the use of nonmaximally entangled two-qudit state for implementing the nonlocal generalized quantum CNOT gate leads to the problem of 'the general optimal information extraction'.[28] Finally, as a by-product, we point out that the nonlocal generalized quantum CNOT gate can be applied to the entanglement swapping between parties belonging to widely separated nodes of a quantum network.

2. Implementing a nonlocal generalized CNOT gate by an entangled state of two qudits

Assume the Hilbert spaces of two remote observers, Alice and Bob, to be denoted by $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and $\dim(\mathcal{H}_A)=\dim(\mathcal{H}_B)=d$, further assume that Alice and Bob are able to perform local operations, and they also have additional resources, namely, they share entangled state, and they are able to communicate classically.

To start with, we propose a scheme for implementing the nonlocal generalized CNOT gate $U_{\mathrm{GCNOT}}(A;B)$ by using a maximally entangled two-qudit state as a quantum channel. Suppose that Alice holds a message particle A in an arbitrary state $\left|A\right\rangle_{A}=\sum_{j=0}^{d-1}\alpha_{j}\left|j\right\rangle_{A}\in\mathcal{H}_{A}$, and Bob holds B in a state $\left|B\right\rangle_{B}=\sum_{j=0}^{d-1}\beta_{j}\left|j\right\rangle_{B}\in\mathcal{H}_{B}$, where α_j and β_j $(j=0,1,2\cdots,d-1)$ are unknown real numbers, and $\sum_{j=0}^{d-1}\alpha_{j}^{2}=1$ and $\sum_{j=0}^{d-1}\beta_{j}^{2}=1$. We will show how to perform a nonlocal operation $U_{\mathrm{GCNOT}}(A;B)$ on the state $\left|A\right\rangle_{A}\left|B\right\rangle_{B}$, i.e. transform it into

$$\begin{aligned}
&U_{\mathrm{GCNOT}}(A;B)\left|A\right\rangle_{A}\left|B\right\rangle_{B}\\
=\;&\alpha_{0}\left|0\right\rangle_{A}\sum_{j=0}^{d-1}\beta_{j}\left|(0-j) \bmod d\right\rangle_{B}\\
&+\alpha_{1}\left|1\right\rangle_{A}\sum_{j=0}^{d-1}\beta_{j}\left|(1-j) \bmod d\right\rangle_{B}+\cdots\\
&+\alpha_{d-1}\left|d-1\right\rangle_{A}\sum_{j=0}^{d-1}\beta_{j}\left|[(d-1)-j] \bmod d\right\rangle_{B}, \quad (2)
\end{aligned}$$

where the $U_{\mathrm{GCNOT}}(A;B)$ gate has $|A\rangle_A$ as its control, and $|B\rangle_B$ as its target.

To illustrate this, we first demonstrate the process for the case $d=4$. For this case $|A_4\rangle_A = \sum_{j=0}^{3}\alpha_j|j\rangle_A = (\alpha_0|0\rangle+\alpha_1|1\rangle+\alpha_2|2\rangle+\alpha_3|3\rangle)_A$, and $|B_4\rangle_B = \sum_{j=0}^{3}\beta_j|j\rangle_B = (\beta_0|0\rangle+\beta_1|1\rangle+\beta_2|2\rangle+\beta_3|3\rangle)_B$. In order to complete the transformation of expression (2), an ancillary maximally entangled state of two qudits

$$|\psi_4\rangle_{ab} = (1/\sqrt{4})\sum_{k=0}^{3}|kk\rangle_{ab} = (1/\sqrt{4})(|00\rangle+|11\rangle+|22\rangle+|33\rangle)_{ab} \quad (3)$$

must be prepared and shared by Alice with a and Bob with b, and the following operation must be performed.

Firstly, taking b as the control qudit and B as the target qudit, Bob applies a local generalized CNOT gate $U^4_{\mathrm{GCNOT}}(b;B)$ to qutdits b and B, followed by a generalized Hadamard transformation

$$H^4 = \frac{1}{\sqrt{4}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{e}^{\mathrm{i}2\pi\times1\times1/4} & \mathrm{e}^{\mathrm{i}2\pi\times2\times1/4} & \mathrm{e}^{\mathrm{i}2\pi\times3\times1/4} \\ 1 & \mathrm{e}^{\mathrm{i}2\pi\times1\times2/4} & \mathrm{e}^{\mathrm{i}2\pi\times2\times2/4} & \mathrm{e}^{\mathrm{i}2\pi\times3\times2/4} \\ 1 & \mathrm{e}^{\mathrm{i}2\pi\times1\times3/4} & \mathrm{e}^{\mathrm{i}2\pi\times2\times3/4} & \mathrm{e}^{\mathrm{i}2\pi\times3\times3/4} \end{pmatrix} \quad (4)$$

on qudit b. After these operations, the state $|A_4\rangle_A|B_4\rangle_B|\psi_4\rangle_{ab}$ becomes

$$\begin{aligned}&(1/4)|A_4\rangle_A\sum_{k=0}^{3}\{[|0\rangle_a(\beta_0|0\rangle+\beta_1|3\rangle+\beta_2|2\rangle+\beta_3|1\rangle)_B+\mathrm{e}^{\mathrm{i}2\pi\times1\times k/4}|1\rangle_a(\beta_0|1\rangle+\beta_1|0\rangle+\beta_2|3\rangle+\beta_3|2\rangle)_B\\&+\mathrm{e}^{\mathrm{i}2\pi\times2\times k/4}|2\rangle_a(\beta_0|2\rangle+\beta_1|1\rangle+\beta_2|0\rangle+\beta_3|3\rangle)_B\\&+\mathrm{e}^{i2\pi\times3\times k/4}|3\rangle_a(\beta_0|3\rangle+\beta_1|2\rangle+\beta_2|1\rangle+\beta_3|0\rangle)_B]|k\rangle_b\}. \quad (5)\end{aligned}$$

Now Bob performs a computation basis measurement on qudit b. When the result is $|k\rangle_b$ $(k=1,2,3)$, the state of expression (5) collapses into

$$\begin{aligned}&(1/4)[|0\rangle_a(\beta_0|0\rangle+\beta_1|3\rangle+\beta_2|2\rangle+\beta_3|1\rangle)_B+\mathrm{e}^{\mathrm{i}2\pi\times1\times k/4}|1\rangle_a(\beta_0|1\rangle+\beta_1|0\rangle+\beta_2|3\rangle+\beta_3|2\rangle)_B\\&+\mathrm{e}^{\mathrm{i}2\pi\times2\times k/4}|2\rangle_a(\beta_0|2\rangle+\beta_1|1\rangle+\beta_2|0\rangle+\beta_3|3\rangle)_B\\&+\mathrm{e}^{\mathrm{i}2\pi\times3\times k/4}|3\rangle_a(\beta_0|3\rangle+\beta_1|2\rangle+\beta_2|1\rangle+\beta_3|0\rangle)_B]|A_4\rangle_A. \quad (6)\end{aligned}$$

Alice needs to perform the following unitary transformation

$$Z_k = |0\rangle_{a\ a}\langle 0| + \mathrm{e}^{\mathrm{i}2\pi\times3\times k/4}|1\rangle_{a\ a}\langle 1| + \mathrm{e}^{\mathrm{i}2\pi\times2\times k/4}|2\rangle_{a\ a}\langle 2| + \mathrm{e}^{\mathrm{i}2\pi\times1\times k/4}|3\rangle_{a\ a}\langle 3| \quad (7)$$

on her qudit a to obtain

$$\begin{aligned}&(1/4)[|0\rangle_a(\beta_0|0\rangle+\beta_1|3\rangle+\beta_2|2\rangle+\beta_3|1\rangle)_B+|1\rangle_a(\beta_0|1\rangle+\beta_1|0\rangle+\beta_2|3\rangle+\beta_3|2\rangle)_B\\&+|2\rangle_a(\beta_0|2\rangle+\beta_1|1\rangle+\beta_2|0\rangle+\beta_3|3\rangle)_B\\&+|3\rangle_a(\beta_0|3\rangle+\beta_1|2\rangle+\beta_2|1\rangle+\beta_3|0\rangle)_B]|A_4\rangle_A. \quad (8)\end{aligned}$$

Otherwise no operation is performed on it. Here one cdit (dit of classical communication) must be sent from Bob to Alice. Notice that the four results appear with an equal probability of 1/4, hence the classical dit is unbiased.

Secondly, taking A as control qudit and a as target qudit, Alice applies a local generalized CNOT gate $U^4_{\mathrm{GCNOT}}(A;a)$ to qudits A and a to obtain

$$
\begin{aligned}
&(1/4)\{\alpha_0 |0\rangle_A [|0\rangle_a (\beta_0 |0\rangle + \beta_1 |3\rangle + \beta_2 |2\rangle + \beta_3 |1\rangle)_B + |3\rangle_a (\beta_0 |1\rangle + \beta_1 |0\rangle + \beta_2 |3\rangle + \beta_3 |2\rangle)_B \\
&+ |2\rangle_a (\beta_0 |2\rangle + \beta_1 |1\rangle + \beta_2 |0\rangle + \beta_3 |3\rangle)_B + |1\rangle_a (\beta_0 |3\rangle + \beta_1 |2\rangle + \beta_2 |1\rangle + \beta_3 |0\rangle)_B] \\
&+\alpha_1 |1\rangle_A [|1\rangle_a (\beta_0 |0\rangle + \beta_1 |3\rangle + \beta_2 |2\rangle + \beta_3 |1\rangle)_B + |0\rangle_a (\beta_0 |1\rangle + \beta_1 |0\rangle + \beta_2 |3\rangle + \beta_3 |2\rangle)_B \\
&+ |3\rangle_a (\beta_0 |2\rangle + \beta_1 |1\rangle + \beta_2 |0\rangle + \beta_3 |3\rangle)_B + |2\rangle_a (\beta_0 |3\rangle + \beta_1 |2\rangle + \beta_2 |1\rangle + \beta_3 |0\rangle)_B] \\
&+\alpha_2 |2\rangle_A [|2\rangle_a (\beta_0 |0\rangle + \beta_1 |3\rangle + \beta_2 |2\rangle + \beta_3 |1\rangle)_B + |1\rangle_a (\beta_0 |1\rangle + \beta_1 |0\rangle + \beta_2 |3\rangle + \beta_3 |2\rangle)_B \\
&+ |0\rangle_a (\beta_0 |2\rangle + \beta_1 |1\rangle + \beta_2 |0\rangle + \beta_3 |3\rangle)_B + |3\rangle_a (\beta_0 |3\rangle + \beta_1 |2\rangle + \beta_2 |1\rangle + \beta_3 |0\rangle)_B] \\
&+\alpha_3 |3\rangle_A [|3\rangle_a (\beta_0 |0\rangle + \beta_1 |3\rangle + \beta_2 |2\rangle + \beta_3 |1\rangle)_B + |2\rangle_a (\beta_0 |1\rangle + \beta_1 |0\rangle + \beta_2 |3\rangle + \beta_3 |2\rangle)_B \\
&+ |1\rangle_a (\beta_0 |2\rangle + \beta_1 |1\rangle + \beta_2 |0\rangle + \beta_3 |3\rangle)_B + |0\rangle_a (\beta_0 |3\rangle + \beta_1 |2\rangle + \beta_2 |1\rangle + \beta_3 |0\rangle)_B]\}. \qquad (9)
\end{aligned}
$$

After that, Alice performs a measurement to collapse the state into one of the states $|0\rangle_a$, $|1\rangle_a$, $|2\rangle_a$, and $|3\rangle_a$. Notice that, as shown in the first step, the results are again unbiased. She then sends one cdit to Bob and informs him of her measurement results. For the cases in which Alice obtains $|1\rangle_a$, $|2\rangle_a$, or $|3\rangle_a$, Bob must perform the operations

$$
X_1 = \sum_{j=0}^{3} |(j+1) \bmod 4\rangle_B \, {}_B\langle j| = |1\rangle_B \, {}_B\langle 0| + |2\rangle_B \, {}_B\langle 1| + |0\rangle_B \, {}_B\langle 2| + |0\rangle_B \, {}_B\langle 3|\,, \qquad (10a)
$$

$$
X_2 = \sum_{j=0}^{3} |(j-2) \bmod 4\rangle_B \, {}_B\langle j| = |2\rangle_B \, {}_B\langle 0| + |3\rangle_B \, {}_B\langle 1| + |0\rangle_B \, {}_B\langle 2| + |1\rangle_B \, {}_B\langle 3|\,, \qquad (10b)
$$

or

$$
X_3 = \sum_{j=0}^{3} |(j+3) \bmod 4\rangle_B \, {}_B\langle j| = |3\rangle_B \, {}_B\langle 0| + |0\rangle_B \, {}_B\langle 1| + |1\rangle_B \, {}_B\langle 2| + |2\rangle_B \, {}_B\langle 3| \qquad (10c)
$$

on qudit B respectively to obtain

$$
\begin{aligned}
&(1/4)[\alpha_0 |0\rangle_A (\beta_0 |0\rangle + \beta_1 |3\rangle + \beta_2 |2\rangle + \beta_3 |1\rangle)_B + \alpha_1 |1\rangle_A (\beta_0 |1\rangle + \beta_1 |0\rangle + \beta_2 |3\rangle + \beta_3 |2\rangle)_B \\
&+\alpha_2 |2\rangle_A (\beta_0 |2\rangle + \beta_1 |1\rangle + \beta_2 |0\rangle + \beta_3 |3\rangle)_B] + \alpha_3 |3\rangle_A (\beta_0 |3\rangle + \beta_1 |2\rangle + \beta_2 |1\rangle + \beta_3 |0\rangle)_B \\
&= (1/4) U^4_{\mathrm{GCNOT}}(A;B) \, |A_4\rangle_A \, |B_4\rangle_B \,. \qquad (11)
\end{aligned}
$$

This completes the procedure of implementing a nonlocal generalized CNOT gate $U^4_{\mathrm{GCNOT}}(A;B)$. Combining all conditions together, we obtain a probability of successful implementation, $P = (1/4)^2 \times 4 \times 4 = 1$.

The above procedure can be applied to an arbitrary d-dimensional system. The maximally entangled state of two 4-dimensional qudits $|\psi_4\rangle_{ab}$ is then replaced by a maximally entangled pair of d-dimensional systems $|\psi\rangle_{ab} = (1/\sqrt{d}) \sum_{k=0}^{d-1} |kk\rangle_{ab}$. After applying the local generalized CNOT gate $U_{\mathrm{GCNOT}}(b;B)$, the initial state of the system $|\psi\rangle_{ab} |A\rangle_A |B\rangle_B$ becomes

$$
\begin{aligned}
|\psi\rangle_1 &= U_{\mathrm{GCNOT}}(b;B) \, |\psi\rangle_{ab} \, |A\rangle_A \, |B\rangle_B \\
&= (1/\sqrt{d})[|00\rangle_{ab} \sum_{j=0}^{d-1} \beta_j \, |(0-j) \bmod d\rangle_B + |11\rangle_{ab} \sum_{j=0}^{d-1} \beta_j \, |(1-j) \bmod d\rangle_B + \cdots \\
&\quad + |(d-1)(d-1)\rangle_{ab} \sum_{j=0}^{d-1} \beta_j \, |[(d-1)-j] \bmod d\rangle_B] \, |A\rangle_A . \qquad (12)
\end{aligned}
$$

Bob then performs a generalized Hadamard transformation

$$H=\frac{1}{\sqrt{d}}\begin{pmatrix}1 & 1 & 1 & \cdots & 1 & \cdots & 1\\ 1 & \mathrm{e}^{\mathrm{i}2\pi\times1\times1/d} & \mathrm{e}^{\mathrm{i}2\pi\times2\times1/d} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\times k\times1/d} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\times(d-1)\times1/d}\\ 1 & \mathrm{e}^{\mathrm{i}2\pi\times1\times2/d} & \mathrm{e}^{\mathrm{i}2\pi\times2\times2/d} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\times k\times2/d} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\times(d-1)\times2/d}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & \mathrm{e}^{\mathrm{i}2\pi\times1\times k'/d} & \mathrm{e}^{\mathrm{i}2\pi\times2\times k'/d} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\times k\times k'/d} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\times(d-1)\times k'/d}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & \mathrm{e}^{\mathrm{i}2\pi\times1\times(d-1)/d} & \mathrm{e}^{\mathrm{i}2\pi\times2\times(d-1)/d} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\times k\times(d-1)/d} & \cdots & \mathrm{e}^{\mathrm{i}2\pi\times(d-1)\times(d-1)/d}\end{pmatrix} \tag{13}$$

on qudit b, followed by a computation basis measurement on qudit b. Bob sends one cdit to Alice to inform her which of the d-possible outcomes has been obtained. Alice on her side operates the relevant unitary operation

$$Z_k=|0\rangle_{a\,a}\langle 0|+\sum_{k'=0}^{d-1}\exp[\mathrm{i}2\pi(d-k')k/d]\,|k'\rangle_{a\,a}\langle k'| \tag{14}$$

on qudit a , where $k=1,\cdots,d-1$. After this, the state (12) collapses into

$$\begin{aligned}|\psi\rangle_2 &= (Z_k)_{a\,b}\langle u|\,H_b\,|\psi\rangle_1\\ &= (1/d)[|0\rangle_a\sum_{j=0}^{d-1}\beta_j\,|(0-j)\bmod d\rangle_B+|1\rangle_a\sum_{j=0}^{d-1}\beta_j\,|(1-j)\bmod d\rangle_B+\cdots\\ &\quad+|d-1\rangle_a\sum_{j=0}^{d-1}\beta_j\,|[(d-1)-j]\bmod d\rangle_B]\,|A\rangle_A,\end{aligned} \tag{15}$$

where $|u\rangle_b=|0\rangle_b,|1\rangle_b,\cdots,|d-1\rangle_b$ is the computation basis in which the particle b is measured.

Now Alice applies a local generalized CNOT gate $U_{\mathrm{GCNOT}}(A;a)$ to her qudits a and A, and then performs a computation basis measurement on a. The measurement result is sent to Bob through a classical channel. If the result is $|0\rangle_a$, Alice does nothing; and if the result is $|k\rangle_a\quad(k=1,2,\cdots,d-1)$, Bob performs the X_k operation on qudit B. It can be shown that for $k>0$ the relevant operator is

$$X_k=\sum_{j=0}^{d-1}|(j+k)\bmod d\rangle_{B\;B}\langle j| \tag{16a}$$

for odd k, and

$$X_k=\sum_{j=0}^{d-1}|(j-k)\bmod d\rangle_{B\;B}\langle j| \tag{16b}$$

for even k. This completes the process

$$\begin{aligned}(X_k)_{B\,a}\langle u|\,U_{\mathrm{GCNOT}}(A;a)\,|\psi\rangle_2 &= (1/d)[\alpha_0\,|0\rangle_A\sum_{j=0}^{d-1}\beta_j\,|(0-j)\bmod d\rangle_B+\alpha_1\,|1\rangle_A\sum_{j=0}^{d-1}\beta_j\,|(1-j)\bmod d\rangle_B\\ &\quad+\cdots+\alpha_{d-1}\,|d-1\rangle_A\sum_{j=0}^{d-1}\beta_j\,|[(d-1)-j]\bmod d\rangle_B]\\ &= (1/d)U_{\mathrm{GCNOT}}(A;B)\,|A\rangle_A\,|B\rangle_B\,.\end{aligned} \tag{17}$$

The total probability of successful implementation is $P=(1/d)^2\times d\times d=1$.

To summarize, for d-dimensional systems, we use the resources of one pair of maximally entangled qudits and a two-way classical communication of one dit in each direction.

A particularly interesting situation arises when the pair of particles is initially in a partially entan-

gled state. This means that the quantum channel will be imperfect and the scheme described above will no longer work well. To complete the nonlocal generalized CNOT gate $U_{\mathrm{GCNOT}}(A;B)$ via a partially entangled channel, we must correct the state distorted by the imperfect quantum channel. Here we provide the following strategy.

We suppose the ancillary entangled state shared by Alice and Bob is

$$|\psi'\rangle_{ab}=\sum_{k=0}^{d-1}c_k\,|kk\rangle_{ab}, \tag{18}$$

where the coefficients $c_k(k=0,1,2\cdots,d-1)$ are the known real numbers and $\sum\limits_{k=0}^{d-1}c_k^2=1$. The initial state of the whole system is $|\psi'\rangle_{ab}\,|A\rangle_A\,|B\rangle_B$. The first step of this scheme is exactly the same as the one shown above. After this, the state $|\psi'\rangle_{ab}\,|A\rangle_A\,|B\rangle_B$ is transformed into

$$\begin{aligned}|\psi\rangle_2'=&\,[c_0\,|0\rangle_a\sum_{j=0}^{d-1}\beta_j\,|(0-j)\bmod d\rangle_B\\&+c_1\,|1\rangle_a\sum_{j=0}^{d-1}\beta_j\,|(1-j)\bmod d\rangle_B+\cdots\\&+c_{d-1}\,|d-1\rangle_a\sum_{j=0}^{d-1}\beta_j\\&\times|[(d-1)-j]\bmod d\rangle_B]\,|A\rangle_A\,.\end{aligned} \tag{19}$$

Different from the state of expression (15), the above state includes the parameters of the imperfect quantum channel c_0, $c_1,\cdots$, and c_{d-1}, and the states of particles a, A, and B have been distorted by the imperfect channel. Alice must remove the channel dependence to recover the state before performing her local operations to complete the nonlocal generalized CNOT gate $U_{\mathrm{GCNOT}}(A;B)$. Now Alice's task is to extract the unknown state $|\psi\rangle_2$ from $|\psi\rangle_2'$ as much as possible by means of her limited information. Since Alice cannot measure the unknown qudit directly, the strategy to fulfill Alice's task is to use auxiliary qubit as we probe together with $|\psi\rangle_2'$ and measure the probe qubit following a collective unitary transformation on these qudits. In Li *et al*'s[29] work on optimal probabilistic teleportation in the two-dimensional case, the authors considered the extraction of the unknown qubit $|\phi\rangle=\alpha_0\,|0\rangle+\alpha_1\,|1\rangle$ from the qubit $|\varphi\rangle=(1/\sqrt{N})(\alpha_0\beta_0\,|0\rangle+\alpha_1\beta_1\,|1\rangle)$. In the present paper, we consider the extraction in the d-dimensional case.

To do this, a two-dimensional auxiliary particle a' with the initial state $|0\rangle_{a'}$ is introduced by Alice, and a collective unitary transformation based on the basis $\{|00\rangle_{aa'},|10\rangle_{aa'},|20\rangle_{aa'},\cdots,|(d-1)0\rangle,|01\rangle_{aa'},|11\rangle_{aa'},|21\rangle_{aa'},\cdots,|(d-1)1\rangle\}$ is constructed

$$U=\begin{pmatrix}A_1 & A_2\\ A_2 & -A_1\end{pmatrix}, \tag{20a}$$

where $A_i(i=1,2)$ is a $d\times d$ matrix and may be expressed as

$$A_1=\mathrm{diag}(a_0,a_1,\cdots,a_i,\cdots,a_{d-1}), \tag{20b}$$

$$\begin{aligned}A_2=\mathrm{diag}(&\sqrt{1-a_0^2},\sqrt{1-a_1^2},\cdots,\\&\sqrt{1-a_i^2},\cdots,\sqrt{1-a_{d-1}^2}).\end{aligned} \tag{20c}$$

Here $|a_i|\le 1$ $(i=0,1,2,\cdots,d-1)$ can be determined by dividing $c_{\min}(c_{\min}=\min\{c_k,k=0,1,\cdots,d-1\}$ is the smallest coefficient in the state of expression (18)) by the coefficient of the ith term shown in expression (19), for instance, $a_3=\dfrac{c_{\min}}{c_3}$. It is obvious that a_i, α_k and β_k are all independent. Alice applies U to the qudit a and auxiliary qubit a', and the nonnormalized product state $|\psi_2\rangle'\otimes|0\rangle_{a'}$ is transformed into

$$\left(1/\sqrt{d}\right)c_{\min}(|\psi\rangle_2\,|0\rangle_{a'}+|W\rangle_{aAB}\,|1\rangle_{a'}), \tag{21}$$

where $|\psi\rangle_2$ is a desired state of expression (15) and $|W\rangle_{aAB}$ is a wrong state. Then a measurement on the auxiliary qubit follows. If the result is$|1\rangle_{a'}$, the correction fails and no information about the initial state is left. If $|0\rangle_{a'}$ is obtained, the correction can be successfully made. The maximal probability of successfully extracting the expected state of expression (15) is $(1/d)\,|c_{\min}|^2$. Hence, following the second step we have given after expression (15), the implementation of nonlocal generalized CNOT gate $U_{\mathrm{GCNOT}}(A;B)$ using the partially entangled channel can now be easily completed. We thus see that, by using an auxiliary qubit and performing some specific collective unitary operations, Alice and Bob are able to complete the nonlocal generalized CNOT gate $U_{\mathrm{GCNOT}}(A;B)$ via a partially entangled quantum channel with a probability$P=(1/d)\,|c_{\min}|^2\times d^2=d\,|c_{\min}|^2$. One thing to note is that the auxiliary particle is not necessarily two-dimensional. However, it has been proved that we cannot extract more quantum information with more probe qubits.[28] That is, the maximal probability is independent of the resources used. Therefore, we can just use a two-dimensional qubit as well as we probe.

Finally, we point out that the nonlocal generalized CNOT gate $U_{\mathrm{GCNOT}}(A;B)$ can be applied to the entanglement swapping between parties belonging to widely separated nodes of a quantum network. For example, suppose that Alice and Bob prepare two entangled pairs locally

$$|A\rangle_{A_1,A_2} = (1/\sqrt{d})(|00\rangle + |11\rangle + |22\rangle + \cdots + |(d-1)(d-1)\rangle)_{A_1,A_2},$$

and

$$|B\rangle_{B_1,B_2} = (1/\sqrt{d})(|00\rangle + |11\rangle + |22\rangle + \cdots + |(d-1)(d-1)\rangle)_{B_1,B_2},$$

where Alice's particles are labelled with A_1 and A_2, and Bob's, with B_1 and B_2. The entanglement swapping can be succinctly described by the following identity:

$$U_{\mathrm{GCNOT}}(A_2;B_1)\,|A\rangle_{A_1,A_2}\,|B\rangle_{B_1,B_2} = (1/d)\sum_{l,u}^{d-1}|\psi_{l,u}\rangle_{A_1B_2}F^{-1}\,|l\rangle_{A_2}\,|u\rangle_{B_1}. \quad (22)$$

The maximally entangled states $|\psi_{l,u}\rangle$ with $l,u = 0,1,\cdots,d-1$ are a generalization of Bell states to two d-dimensional quantum systems defined as

$$|\psi_{l,u}\rangle_{A_1B_2} = (1/\sqrt{d})\sum_{q=0}^{d-1}\exp\left(\frac{\mathrm{i}2\pi}{d}ql\right)|q\rangle_{A_1} \times |(q-u) \bmod d\rangle_{B_2}. \quad (23)$$

F is the Fourier transform, i.e. $F\,|l\rangle = (1/\sqrt{d})\sum_{k=0}^{d-1}\exp\left(\frac{\mathrm{i}2\pi}{d}kl\right)|k\rangle$. The protocol for entanglement swapping can readily be read from Eq.(22). After performing a nonlocal generalized CNOT gate $U_{\mathrm{GCNOT}}(A_2;B_1)$ on qudits A_2 and B_1, and measuring them separately in bases $F^{-1}\,|l\rangle_{A_2}$ and $|u\rangle_{B_1}$, the joint state of qudits A_1 and B_2 will be projected onto the maximally entangled state $|\psi_{l,u}\rangle_{A_1B_2}$. Thereby, the entanglement contained in states $|A\rangle_{A_1,A_2}$ and $|B\rangle_{B_1,B_2}$ is transferred to state $|\psi_{l,u}\rangle_{A_1B_2}$. This state can be deterministically transformed into the state $|\psi\rangle_{A_1,B_2} = 1/\sqrt{d}(|00\rangle + |11\rangle + |22\rangle + \cdots + |(d-1)(d-1)\rangle)_{A_1,B_2}$ by Alice and Bob through appropriate local unitary transformations conditional on the outcome of the measurement procedure.

3. Conclusion

In conclusion, we have presented a systematic simple method of constructing a generalized quantum CNOT gate on two d-dimensional distributed systems. Firstly, we have presented explicit quantum construction for implementing the nonlocal generalized CNOT gate with unity fidelity and unity probability. Secondly we have put forward a scheme for the probabilistic implementing of the nonlocal operation with unity fidelity by using a partially entangled pair of qudits as a quantum channel. This will be useful in some practical cases of distributed quantum information processing where the quantum channel is partially entangled. It is shown that, under some specific collective unitary transformation U on the state distorted by the imperfect channel and auxiliary qubit as probe, the maximal probability of successfully extracting the expected operation is $P = d\,|c_{\min}|^2$. It is also shown that the entries of such U are independent of all unknown α_k and β_k. The required physical resources for the implementation of the nonlocal operation in these two cases are discussed. The nonlocal generalized quantum CNOT gate can be used as a manipulation protocol to generate entanglement swapping between particles belonging to distant users in a communication network and distributed quantum computer.

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