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Exact periodic wave and soliton solutions in two-component Bose–Einstein condensates*

Li Hua-Mei (李画眉)†

Department of Physics, Zhejiang Normal University, Jinhua 321004, *China*

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We present several families of exact solutions to a system of coupled nonlinear Schrödinger equations. The model describes a binary mixture of two Bose–Einstein condensates in a magnetic trap potential. Using a mapping deformation method, we find exact periodic wave and soliton solutions, including bright and dark soliton pairs.

Keywords: two-component Bose–Einstein condensates, exact solutions, mapping deformation method

PACC: 0340K, 0290, 0230

1. Introduction

The recent progress in experimental and theoretical studies of Bose–Einstein condensates(BECs)[1−7] has made the matter–wave soliton a physically relevant object of research. In particular, the dynamics of multicomponent BECs has been studied in detail. Such multispecies condensates have arisen as mixtures of different spin states in ^{87}Rb[8] and ^{23}Na[9] condensates. In addition, many theoretical studies were devoted to mixtures of different bosonic atomic species, such as Na–Rb,[10] K–Rb,[11,12] Cs–Rb,[13] and Li–Rb.[14] Experimental realization of a two-species BEC, namely the ^{41}K–^{87}Rb mixture, has been reported.[15]

Typically, the relevant model for two coupled BECs is based on a system of coupled Gross–Pitaevskii(GP) equations for the mean-field single-atom wave function of each component, which takes into account the self- and cross-interactions between the species. Assuming that the nonlinear interactions are weak with respect to the confinement in the transverse dimensions, the transverse size of the condensate is much smaller than its length. In this case, the BEC is of a 'cigar-shape', and the GP equations take an effective 1D form[16]

$$\mathrm{i}\frac{\partial u_j}{\partial t} = -\frac{m_1}{m_j}\frac{\partial^2 u_j}{\partial x^2} + \sum_{k=1}^{2} g_{jk}|u_k|^2 u_j + V_j(x)u_j, \quad (j=1,2), \tag{1}$$

where $u_j(x,t)(j=1,2)$ are the mean-field wave functions of the two species, t and x are, respectively, measured in units of $2/\omega_{1\perp}$ and the transverse harmonic-oscillator length $g_{1\perp} \equiv \sqrt{\hbar/(m_1\omega_1)}$ (m_j and $\omega_{j\perp}$ are the atomic mass and transverse confining frequency of each species), while $\hbar\omega_{1\perp}/2$ is the energy unit. The coefficients g_{jk} in Eqs.(1), related to the three scattering lengths α_{jk}(note that $\alpha_{12} = \alpha_{21}$) through $g_{jk} = 4\pi m_1(\alpha_{jk}/g_{1\perp})(m_j + m_k)/(m_j m_k)$, allow for collisions between atoms belonging to the same (g_{jj}) and different ($g_{jk}, j \neq k$) species. In the BEC context the interactions are tunable,[17] especially because they can be modified by means of the Feshbach resonance (i.e. by using magnetic field to affect the sign and magnitude of the scattering length of the interatomic collisions).[18] The Feshbach resonance allows one to switch between attractive and repulsive interaction,[19] and even to switch it periodically in time, by means of an ac magnetic field, which allows one to create a self-confined 2D BECs without using the magnetic trap.[20] Numerous issues have been considered in this framework. Among them are ground-state solutions,[10,21] small-amplitude excitations,[22] formation of domain walls(DWs) between immiscible species,[23−25] bound states of dark–bright[26] and dark–dark,[27] dark–grey, bright–grey, bright–antidark and dark–antidark[28] complexes of solitary waves, spatially periodic states.[29]

It is very difficult to solve a variable coefficient

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† E-mail: lihuamei@zjnu.cn

nonlinear system. Exact solutions in the explicit form for Eqs.(1) are not often available. To our knowledge, only stationary trigonometric solutions of Eqs.(1) have been obtained when $V = V_0 \sin^2(mx)$.[29] However, the construction of the exact solutions of nonlinear partial differential equations (NLPDEs) is one of the most important and essential tasks in nonlinear science. With the help of exact solutions, when they exist, phenomena modelled by these NLPDEs can be better understood. They can also help to select the experimental parameters, to analyse the stability of the solutions and to check numerical analysis for these NLPDEs. In this paper, we aim to study the exact analytical solutions of Eqs.(1). The rest of this paper is organized as follows. In Section 2, we simply introduce the extended mapping deformation method.[30−33] Applying the method, with the aid of Maple, we obtain Jacobian elliptic function solutions of Eqs.(1) in a safe range of parameters for the first time. In Section 3, when the elliptic modulus equals 1, various soliton solutions can be obtained. The last section is devoted to a short summary and discussion.

2. Periodic wave solutions for two-component BECs in a magnetic trap potential

In this work, we consider the case with $m_1 = m_2 \equiv m$, which corresponds to a most experimentally feasible mixture of two different hyperfine states of the same atom species, or, approximately, to different isotopes of the same alkali metal. The trapping potential is assumed to be of the usual parabolic type, customarily used in BEC experiments[1,34,35]

$$V_1(x) = V_2(x) \equiv V(x) = \frac{1}{2}\Omega^2 x^2. \tag{2}$$

In Eq.(2), $\Omega^2 \equiv 2\omega_x^2/\omega_\perp^2 (\omega_{1x} = \omega_{2x} \equiv \omega_x$ are the confining frequencies in the axial direction).

In Ref.[24], we present an extended mapping deformation method to construct periodic wave and soliton solutions of NLPDEs. The key idea in this method is to take full advantage of the solutions of an auxiliary ordinary differential equation. The auxiliary equation reads

$$\phi'^2 = b_0 + b_2\phi^2 + b_4\phi^4, \tag{3}$$

where $\phi' = \mathrm{d}\phi/\mathrm{d}\xi, \xi = \xi(x,t)$, and b_0, b_2 and b_4 are constants. In order to obtain some exact solutions of Eqs.(1), first we make the transformation

$$\begin{aligned} u_1 &= v_1 \exp(\mathrm{i}\theta_1), \\ u_2 &= v_2 \exp(\mathrm{i}\theta_2). \end{aligned} \tag{4}$$

Then substituting expressions (4) into Eqs.(1) and setting the real and imaginary parts of the resulting equations to be zero, we obtain the following sets of partial differential equations:

$$\begin{aligned} &-v_1\theta_{1t} + v_{1xx} - v_1\theta_{1x}^2 - g_{11}v_1^3 - g_{12}v_2^2 v_1 \\ &-\frac{1}{2}\Omega^2 x^2 v_1 = 0 , \end{aligned} \tag{5}$$

$$v_{1t} + v_1\theta_{1xx} + 2v_{1x}\theta_{1x} = 0 , \tag{6}$$

$$\begin{aligned} &-v_2\theta_{2t} + v_{2xx} - v_2\theta_{2x}^2 - g_{22}v_2^3 - g_{12}v_1^2 v_2 \\ &-\frac{1}{2}\Omega^2 x^2 v_2 = 0 , \end{aligned} \tag{7}$$

$$v_{2t} + v_2\theta_{2xx} + 2v_{2x}\theta_{2x} = 0 . \tag{8}$$

Using the standard leading order analysis, the solutions of Eqs.(5)–(8) can be chosen to be in the following form:

$$\begin{aligned} v_1 &= f_1(t) + h_1(t)\phi(\xi), \\ v_2 &= f_2(t) + h_2(t)\phi(\xi), \\ \xi &= k(t)x + \eta(t) , \\ \theta_1 &= \Delta_1(t)x^2 + \Gamma_1(t)x + \Omega_1(t), \\ \theta_2 &= \Delta_2(t)x^2 + \Gamma_2(t)x + \Omega_2(t), \end{aligned} \tag{9}$$

where $f_j(t), h_j(t)$, $k(t)$, $\eta(t)$, $\Delta_j(t)$, $\Gamma_j(t)$ and $\Omega_j(t)$ $(j = 1,2)$ are functions of t to be determined, with $\phi(\xi)$ satisfying Eq.(3). Substituting Eqs.(3) and (9) into Eqs.(5)–(8) and collecting coefficients of power of $x^k\phi^l$, and $\sqrt{b_0 + b_2\phi^2 + b_4\phi^4}$ $(k = 0,1,2;\ l = 0,1,2,3)$ with the aid of Maple, then setting each of the coefficients to be zero, we can obtain a set of overdetermined partial differential equations with respect to unknown derivative functions (g_{jj}, g_{12}, $f_j(t)$, $h_j(t)$, $k(t)$, $\eta(t)$, $\Delta_j(t)$, $\Gamma_j(t)$, $\Omega_j(t)$ $(j = 1,2)$). Solve them by means of Maple, we obtain the following results:

$$\begin{aligned} f_1(t) &= f_2(t) = 0, \\ k(t) &= C_1 s(t), \\ \eta(t) &= -2C_1C_2 \int s(t)^2 \mathrm{d}t + C_3, \\ \Gamma_1(t) &= \Gamma_2(t) = C_2 s(t), \\ \Delta_1(t) &= \Delta_2(t) = -\frac{s'(t)}{4s(t)}, \\ \Omega_1(t) &= (b_2C_1^2 - C_2^2) \int s(t)^2 \mathrm{d}t + C_4, \end{aligned}$$

$$\begin{aligned}
\Omega_2(t) &= (b_2C_1^2 - C_2^2)\int s(t)^2\mathrm{d}t + C_5,\\
h_1(t) &= \pm C_1\sqrt{2b_4 s(t)},\\
h_2(t) &= \pm C_1\sqrt{2b_4 c s(t)},\\
s(t) &= \frac{g_{12}^2 - g_{11}g_{22}}{g_{12} - g_{22}}\\
&= \frac{\sqrt{2}\Omega}{C_6\sin(\sqrt{2}\Omega t) - C_7\cos(\sqrt{2}\Omega t)},\\
c &= \frac{g_{12} - g_{11}}{g_{12} - g_{22}}\quad (c>0),
\end{aligned}\tag{10}$$

where C_1, C_2, C_3, C_4, C_5, C_6 and C_7 are arbitrary constants, and $s'(t) = \dfrac{\mathrm{d}s(t)}{\mathrm{d}t}$. Hence, the solution of Eqs.(1) reads

$$\begin{aligned}
u_1 &= \pm C_1\sqrt{2b_4 s(t)}\phi(\xi)\exp(\mathrm{i}\theta_1),\\
u_2 &= \pm C_1\sqrt{2b_4 c s(t)}\phi(\xi)\exp(\mathrm{i}\theta_2),
\end{aligned}\tag{11}$$

where

$$\begin{aligned}
\xi &= C_1 s(t)x - 2C_1C_2\int s(t)^2\mathrm{d}t + C_3,\\
\theta_1 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x + (b_2C_1^2 - C_2^2)\int s(t)^2\mathrm{d}t\\
&\quad + C_4,\\
\theta_2 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x + (b_2C_1^2 - C_2^2)\int s(t)^2\mathrm{d}t\\
&\quad + C_5,\\
s(t) &= \frac{\sqrt{2}\Omega}{C_6\sin(\sqrt{2}\Omega t) - C_7\cos(\sqrt{2}\Omega t)},
\end{aligned}\tag{12}$$

with ϕ being the solution of Eq.(3). Considering that Eq.(3) has different solutions, so we ought to discuss solutions of Eqs.(1) in several cases.

Family 1 sn wave in BECs with $s(t) > 0$:
As $b_4 = m^2$, $b_2 = -(m^2+1)$, $b_0 = 1$, and $\phi = \mathrm{sn}(\xi, m)$, the periodic sn solution is obtained as

$$\begin{aligned}
u_{11} &= \pm C_1 m\sqrt{2s(t)}\mathrm{sn}(\xi, m)\exp(\mathrm{i}\theta_1)\ ,\\
u_{21} &= \pm C_1 m\sqrt{2cs(t)}\mathrm{sn}(\xi, m)\exp(\mathrm{i}\theta_2)\ ,
\end{aligned}\tag{13}$$

where

$$\begin{aligned}
\xi(x,t) &= C_1 s(t)x - 2C_1C_2\int s(t)^2\mathrm{d}t + C_3,\\
\theta_1 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x\\
&\quad -((1+m^2)C_1^2 + C_2^2)\int s(t)^2\mathrm{d}t + C_4,\\
\theta_2 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x\\
&\quad -((1+m^2)C_1^2 + C_2^2)\int s(t)^2\mathrm{d}t + C_5.
\end{aligned}$$

Family 2 cn wave in BECs with $s(t) < 0$:
As $b_4 = -m^2$, $b_2 = 2m^2 - 1$, $b_0 = 1 - m^2$, and $\phi = \mathrm{cn}(\xi, m)$, the periodic cn solution is obtained as

$$\begin{aligned}
u_{12} &= \pm C_1 m\sqrt{-2s(t)}\mathrm{cn}(\xi, m)\exp(\mathrm{i}\theta_1)\ ,\\
u_{22} &= \pm C_1 m\sqrt{-2cs(t)}\mathrm{cn}(\xi, m)\exp(\mathrm{i}\theta_2)\ ,
\end{aligned}\tag{14}$$

where

$$\begin{aligned}
\xi &= C_1 s(t)x - 2C_1C_2\int s(t)^2\mathrm{d}t + C_3,\\
\theta_1 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x\\
&\quad +((2m^2-1)C_1^2 - C_2^2)\int s(t)^2\mathrm{d}t + C_4,\\
\theta_2 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x\\
&\quad +((2m^2-1)C_1^2 - C_2^2)\int s(t)^2\mathrm{d}t + C_5,
\end{aligned}$$

with $g_{12} < 0$ and $g_{jj} < 0$ $(j = 1, 2)$.

Family 3 dn wave in BECs with $s(t) < 0$:
As $b_4 = -1$, $b_2 = 2 - m^2$, $b_0 = m^2 - 1$, and $\phi = \mathrm{dn}(\xi, m)$, the periodic dn solution is obtained as

$$\begin{aligned}
u_{13} &= C_1\sqrt{-2s(t)}\mathrm{dn}(\xi, m)\exp(\mathrm{i}\theta_1)\ ,\\
u_{23} &= \pm C_1\sqrt{-2cs(t)}\mathrm{dn}(\xi, m)\exp(\mathrm{i}\theta_2)\ ,
\end{aligned}\tag{15}$$

where

$$\begin{aligned}
\xi &= C_1 s(t)x - 2C_1C_2\int s(t)^2\mathrm{d}t + C_3,\\
\theta_1 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x\\
&\quad +((2-m^2)C_1^2 - C_2^2)\int s(t)^2\mathrm{d}t + C_4,\\
\theta_2 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x\\
&\quad +((2-m^2)C_1^2 - C_2^2)\int s(t)^2\mathrm{d}t + C_5,
\end{aligned}$$

with $g_{12} < 0$ and $g_{jj} < 0$ $(j = 1, 2)$.

In families 1–3, $s'(t) = \dfrac{\mathrm{d}s(t)}{\mathrm{d}t}$; $s(t) = \dfrac{\sqrt{2}\Omega}{C_6\sin(\sqrt{2}\Omega t) - C_7\cos(\sqrt{2}\Omega t)}$; C_1, C_2, C_3, C_4, C_5, C_6 and C_7 are arbitrary constants; $m(0 \le m \le 1)$ is the modulus of the elliptic function.

3. Soliton solutions for two-component BECs in a magnetic trap potential

When $m \to 1$ we can obtain the soliton solutions:

Family 1 bright–bright soliton pairs with $s(t) < 0$:

$$\begin{aligned} u_{18} &= \pm C_1\sqrt{-2s(t)}\mathrm{sech}(\xi)\exp(\mathrm{i}\theta_1)\ , \\ u_{28} &= \pm C_1\sqrt{-2cs(t)}\mathrm{sech}(\xi)\exp(\mathrm{i}\theta_2)\ , \end{aligned} \quad (16)$$

where

$$\begin{aligned} \xi &= C_1 s(t)x - 2C_1C_2\int s(t)^2\mathrm{d}t + C_3, \\ \theta_1 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x + (C_1^2 - C_2^2)\int s(t)^2\mathrm{d}t \\ &\quad + C_4, \\ \theta_2 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x + (C_1^2 - C_2^2)\int s(t)^2\mathrm{d}t \\ &\quad + C_5, \end{aligned}$$

with $g_{12} < 0$ and $g_{jj} < 0$ $(j = 1, 2)$.

Family 2 dark–dark soliton pairs with $s(t) > 0$:

$$\begin{aligned} u_{19} &= \pm C_1\sqrt{2s(t)}\tanh(\xi)\exp(\mathrm{i}\theta_1)\ , \\ u_{29} &= \pm C_1\sqrt{2cs(t)}\tanh(\xi)\exp(\mathrm{i}\theta_2)\ , \end{aligned} \quad (17)$$

where

$$\begin{aligned} \xi(x,t) &= C_1 s(t)x - 2C_1C_2\int s(t)^2\mathrm{d}t + C_3, \\ \theta_1 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x \\ &\quad -(2C_1^2 + C_2^2)\int s(t)^2\mathrm{d}t + C_4, \\ \theta_2 &= -\frac{s'(t)}{4s(t)}x^2 + C_2 s(t)x \\ &\quad -(2C_1^2 + C_2^2)\int s(t)^2\mathrm{d}t + C_5. \end{aligned}$$

4. Summary

In this paper, we have presented several families of exact solutions for two-component nonlinear Schrödinger/Gross–Pitaevskii equations in the presence of a magnetic trap potential. Using a mapping deformation method, we have successfully obtained three families: typical Jacobian elliptic function solutions of Eqs.(1) in a safe range of parameters. Particular cases of these solutions, where the elliptic modulus equals 1, are bright and dark soliton pairs. Our results describe the dynamics of the Feshbach resonance managed bright and dark soliton pairs of two-component BECs in a magnetic trap potential.

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