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Lie symmetrical perturbation and adiabatic invariants of generalized Hojman type for Lagrange systems*

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Based on the invariance of differential equations under infinitesimal transformations of group, Lie symmetries, exact invariants, perturbation to the symmetries and adiabatic invariants in form of non-Noether for a Lagrange system are presented. Firstly, the exact invariants of generalized Hojman type led directly by Lie symmetries for a Lagrange system without perturbations are given. Then, on the basis of the concepts of Lie symmetries and higher order adiabatic invariants of a mechanical system, the perturbation of Lie symmetries for the system with the action of small disturbance is investigated, the adiabatic invariants of generalized Hojman type for the system are directly obtained, the conditions for existence of the adiabatic invariants and their forms are proved. Finally an example is presented to illustrate these results.

Keywords: Lagrange system, Lie symmetrical perturbation, exact invariant, adiabatic invariant of generalized Hojman type

PACC: 0320, 0220, 0350, 1130

1. Introduction

Symmetries and conserved quantities of mechanical systems play an important role in the fields of mathematics, mechanics and physics. People have paid more and more attention to the research and many results have been obtained. These results develop our knowledge about the essence of mechanical system.[1−14] Lie symmetry is an invariance of the differential equations under the infinitesimal transformations of group, and the study of it has been very active in recent years.[15−20] In general, a Lie symmetry does not imply a conserved quantity. If and only if the infinitesimal transformations of Lie symmetries satisfy the structure equation, i.e. Noether identity, the Lie symmetries can lead to a conserved quantity, which is then called the Noetherian. Pioneering in this area, Hojman gave the direct method of finding conserved quantities by using Lie symmetries.[21] This direct method is generalized by Gonálze-Gascón[22] and Lutzky.[23,24] Pillay and Leach[25] demonstrated that the invariant derived from Hojman's conservation law is trivial for all Noether symmetries. Therefore, by using Hojman's method one can find the non-trivial conserved quantity which is derived by the non-Noether symmetries.[26,27] The conserved quantity is called the non-Noetherian. In recent years, the Lie symmetrical non-Noether conserved quantities of some dynamical systems have been studied by using this direct method in Refs.[28–33], and a series of important results are obtained.

It is well known that even tiny changes in symmetries, so-called perturbation to symmetries, are of great importance for physical systems.[34−36] Pioneering in this area, Burgers proposed adiabatic invariants for a special kind of Hamilton systems.[37] A classical adiabatic invariant is a certain physical quantity that changes more slowly than some other parameter of the system which varies very slowly. For a mechanical system, there exists an intimate relation

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between the integrability of the system and the variations of its symmetries and invariants under the action of small disturbance, and therefore the researches on perturbation of symmetries and adiabatic invariants are of great significance. Kruskal[38] and Djukic[39] also made great progresses and obtained a series of important results. Later, studies of adiabatic invariants have become very active in many areas such as mechanics, atomic and molecular physics, Astro- physics and engineering (see Refs.[40–47]). But the adiabatic invariants obtained by all these studies belong to the form of Noether. Recently, Zhang and Mei gave the adiabatic invariants of a Hojman type of a dynamical system.[48−51]

In this paper, the authors will discuss Lie symmetries, exact invariants, perturbation to symmetries and adiabatic invariants in the form of non-Noether for a Lagrange system. Firstly, we give the exact invariants of generalized Hojman type directly induced by Lie symmetries for a Lagrange system without perturbations. Then, on the basis of the concepts of Lie symmetries and higher order adiabatic invariants of a mechanical system, we investigate the perturbation of Lie symmetries for the system with the action of small disturbance, directly obtain a new adiabatic invariant of generalized Hojman type for the system, and prove the conditions for existence of the adiabatic invariants and their forms. Finally, we present an example to illustrate these results.

2. Exact invariants of generalized Hojman type for a Lagrange system

Suppose that the position of a mechanical system is determined by n generalized coordinates $q_s (s = 1, \cdots, n)$. If the differential equations of motion of the mechanical system can be expressed in the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0 \qquad (s = 1, \cdots, n), \tag{1}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is Lagrangian function, then the system is called Lagrange system.

Suppose that the system is not singular, then all the generalized accelerations can be solved from Eq.(1)

$$\ddot{q}_s = h_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \cdots, n). \tag{2}$$

Introduce a single-parameter infinitesimal transformation with respect to time and coordinates

$$\begin{aligned} t^* &= t + \varepsilon\tau^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \end{aligned} \tag{3}$$

where ε is an infinitesimal parameter, τ^0 and ξ_s^0 are infinitesimal generators. Introduce an infinitesimal generator vector

$$\mathrm{X}^{(0)} = \tau^0\frac{\partial}{\partial t} + \xi_s^0\frac{\partial}{\partial q_s}, \tag{4}$$

and its first extension

$$\mathrm{X}^{(1)} = \tau^0\frac{\partial}{\partial t} + \xi_s^0\frac{\partial}{\partial q_s} + (\dot{\xi}_s^0 - \dot{q}_s\dot{\tau}^0)\frac{\partial}{\partial \dot{q}_s}. \tag{5}$$

The criteria of Lie symmetries for Lagrange system are

$$\ddot{\xi}_s^0 - \dot{q}_s\ddot{\tau}^0 - 2\dot{\tau}^0 h_s = \mathrm{X}^{(1)}(h_s) \quad (s = 1, \cdots, n). \tag{6}$$

The infinitesimal generators τ^0 and ξ_s^0 have Lie symmetries, when τ^0 and ξ_s^0 satisfy the determining equations (6).

Lie symmetries do not always imply exact invariants. The following theorem gives the conditions of the existence of exact invariants of generalized Hojman type directly induced by Lie symmetries for a Lagrange system without perturbations and their forms.

Theorem 1 For the infinitesimal generators τ^0 and ξ_s^0 satisfying the determining equations (6), if there exists a function $\lambda_0 = \lambda_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\frac{\partial h_s}{\partial \dot{q}_s} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda_0 = 0 \quad (s = 1, \cdots n), \tag{7}$$

then the Lagrange system possesses the following exact invariants

$$\begin{aligned} I_0 = {} & \frac{1}{\lambda_0}\frac{\partial}{\partial t}(\lambda_0\tau^0) + \frac{1}{\lambda_0}\frac{\partial}{\partial q_s}(\lambda_0\xi_s^0) \\ & + \frac{1}{\lambda_0}\frac{\partial}{\partial \dot{q}_s}[\lambda_0(\dot{\xi}_s^0 - \dot{q}_s\dot{\tau}^0)] - \dot{\tau}^0. \end{aligned} \tag{8}$$

Proof Differentiating I_0 with respect to time t, and taking the partial derivations of determining equations (6) for $\dot{q}_s$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}I_0 = {} & \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial t}\tau^0 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_s^0}{\partial q_s} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}(\dot{\xi}_s^0 \\ & - \dot{q}_s\dot{\tau}^0) - \ddot{\tau}^0 + \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{X}_0^{(1)}(\ln\lambda_0) \\ = {} & \mathrm{X}_0^{(1)}\left(\frac{\partial h_s}{\partial \dot{q}_s} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda_0\right) \\ & + \dot{\tau}^0\left(\frac{\partial h_s}{\partial \dot{q}_s} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda_0\right) \\ = {} & 0. \end{aligned}$$

Therefore, the system has the exact invariants (8).

Theorem 2[1] For the infinitesimal generators τ^0 and ξ_s^0 satisfying the determining equations (6), if there exists a gauge function $G^0 = G^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following structure equation

$$L\dot{\tau}^0 + \mathrm{X}^{(1)}(L) + \dot{G}^0 = 0, \tag{9}$$

and then the Lie symmetries of the Lagrange system are as the Noetherian one.

Theorem 3[31] For the infinitesimal generators τ^0 and ξ_s^0 satisfying the determining equations (6) and (9), if there exists a function $\lambda_0 = \lambda_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\lambda_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}, \tag{10}$$

then the exact invariants (8) is trivial.

3. Perturbation to symmetries and adiabatic invariants of generalized Hojman type for a Lagrange system

Small forces acting on the Lagrange system may result in a small change in its symmetries, defined as symmetrical perturbation. The conserved quantity associated with the symmetries, under the corresponding change, is an adiabatic invariant. Now, we present the concept of higher order adiabatic invariant.

Definition 1 If $I_z(t, \boldsymbol{q}, \dot{q}, \varepsilon)$ is a physical quantity including ε in which the highest power is z in a mechanical system, and its derivative with respect to time t is in direct proportion to ε^{z+1}, then I_z is called a zth order adiabatic invariant of the mechanical system.

Suppose the Lagrange system corresponding to Eq.(1) is perturbed by small quantities εW_s, then the equations of motion of the system become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \varepsilon W_s \quad (s = 1, \cdots, n). \tag{11}$$

Expanding Eqs.(11), we have

$$\ddot{q}_s = h_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon \frac{M_{sk}}{D} W_k, \tag{12}$$

where $D = \det\left[\partial^2 L/\partial \dot{q}_s \partial \dot{q}_k\right]$, M_{sk} are the cofactors of the matrix element $\partial^2 L/\partial \dot{q}_s \partial \dot{q}_k$.

Due to the action of εW_s, the primary symmetries and invariants of the system may vary. Suppose the variation is a small perturbation based on the symmetrical transformations of the system without perturbation, and $\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ express the generators of time and space respectively after being perturbed, then

$$\begin{aligned} \tau &= \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots, \\ \xi_s &= \xi_s^0 + \varepsilon\xi_s^1 + \varepsilon^2\xi_s^2 + \cdots, \end{aligned} \tag{13}$$

The infinitesimal generator vector and its first extended vector are

$$\mathrm{X}^{(0)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s}, \tag{14}$$

$$\mathrm{X}^{(1)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s}. \tag{15}$$

Substituting Eqs.(13) into vector (15), we have

$$\mathrm{X}^{(1)} = \varepsilon^m \mathrm{X}_m^{(1)} \quad (m = 0, 1, 2, \cdots). \tag{16}$$

where

$$\begin{aligned} \mathrm{X}_m^{(1)} = {} & \tau^m\frac{\partial}{\partial t} + \xi_s^m\frac{\partial}{\partial q_s} + (\dot{\xi}_s^m - \dot{q}_s\dot{\tau}^m)\frac{\partial}{\partial \dot{q}_s} \\ & (m = 0, 1, 2, \cdots). \end{aligned} \tag{17}$$

After the action of small forces of perturbation, the invariance of Eqs.(12), under the infinitesimal transformations leads to the following Lie symmetrical determining equations

$$\begin{aligned} & \ddot{\xi}_s - \dot{q}_s\ddot{\tau} - 2\dot{\tau}\left(h_s + \varepsilon\frac{M_{sk}}{D}W_k\right) \\ = {} & \mathrm{X}^{(1)}(h_s) + \varepsilon\mathrm{X}^{(1)}\left(\frac{M_{sk}}{D}W_k\right). \end{aligned} \tag{18}$$

Substituting Eqs.(13) and (16) into Eqs.(18), and comparing the coefficients of ε^m on the left side of equality with the right side, we obtain

$$\begin{aligned} & \ddot{\xi}_s^m - \dot{q}_s\ddot{\tau}^m - 2\dot{\tau}^m h_s - 2\dot{\tau}^{m-1}\frac{M_{sk}}{D}W_k \\ = {} & \mathrm{X}_m^{(1)}(h_s) + \mathrm{X}_{m-1}^{(1)}\left(\frac{M_{sk}}{D}W_k\right), \end{aligned} \tag{19}$$

where we appoint $\tau^{-1} = 0$ and $\xi_s^{-1} = 0$ when $m = 0$.

Definition 2 For a mechanical system disturbed by small forces of perturbation εW_s, if the infinitesimal generators τ^m and ξ_s^m satisfy the determining equations (19), then the invariances are called the Lie symmetries of the corresponding disturbed system.

Theorem 4 For a Lagrange system perturbed by small quantities εW_s, if the generators $\tau^m(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\xi_s^m(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ of infinitesimal transformations satisfy the determining equations (19), and there exists a function $\lambda=\lambda(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ such that

$$\frac{\partial h_s}{\partial\dot{q}_s}+\varepsilon\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)+\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda=0, \tag{20}$$

then the corresponding disturbed system possesses the following adiabatic invariants:

$$I_z=\sum_{m=0}^{z}\varepsilon^m\left[\frac{1}{\lambda}\frac{\partial}{\partial t}(\lambda\tau^m)+\frac{1}{\lambda}\frac{\partial}{\partial q_s}(\lambda\xi_s^m)+\frac{1}{\lambda}\frac{\partial}{\partial\dot{q}_s}[\lambda(\dot{\xi}_s^m-\dot{q}_s\dot{\tau}^m)]-\dot{\tau}^m\right], \tag{21}$$

where we appoint $\lambda=\lambda_0$ when $m=0$.

Proof Differentiating I_z with respect to time t, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}I_z=\sum_{m=0}^{z}\varepsilon^m\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial t}\tau^m+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s^m}{\partial q_s}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s^m-\dot{q}_s\dot{\tau}^m)-\ddot{\tau}^m+\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{X}_m^{(1)}(\ln\lambda)\right]. \tag{22}$$

For an arbitrary function $\phi(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, if the infinitesimal generators τ and ξ_s satisfy the determining equations (18) of the system, we can easily verify that there exists a relation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{X}^{(1)}(\phi)=\mathrm{X}^{(1)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi\right)+\dot{\tau}\frac{\mathrm{d}}{\mathrm{d}t}\phi. \tag{23}$$

Substituting Eqs.(13) and (16) into Eq.(23), and letting $\phi=\ln\lambda$ and comparing the coefficients of ε^m on the left side of equality with the right side, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{X}_m^{(1)}(\ln\lambda)=\mathrm{X}_m^{(1)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda\right)+\dot{\tau}^m\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda \quad (m=0,1,2,\cdots). \tag{24}$$

Taking the partial derivations of the determining equations (18) for $\dot{q}_s$, we can obtain, by way of simple deduction,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\tau}{\partial t}+\frac{\partial\xi_s}{\partial q_s}+\frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\tau})\right]-\ddot{\tau}=\dot{\tau}\frac{\partial}{\partial\dot{q}_s}\left(h_s+\varepsilon\frac{M_{sk}}{D}W_k\right)+\mathrm{X}^{(1)}\left(\frac{\partial h_s}{\partial\dot{q}_s}\right)+\varepsilon\mathrm{X}^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]. \tag{25}$$

Substituting Eqs.(13) and (16) into Eq.(25), and letting the coefficients of ε^m on the two sides of equality be equal each other, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\tau^m}{\partial t}+\frac{\partial\xi_s^m}{\partial q_s}+\frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s^m-\dot{q}_s\dot{\tau}^m)\right]-\ddot{\tau}^m\\&=\dot{\tau}^m\frac{\partial h_s}{\partial\dot{q}_s}+\dot{\tau}^{m-1}\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)+\mathrm{X}_m^{(1)}\left(\frac{\partial h_s}{\partial\dot{q}_s}\right)+\mathrm{X}_{m-1}^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]\quad(m=0,1,2,\cdots).\end{aligned} \tag{26}$$

Substituting Eqs.(24) and (25) into Eq.(22), and using Eq.(20), we obtain

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t}&=\sum_{m=0}^{z}\varepsilon^m\left\{\dot{\tau}^m\frac{\partial h_s}{\partial\dot{q}_s}+\dot{\tau}^{m-1}\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)+\mathrm{X}_m^{(1)}\left(\frac{\partial h_s}{\partial\dot{q}_s}\right)+\mathrm{X}_{m-1}^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]\right.\\&\quad\left.+\mathrm{X}_m^{(1)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda\right)+\dot{\tau}^m\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda\right\}\\&=\sum_{m=0}^{z}\varepsilon^m\left\{\mathrm{X}_{m-1}^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]-\varepsilon\mathrm{X}_m^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]+\dot{\tau}^{m-1}\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right.\\&\quad\left.-\varepsilon\dot{\tau}^m\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right\}\\&=-\varepsilon^{z+1}\left\{\mathrm{X}_z^{(1)}\left[\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right]+\dot{\tau}^z\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{sk}}{D}W_k\right)\right\}.\end{aligned} \tag{27}$$

This shows that $\mathrm{d}I_z/\mathrm{d}t$ is in direct proportion to ε^{z+1}, so I_z is a zth order adiabatic invariant of the system.

By using Theorems 2 and 3, we can give the following theorems.

Theorem 5 For a nonsingular Lagrange system perturbed by small quantities εW_s, if the generators $\tau^m(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\xi_s^m(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ of infinitesimal transformations satisfy the determining equations (19), and there exists a gauge function $G^m(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfying the following structure equation:

$$L\dot{\tau}^m + \mathrm{X}_m^{(1)}(L) + W_s(\xi_s^{m-1} - \dot{q}_s\tau^{m-1}) + \dot{G}^m = 0 \quad (m = 0,1,2,\cdots), \tag{28}$$

then the Lie symmetries of Lagrange system perturbed by small quantities εW_s are as the Noetherian one.

Theorem 6 For a nonsingular Lagrange system perturbed by small quantities εW_s, if the generators $\tau^m(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ and $\xi_s^m(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ of infinitesimal transformations satisfy the determining equations (19) and (28), and there exists a function $\lambda = \lambda(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ such that

$$\lambda(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_k}, \tag{29}$$

then the zth order adiabatic invariants (21) is trivial.

4. An illustrative example

We try to study the exact invariants and adiabatic invariants of the planar Kepler problem. The Lagrangian function of the planar motion of Kepler problem is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \mu(q_1^2 + q_2^2)^{-1/2}, \tag{30}$$

and the equations of motion are

$$\begin{aligned}\ddot{q}_1 &= -\mu q_1(q_1^2+q_2^2)^{-3/2} = h_1,\\ \ddot{q}_2 &= -\mu q_2(q_1^2+q_2^2)^{-3/2} = h_2.\end{aligned} \tag{31}$$

It is a holonomic conservative system.

Firstly, we seek the exact invariant. The determining equation (6) gives

$$\begin{aligned}\ddot{\xi}_1^0 - \dot{q}_1\ddot{\tau}^0 - 2\dot{\tau}^0 h_1 &= \frac{\partial h_1}{\partial q_1}\xi_1^0 + \frac{\partial h_1}{\partial q_2}\xi_2^0,\\ \ddot{\xi}_2^0 - \dot{q}_2\ddot{\tau}^0 - 2\dot{\tau}^0 h_2 &= \frac{\partial h_2}{\partial q_1}\xi_1^0 + \frac{\partial h_2}{\partial q_2}\xi_2^0.\end{aligned} \tag{32}$$

They have the following solutions:

$$\tau^0 = -1,\ \xi_1^0 = \xi_2^0 = 0, \tag{33}$$

$$\tau^0 = 0,\ \xi_1^0 = -q_2, \quad \xi_2^0 = q_1, \tag{34}$$

$$\tau^0 = t,\ \xi_1^0 = \frac{2}{3}q_1, \quad \xi_2^0 = \frac{2}{3}q_2. \tag{35}$$

The condition (7) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda_0 = 0; \tag{36}$$

it has a solution

$$\lambda_0 = (q_1\dot{q}_2 - q_2\dot{q}_1). \tag{37}$$

So Eqs.(35) and (37) give the following exact invariant according to Theorem 1,

$$I_0 = 2 - 2(q_1\dot{q}_2 - q_2\dot{q}_1)^{-1} = \text{const.} \tag{38}$$

Equations (33), (34) and (35) correspond to Lie symmetries of the system; Equations (33) and (34) satisfy the structure equation (9), i.e. the Lie symmetries are Noetherian ones. But, Eq.(37) does not satisfy Theorem 3, therefore the corresponding exact invariants (8) are all non-trivial. We studied only the exact invariant and adiabatic invariants of the system corresponding to generators (35).

Finally, we study the adiabatic invariants of the system corresponding to generators (35). Suppose the system is perturbed by the small quantities as follows:

$$\varepsilon W_1 = -\varepsilon q_1, \quad \varepsilon W_2 = -\varepsilon q_2. \tag{39}$$

The determining equation (19) gives

$$\begin{aligned}&\ddot{\xi}_1^m - \dot{q}_1\ddot{\tau}^m - 2\dot{\tau}^m h_1 - 2\dot{\tau}^{m-1}W_1\\ &= \frac{\partial h_1}{\partial q_1}\xi_1^m + \frac{\partial h_1}{\partial q_2}\xi_2^m + \frac{\partial W_1}{\partial q_1}\xi_1^{m-1},\\ &\ddot{\xi}_2^m - \dot{q}_2\ddot{\tau}^m - 2\dot{\tau}^m h_2 - 2\dot{\tau}^{m-1}W_2\\ &= \frac{\partial h_2}{\partial q_1}\xi_1^m + \frac{\partial h_2}{\partial q_2}\xi_2^m + \frac{\partial W_2}{\partial q_2}\xi_2^{m-1}.\end{aligned} \tag{40}$$

They have the following solutions:

$$\begin{aligned}&\tau^0 = \tau^1 = t, \ \ \xi_1^0 = \xi_1^1 = \frac{2}{3}q_1, \quad \xi_2^0 = \frac{2}{3}q_2,\\ &\xi_2^1 = \frac{2}{3}q_2 + \frac{1}{q_1}\sin\sqrt{\mu}(q_1^2+q_2^2)^{-3/4}t.\end{aligned} \tag{41}$$

The condition (20) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda = 0; \tag{42}$$

it has a solution

$$\lambda = \varepsilon(q_1\dot{q}_2 - q_2\dot{q}_1). \tag{43}$$

So Eqs.(41) and (43) give the following first-order adiabatic invariant according to Theorem 4:

$$I_1 = 2 - 2(q_1\dot{q}_2 - q_2\dot{q}_1)^{-1} - \varepsilon\{2\dot{q}_1 q_1^{-1}(q_1\dot{q}_2 - q_2\dot{q}_1)^{-1}\sin\sqrt{\mu}(q_1^2+q_2^2)^{-3/4}t$$
$$+2\sqrt{\mu}(q_1^2+q_2^2)^{1/4}t\left[q_1 q_2^{-1}/4 + (q_1\dot{q}_1 + q_2\dot{q}_2)(q_1\dot{q}_2 - q_2\dot{q}_1)^{-1}\right]\cos\sqrt{\mu}(q_1^2+q_2^2)^{-3/4}t$$
$$-\sqrt{\mu}(q_1^2+q_2^2)^{-3/4}(q_1\dot{q}_2 - q_2\dot{q}_1)^{-1} + 3\}. \tag{44}$$

Moreover we can obtain more higher-order adiabatic invariants.

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