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Exact solutions for four coupled complex nonlinear differential equations

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In this paper, exact solutions are derived for four coupled complex nonlinear different equations by using simplified transformation method and algebraic equations.

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1. Introduction

Exact solutions to nonlinear differential equations in mathematical physics have received much attention of many mathematicians and physicists. Much work in this respect has been done,[1−9] but the work on coupled nonlinear differential equations,[10−12] particularly on complex coupled nonlinear differential equations has been little done.

Generally speaking, for complex coupled nonlinear differential equations(CCNDEs):

$$\begin{aligned} &F_1(\Psi, \Phi, \Psi_t, \Psi_x, \Phi_t, \Phi_x, \cdots) = 0, \\ &F_2(\Psi, \Phi, \Psi_t, \Psi_x, \Phi_t, \Phi_x, \cdots) = 0, \end{aligned} \tag{1}$$

we must find suitable transformations to easily solve them. Tentatively, we first start with the transformation

$$\begin{aligned} &\Psi = H_1(u, v) \\ \text{and} \quad &\Phi = H_2(u, v) \end{aligned} \tag{2}$$

to obtain the following real coupled nonlinear differential equations of real variables u and v:

$$\begin{aligned} &G_1(u, v, u_\xi, u_\eta, v_\xi, v_\eta, \cdots) = 0, \\ &G_2(u, v, u_\xi, u_\eta, v_\xi, v_\eta, \cdots) = 0, \end{aligned} \tag{3}$$

where u and v are real functions of ξ and η, which are linear in x and t. If we can decouple Eqs.(3) to obtain the solutions of u and v, then we will have the solutions of Ψ and Φ via Eqs.(2).

Following this line, in this paper we employ some simple transformations $\Psi = u(\xi)\mathrm{e}^{\mathrm{i}\eta}$ and $\Phi = v(\xi)\mathrm{e}^{\mathrm{i}\sigma}$ with u and v being real variables, and ξ, η and σ being linear functions of t and x and obtain many solutions of the following four pairs of CCNDEs, i.e. Eqs.(A), (B), (C) and (D).

Equations (A)

$$\mathrm{i}\Psi_z - \beta\Psi_{tt} + K\Phi + (|\Psi|^2 + \sigma|\Phi|^2)\Psi = 0, \tag{4}$$

$$\mathrm{i}\Phi_z - \beta\Phi_{tt} + K\Psi + (|\Phi|^2 + \sigma|\Psi|^2)\Phi = 0. \tag{5}$$

This pair of equations describes single-mode fibres and supports co-propagation of two different modes with orthogonal polarizations. Here, σ is the relative cross-phase modulation coefficient and $\sigma > 0$, and K is the linear coupling coefficient accounting for possible twist of the fibre in the case where Ψ and Φ represent orthogonal linear polarizations or elliptic deformation of the fibre's cross-section if Ψ and Φ correspond to circular polarizations.[13]

Equations (B)

$$\mathrm{i}\frac{\partial\Psi}{\partial t} + \alpha\nabla^2\Psi + \rho\Psi\Phi = 0, \tag{6}$$

$$\frac{\partial^2\Phi}{\partial t^2} - C_0^2\nabla^2\Psi + \mu^2\Phi - \beta|\Psi|^2 = 0, \tag{7}$$

which describe a system of conserved scalar nucleons interacting with neutral scalar mesons. Here, Ψ represents a complex scalar nucleon field and Φ denotes a real scalar meson field. The real constants μ, ρ and β describe the mass of a meson and coupled constants respectively, the real constants α and C_0^2 are parametric constants, and $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.[14]

Equations (C)

$$\mathrm{i}\Psi_t + \Psi_{xx} - 2\lambda|\Psi|^\alpha\Psi + 2\Psi\Phi = 0, \tag{8}$$

$$\Phi_{tt} - \Phi_{xx} + (|\Psi|^\alpha)_{xx} = 0, \tag{9}$$

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which are the generalized Zakharov equations. Here, the complex envelope $\Psi(x,t)$ is the high-frequency wave and $\Phi(x,t)$ is the real low-frequency field.[15]

Equations (D)

$$\Psi_{tt} - C_S^2 \Psi_{xx} = \beta \frac{\partial^2(|\Phi|^\alpha)}{\partial x^2}, \tag{10}$$

$$\mathrm{i}\Phi_t + \alpha \Phi_{xx} - \delta \Psi \Phi = 0, \tag{11}$$

which are another generalized Zakharov equations.[16]

2. Exact solutions to Eqs.(A)

In this part we work out the solutions to Eqs.(A). We first make the following transformation:

$$\Psi(z,t) = u(t)\mathrm{e}^{\mathrm{i}\eta z}, \tag{12}$$

$$\Phi(z,t) = v(t)\mathrm{e}^{\mathrm{i}\eta z}. \tag{13}$$

Substituting the above transformation into Eqs.(4) and (5) leads to

$$-\eta u - \beta u'' + Kv + u^3 + \sigma u v^2 = 0, \tag{14}$$

$$-\eta v - \beta v'' + Ku + v^3 + \sigma u^2 v = 0. \tag{15}$$

Suppose that u is related to v by the relationship

$$u = av^\alpha + b, \tag{16}$$

where a, b, α are determined later. Inserting expression (16) into Eq.(15) yields

$$\begin{aligned}\beta v'' =\ & v^3 + \sigma a^2 v^{2\alpha+1} + 2\sigma a b v^{\alpha+1} \\ & + Kav^\alpha + (\sigma b^2 - \eta)v + Kb\end{aligned} \tag{17}$$

and

$$\begin{aligned}\frac{1}{2}\beta(v')^2 =\ & \frac{1}{4}v^4 + \frac{\sigma a^2}{2\alpha+2}v^{2\alpha+2} + \frac{2\sigma ab}{\alpha+2}v^{\alpha+2} \\ & + \frac{Ka}{\alpha+1}v^{\alpha+1} + \frac{\sigma b - \eta}{2}v^\alpha + Kbv.\end{aligned} \tag{18}$$

Insert expression (16) and Eqs.(17) and (18) into Eq.(14), then we will have

$$\begin{aligned}&-\eta a v^\alpha - \eta b + Kv + a^3 v^{3\alpha} + 3a^2 b v^{2\alpha} + 3ab^2 v^\alpha + b^3 + \sigma a v^{\alpha+2} + \sigma b v^2 - 2a\alpha(\alpha-1) \\ &\times\left(\frac{a^2\sigma}{2\alpha+2}v^{3\alpha} + \frac{1}{4}v^{\alpha+2} + \frac{2ab\sigma}{\alpha+2}v^{2\alpha} + \frac{Ka}{\alpha+1}v^{2\alpha-1} + \frac{\sigma b^2 - \eta}{2}v^\alpha + Kbv^{\alpha-1}\right) - a\alpha(\sigma a^2 v^{3\alpha} + v^{\alpha+2} \\ &+ 2\sigma a b v^{2\alpha} + Kav^{2\alpha-1} + (\sigma b^2 - \eta)v^\alpha + Kbv^{\alpha-1}) = 0.\end{aligned} \tag{19}$$

Now, we discuss the solutions of Eq.(19).

(I) Let $K \neq 0$ and balance the orders of terms of v, then we will have $\alpha = 1$ and obtain the following algebraic equations:

$$\begin{aligned}&-\eta b + b^3 = 0, \\ &-\eta a + K + 3ab^2 - a(\sigma b^2 - \eta + Ka) = 0, \\ &3a^2 b + \sigma b - 2\sigma a^2 b = 0, \\ &a^3 + \sigma a - a(\sigma a^2 + 1) = 0.\end{aligned} \tag{20}$$

By solving Eq.(20), we have the relation between u and v, i.e. $u = \pm v$, $b = 0$ and $a = \pm 1$. From this relation we obtain the solutions to Eqs.(14) and (15):

(i) When $s = \dfrac{K-\eta}{\beta} > 0$ and $p = \dfrac{\sigma+1}{2\beta}$, then

$$\begin{aligned}u &= \pm\sqrt{\frac{cs}{p}}\frac{2\mathrm{e}^{\sqrt{s}t}}{1 - c\mathrm{e}^{2\sqrt{s}t}}, \\ v &= u,\end{aligned}$$

where $c \neq 0$ is a real number. When $c = -1$, then $u = \pm\sqrt{-\dfrac{s}{p}}\mathrm{sech}(\sqrt{s}t)$ is similar to the expression in the limited form of $\beta = 0$ in Ref.[13].

(ii) When $s = \dfrac{-K-\eta}{\beta} > 0$ and $p = \dfrac{\sigma+1}{2\beta}$, then

$$\begin{aligned}u &= \pm\sqrt{\frac{cs}{p}}\frac{2\mathrm{e}^{\sqrt{s}t}}{1 - c\mathrm{e}^{2\sqrt{s}t}}, \\ v &= -u,\end{aligned}$$

where $c \neq 0$ is a real number. When $c = -1$, then $u = \pm\sqrt{-\dfrac{s}{p}}\mathrm{sech}(\sqrt{s}t)$ is similar to the expression in the limited form of $\beta = 0$ in Ref.[13].

(iii) When $s = \dfrac{K-\eta}{\beta} < 0$ and $p = \dfrac{\sigma+1}{2\beta}$, then

$$\begin{aligned}u &= \pm\sqrt{-\frac{s}{p}}\sec(\sqrt{-s}t + c), \\ v &= u,\end{aligned}$$

where c is a real number. This pair of solutions are new solutions.

(iv) When $s=\dfrac{-K-\eta}{\beta}<0$ and $p=\dfrac{\sigma+1}{2\beta}$, then

$$u=\pm\sqrt{-\frac{s}{p}}\sec(\sqrt{-s}t+c),$$
$$v=-u,$$

where c is a real number. This pair of solutions are new solutions.

(v) When $s=0$ and $p=\dfrac{\sigma+1}{2\beta}$, then

$$u=-\frac{1}{\pm\sqrt{p}t+c},$$
$$v=\pm u,$$

which are new solutions.

From (i) to (v), we have

$$\Psi=u(t)\mathrm{e}^{\mathrm{i}\eta z}\quad\text{and}\quad\Phi=\pm\Psi.$$

(II) Let $K=0$ and balance the orders of term v, then we will have $\alpha=1$ or $\alpha=2$.

(i) when $\alpha=2$, we have

$$-\eta b+b^3=0,$$
$$-\eta a+3ab^2+\sigma b-2a(\sigma b^2-\eta)-2a(\sigma b^2-\eta)=0,$$
$$3a^2b+\sigma a+4a\left(\frac{1}{4}+\frac{1}{2}ab\sigma\right)-2a(1+2ab\sigma)=0,$$
$$a^3-\frac{4a^3\sigma}{6}-2a^3\sigma=0.$$

Using Mathematica yields no solutions.

(ii) when $\alpha=1$, we have

$$-\eta b+b^3=0,$$
$$-\eta a+3ab^2-a(\sigma b^2-\eta)=0,$$
$$3a^2b+\sigma b-2a^2b\sigma=0,$$
$$a^3+a\sigma-a(a^2\sigma+1)=0.$$

Solving the above equations we obtain three groups of solutions as follows:

(a) $b=0$ and $a=\pm1$, σ can be any real number. In this case, solutions are the same as those in case (I).

(b) $b=0$ and $\sigma=1$, and $a\neq0$. There exist the two following cases:

(b.1) when $s=-\dfrac{\eta}{\beta}>0$, then

$$v=\pm\sqrt{-\frac{2c\eta}{a^2+1}}\frac{2\mathrm{e}^{\sqrt{s}t}}{1-c\mathrm{e}^{2\sqrt{s}t}},$$
$$u=av,$$

where $c\neq0$, that is,

$$\Psi=av(t)\mathrm{e}^{\mathrm{i}\eta z}\quad\text{and}\quad\Phi=v(t)\mathrm{e}^{\mathrm{i}\eta z}.$$

(b.2) when $s=-\dfrac{\eta}{\beta}<0$, then

$$v=\pm\sqrt{\frac{2\eta}{a^2+1}}\sec(\sqrt{-s}t+c),$$
$$u=av,$$

that is,

$$\Psi=av(t)\mathrm{e}^{\mathrm{i}\eta z}\quad\text{and}\quad\Phi=v(t)\mathrm{e}^{\mathrm{i}\eta z},$$

which are similar to those discussed in Ref.[17].

(c) when $b=\eta^2,\sigma=3$ and $a=\pm1$, there exist the following solutions:

(c.1) when $\dfrac{3\eta^4-\eta}{\beta}>0$, then

$$v=\frac{\mp4\eta^2(3\eta^4-\eta)\mathrm{sech}^2\left(\sqrt{\frac{3\eta^4-\eta}{4\beta}}t\right)}{16\eta^4-2(3\eta^4-\eta)\left(1-\tanh\left(\pm\sqrt{\frac{3\eta^4-\eta}{4\beta}}t\right)\right)^2},$$
$$u=\pm v+\eta^2;$$

(c.2) when $\dfrac{3\eta^4-\eta}{\beta}>0$ and $\eta>\eta^4$, then

$$v=\frac{2(3\eta^4-\eta)\mathrm{sech}\left(\sqrt{\frac{3\eta^4-\eta}{\beta}}t\right)}{\sqrt{\eta-\eta^4}\mp4\eta^2\mathrm{sech}\left(\sqrt{\frac{3\eta^4-\eta}{\beta}}t\right)},$$
$$u=\pm v+\eta^2.$$

From cases (c.1) to (c.2), we have

$$\Psi=v(t)\mathrm{e}^{\mathrm{i}\eta z}+\eta^2\mathrm{e}^{\mathrm{i}\eta z}\quad\text{and}\quad\Phi=v(t)\mathrm{e}^{\mathrm{i}\eta z}.$$

These pairs of solutions are new solutions. There are other kinds of solutions under different conditions, which are omitted here.

3. Exact solutions to Eqs.(B)

In this section, we first assume

$$\Psi=u(\xi)\mathrm{e}^{\mathrm{i}\sigma t},\quad\Phi=v(\xi)$$
$$\text{and}\quad\xi=kx+ly+mz.\tag{21}$$

Inserting expression (21) into Eqs.(6) and (7) yields

$$\alpha(k^2+l^2+m^2)u''-\sigma u+\rho uv=0, \tag{22}$$
$$-C_0^2(k^2+l^2+m^2)v''+\mu^2 v-\beta u^2=0. \tag{23}$$

Let $n^2=k^2+l^2+m^2$, then

$$\alpha n^2u''-\sigma u+\rho uv=0, \tag{24}$$
$$-C_0^2n^2v''+\mu^2v-\beta u^2=0. \tag{25}$$

Suppose that

$$v=au^\gamma+b, \tag{26}$$

then

$$v''=a\gamma(\gamma-1)u^{\gamma-2}(u')^2+a\gamma u^{\gamma-1}u''. \tag{27}$$

From Eqs.(26), (27) and (25), we obtain

$$-C_0^2n^2a\gamma(\gamma-1)u^{\gamma-2}(u')^2-C_0^2n^2a\gamma u^{\gamma-1}u''$$
$$+\mu^2au^\gamma+\mu^2b-\beta u^2=0. \tag{28}$$

Considering Eqs.(26) and (24), we have

$$\alpha n^2u''=-\rho au^{\gamma+1}+(\sigma-\rho b)u, \tag{29}$$
$$\alpha n^2(u')^2=-\frac{2a\rho}{\gamma+2}u^{\gamma+2}+(\sigma-\rho b)u^2. \tag{30}$$

Insert expressions (29) and (30) into Eq.(28), then we will have

$$\frac{C_0^2}{\alpha}\frac{3}{\gamma+2}a^2\gamma^2\rho u^{2\gamma}-\frac{C_0^2}{\alpha}a\gamma^2(\sigma-\rho b)u^\gamma+\mu^2au^\gamma$$
$$-\beta u^2+\mu^2b=0.$$

Balancing the power of u, we obtain

$$\gamma=1 \ \text{ and } \ b=0,$$

and the following relations:

$$\frac{C_0^2}{\alpha}\frac{3}{\gamma+2}a^2\gamma^2\rho=\beta,$$
$$\frac{C_0^2}{\alpha}a\gamma^2\sigma=\mu^2a,$$

hence we obtain $v=au$ and a satisfying $\beta\sigma=\mu^2a^2\rho$.

Now, Eq.(24) becomes

$$\alpha n^2u''-\sigma u+\rho au^2=0,$$

from which we obtain

$$u=\frac{3\sigma}{2a\rho}\mathrm{sech}^2\left(\frac{1}{2n}\sqrt{\frac{\sigma}{\alpha}}\xi\right),$$

$$v=\frac{3\sigma}{2\rho}\mathrm{sech}^2\left(\frac{1}{2n}\sqrt{\frac{\sigma}{\alpha}}\xi\right),$$

and

$$\Psi=\frac{3\sigma}{2a\rho}\mathrm{sech}^2\left(\frac{1}{2n}\sqrt{\frac{\sigma}{\alpha}}\xi\right)\mathrm{e}^{\mathrm{i}\sigma t},$$
$$\Phi=\frac{3\sigma}{2\rho}\mathrm{sech}^2\left(\frac{1}{2n}\sqrt{\frac{\sigma}{\alpha}}\xi\right),$$

where $\xi=kx+ly+mz$, and a and σ satisfying the relationship $\beta\sigma=\mu^2a^2\rho$. This pair of solutions is very important in scalar nucleons interacting with neutral scalar mesons.

4. Exact solutions to Eqs.(C)

Let

$$\Psi=u(\xi)\mathrm{e}^{\mathrm{i}\eta} \ \text{ and } \ \Phi=v(\xi), \tag{31}$$

where $\eta=ax+bt$ and $\xi=cx+dt$, then

$$\Psi_{xx}=-a^2\mathrm{e}^{\mathrm{i}\eta}u+2\mathrm{i}ac\mathrm{e}^{\mathrm{i}\eta}u'+c^2\mathrm{e}^{\mathrm{i}\eta}u'',$$
$$\Psi_t=\mathrm{i}b\mathrm{e}^{\mathrm{i}\eta}u(\xi)+d\mathrm{e}^{\mathrm{i}\eta}u'. \tag{32}$$

Inserting expressions (31) and (32) into Eq.(8) yields

$$-bu\mathrm{e}^{\mathrm{i}\eta}+\mathrm{i}d\mathrm{e}^{\mathrm{i}\eta}u'-a^2u\mathrm{e}^{\mathrm{i}\eta}+2\mathrm{i}ac\mathrm{e}^{\mathrm{i}\eta}u'+c^2\mathrm{e}^{\mathrm{i}\eta}u''$$
$$-2\lambda u^{\alpha+1}\mathrm{e}^{\mathrm{i}\eta}+2uv\mathrm{e}^{\mathrm{i}\eta}=0,$$

from which we have

$$c^2u''+2uv-(a^2+b)u-2\lambda u^{\alpha+1}=0, \tag{33}$$
$$d=-2ac. \tag{34}$$

From expressions (31), we have

$$\Phi_{tt}=d^2v''=\frac{d^2}{c^2}v_{xx}, \tag{35}$$
$$\Phi_{xx}=c^2v'', \tag{36}$$
$$(|\Psi|^\alpha)_{xx}=(u^\alpha)_{xx}. \tag{37}$$

Inserting expressions (35), (36) and (37) into Eq.(9) gives

$$v=-\frac{c^2}{d^2-c^2}u^\alpha+f, \tag{38}$$

where f is an integral constant. From Eq.(33) and expression (38), we have

$$u(\xi)=\left(-\frac{(\alpha+2)(4a^2-1)(a^2+b-2f)}{4(4a^2\lambda-\lambda-1)}\right)^{\frac{1}{\alpha}}$$
$$\times\mathrm{sech}^{\frac{2}{\alpha}}\left(\frac{\alpha}{2c}\sqrt{a^2+b-2f}\xi\right),$$
$$v(\xi)=\frac{(\alpha+2)(a^2+b-2f)}{4(4a^2\lambda-\lambda-1)}$$
$$\times\mathrm{sech}^2\left(\frac{\alpha}{2c}\sqrt{a^2+b-2f}\xi\right)+f,$$

where $\xi=cx+dt=cx-2act$, that is,

$$\Psi=\mathrm{e}^{\mathrm{i}(ax+bt)}u(\xi) \ \text{ and } \ \Phi=v(\xi).$$

5. Exact solutions to Eqs.(D)

Let $\Psi(x,t) = u(\xi)$ and $\Phi = v(\xi)\mathrm{e}^{\mathrm{i}\eta}$, where $\eta = kx - \omega t$ and $\xi = x - 2\alpha k t$. In a way similar to that in (C), we have

$$(4k^2\alpha^2 - C_\mathrm{s}^2)u'' = \beta(v^\alpha)'',$$

$$\alpha v'' + (\omega - \alpha k^2)v - \delta u v = 0.$$

Then

$$u = \frac{\beta v^\alpha}{4k^2\alpha^2 - C_\mathrm{s}^2} + f,$$

where f is an integral constant. Thus we obtain

$$\alpha v'' + (\omega - k^2\alpha - f\delta)v - \frac{\delta\beta}{4k^2\alpha^2 - C_\mathrm{s}^2}v^{\alpha+1} = 0.$$

Hence

$$v(\xi) = \left(\frac{(f\delta + k^2\alpha - \omega)(4k^2\alpha^2 - C_\mathrm{s}^2)(\alpha+2)}{2\delta\beta}\right)^{\frac{1}{\alpha}} \mathrm{sech}^{\frac{2}{\alpha}}\left(\frac{\alpha}{2}\sqrt{\frac{f\delta + k^2\alpha - \omega}{\alpha}}\xi\right),$$

$$u(\xi) = \frac{\left(f\delta + k^2\alpha - \omega\right)(\alpha+2)}{2\delta}\mathrm{sech}^2\left(\frac{\alpha}{2}\sqrt{\frac{f\delta + k^2\alpha - \omega}{\alpha}}\xi\right) + f,$$

where $\xi = x - 2k\alpha t$, that is,

$$\Psi = u(\xi), \quad \Phi = v(\xi)\mathrm{e}^{\mathrm{i}(kx-\omega t)}.$$

6. Summary

We have obtained some explicit solutions to four CCNDEs by using the above proposed simple but efficient method. Some solutions obtained here are similar to those or generalized solutions in Refs.[13,14,17], and the others are new solutions. We expect that many other CCNDEs can be solved by this method.

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