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Conserved quantities and symmetries related to stochastic Hamiltonian systems*

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In this paper symmetries and conservation laws for stochastic dynamical systems are discussed in detail. Determining equations for infinitesimal approximate symmetries of Ito and Stratonovich dynamical systems are derived. It shows how to derive conserved quantities for stochastic dynamical systems by using their symmetries without recourse to Noether's theorem.

Keywords: stochastic dynamical systems, symmetries and conserved quantities, Ito and Stratanovich dynamical systems

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1. Introduction

The subject of symmetries and conservation theorems are very important in mathematics and physics. Conserved quantities, also known as first integrals, can simplify the solutions of dynamical systems. Sophus Lie introduced the notion of continuous groups, now known as Lie groups. Lie showed that the order of an ordinary differential equation could be reduced by one, if it is invariant under a one-parameter Lie group of point transformations.[1] There has been a growing interest in the literature to extend the Lie's theorems to stochastic differential equations (see for instance Refs.[2–9]). A restricted definition of symmetry for Stratonovich dynamical systems was given in Refs.[2] and [3]. Furthermore, a theorem on the construction of conserved quantities from the symmetries has been given. A more general definition of symmetry for Stratonovich dynamical systems has appeared in Ref.[4]. We noticed that conserved quantities can be obtained from the symmetries without resort to Noether's theorem.

In this paper we shall study the symmetries and their associated conserved quantities of the stochastic Hamiltonian systems. We introduce the motion equation of constrained Hamiltonian systems excited by Gaussian white noise, and then we apply results on symmetries of stochastic differential equations to investigate conserved quantities for stochastic Hamiltonian systems. New conserved quantities can be obtained by applying symmetry operator to known conserved quantities. In order to make the paper self-contained, we will begin summarizing the most important results about symmetries for stochastic dynamical systems and their related conserved quantities. For more detailed information see Refs.[2] and [3]. Finally, the results have been illustrated on some problems.

2. Symmetries and conservation laws from the deterministic dynamical systems to the stochastic dynamical systems

It is well known that Noether's theorem[10] gives a very useful tool to explore Lagrangian symmetries and construct related conserved quantities. Furthermore, the symmetries of the equations of motion constitute a larger set, which includes the symmetries of the Lagrangian, and also allows us to build the conserved quantities in a systematic way.[11] Hojman has proposed a quite different method for deriving a conserved quantity from a symmetry transformation vector of the equations of motion only. Neither a Lagrangian nor a Hamiltonian structure of the dynamical system is required to obtain the conservation law. The Hojman's theorem is delineated as follows. Consider a set

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of second-order differential equations

$$\ddot{q}^i - F^i\left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) = 0, \quad (i = 1, \cdots, n). \tag{1}$$

Assume that there exists a function $\lambda = \lambda(\boldsymbol{q})$ such that

$$\frac{\partial F^i}{\partial \dot{q}^i} + \frac{\mathrm{D}}{\mathrm{D}t}(\ln \lambda) = 0 \tag{2}$$

with

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \dot{q}_i \frac{\partial}{\partial q_i} + F_i \frac{\partial}{\partial \dot{q}_i}. \tag{3}$$

If $\xi^i = \xi^i(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ is a symmetry vector for Eqs.(1), that is, if ξ^i satisfies

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\mathrm{D}}{\mathrm{D}t}\xi^i\right) - \frac{\partial F^i}{\partial q^j}\xi^j - \frac{\partial F^i}{\partial \dot{q}^j}\frac{\mathrm{D}}{\mathrm{D}t}\xi^j = 0, \tag{4}$$

then

$$I = \frac{1}{\lambda}\frac{\partial\left(\lambda \xi^i\right)}{\partial q^i} + \frac{1}{\lambda}\frac{\partial}{\partial \dot{q}^i}\left(\lambda \frac{\mathrm{D}\xi^i}{\mathrm{D}t}\right) \tag{5}$$

is a conserved quantity for Eqs.(1).

Much discussions have been devoted to the theory of conserved quantities and symmetry for deterministic dynamical systems.[12−23] Hence it is natural to formulate these notions for stochastic dynamical systems described by stochastic differential equations. The Lagrangian or the Hamiltonian structures usually fail to exist in the stochastic dynamical systems. Therefore, it is significant to formulate an analogous conservation law to that of Hojman[24] for stochastic dynamical systems. Misawa has proposed an elementary theory of conserved quantities and symmetry for the stochastic dynamical systems.[2] Within the framework, a new conservation law is constructed by the infinitesimal symmetry transformation of stochastic dynamical systems.[3] The new conservation law for stochastic dynamical systems is a similar theorem to that of Hojman's work.

We consider the stochastic dynamical systems of Stratonovich type[25] described by the following n-dimensional vector-valued stochastic differential equations:

$$\begin{aligned} &\mathrm{d}\boldsymbol{x}_t = \boldsymbol{b}(\boldsymbol{x}_t, t)\,\mathrm{d}t + \sum_{r=1}^{m} \boldsymbol{g}_r(\boldsymbol{x}_t, t) \circ \mathrm{d}w_t^r, \\ &\boldsymbol{x}_{t_0} = \boldsymbol{c}, \quad t \in [t_0, T], \end{aligned} \tag{6}$$

where $\boldsymbol{x}_t$ is a $\mathbb{R}^n$-valued stochastic process, $\boldsymbol{w}_t = (w_t^r)_{r=1}^m$ is a $\mathbb{R}^m$-valued standard Wiener process, $\boldsymbol{c}$ is a constant n-valued vector, $\boldsymbol{b} = \left(b^i\right)_{i=1}^n$ and $\boldsymbol{g}_r = \left(g_r^i\right)_{i=1}^n$ are $\mathbb{R}^n$-valued smooth functions, respectively. The meaning of $\boldsymbol{g}_r(\boldsymbol{x}_t, t) \circ \mathrm{d}\omega_t^r$ is also given in Ref.[25]. For convenience, differentiability of any function in this article is assumed to be of sufficiently high order. Now, we repeat here the fundamental notions and conclusions for conserved quantities and symmetry for the system (6).[2,3]

Let I be a smooth function on $\mathbb{R}^n \times \mathbb{R}^1$, then I is called a conserved quantity of a stochastic dynamical system (6) if it satisfies

$$\begin{aligned} &\left(\partial_t + X_0\right) I(\boldsymbol{x}, t) = 0, \quad X_r I(\boldsymbol{x}, t) = 0, \\ &(r = 1, \cdots, m), \end{aligned} \tag{7}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad X_0 = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x_i}, \quad X_r = \sum_{i=1}^{n} g_r^i \frac{\partial}{\partial x_i}. \tag{8}$$

Indeed, using the formula with respect to the change of variables for stochastic differential equations of Stratonovich type and Eqs.(7), we obtain the following equation:

$$\begin{aligned} \mathrm{d}I(\boldsymbol{x}_t, t) =\ & \left(\partial_t + X_0\right) I(\boldsymbol{x}_t, t)\,\mathrm{d}t \\ & + \sum_{r=1}^{m} X_r I(\boldsymbol{x}_t, t) \circ \mathrm{d}w_t^r = 0. \end{aligned} \tag{9}$$

This indicates that '$I(\boldsymbol{x}, t) =$ constant' holds on the diffusion process $\boldsymbol{x}_t$, the constant is independent of t, but possibly depends on the initial condition $\boldsymbol{x}_0$. If the initial condition $\boldsymbol{x}_0$ is taken to be deterministic, then I is a deterministic quantity which is independent of time.

Now we proceed to define the symmetry of stochastic differential equations (6) in a similar way to that of the one of deterministic differential equations. Let $y = \varphi(\boldsymbol{x}, t)$ be a transformation from $\mathbb{R}^n \times \mathbb{R}^1$ to $\mathbb{R}^n$. We say that φ is a symmetry transformation for the stochastic dynamical system (6) if the function satisfies

$$\begin{aligned} &\boldsymbol{b}(\varphi(\boldsymbol{x}, t), t) = \left(\partial_t + X_0\right)\varphi(\boldsymbol{x}, t), \\ &\boldsymbol{g}_r(\varphi(\boldsymbol{x}, t), t) = X_r\,\varphi(\boldsymbol{x}, t), \ (r = 1, \cdots, m). \end{aligned} \tag{10}$$

Then, using the formula for the change of variables and Eqs.(10), we have the following stochastic differential equation describing the process $y_t = \varphi(x_t, t)$

$$\begin{aligned} \mathrm{d}\boldsymbol{y}_t =\ & \left(\partial_t + X_0\right)\varphi(\boldsymbol{x}_t, t)\,\mathrm{d}t + \sum_{r=1}^{m} X_r\,\varphi(\boldsymbol{x}_t, t) \circ \mathrm{d}w_t^r \\ =\ & \boldsymbol{b}(\boldsymbol{y}_t, t)\,\mathrm{d}t + \sum_{r=1}^{m} \boldsymbol{g}_r(\boldsymbol{y}_t, t) \circ \mathrm{d}w_t^r. \end{aligned} \tag{11}$$

We next define the symmetry operators of Eqs.(6) on the basis of the above symmetry transformation. Consider the following operator,

$$Y=\sum_{i=1}^{n}\xi^{i}\left(\boldsymbol{x},t\right)\frac{\partial}{\partial x^{i}}, \tag{12}$$

where $\boldsymbol{\xi}=\left(\xi^{i}\right)_{i=1}^{n}$ is a $\mathbb{R}^{n}$-valued smooth function. The operator Y is said to be a symmetry operator for Eqs.(6) if and only if it satisfies

$$\begin{aligned}&\left[\partial_{t}+X_{0},\ Y\right]=0,\quad \left[X_{r},\ Y\right]=0,\\&\left(r=1,\cdots,m\right),\end{aligned} \tag{13}$$

where $[\cdot\ ,\ \cdot]$ denotes Lie bracket. Moreover, let

$$\boldsymbol{y}=\varphi\left(\boldsymbol{x},t;\varepsilon\right) \tag{14}$$

be a local one-parameter transformation generated by Y, where ε is a parameter appeared in $T=(-a_{0},a_{0})$ and $\varphi\left(x,t;0\right)=x$. It is also said that the one-parameter transformation (14) generated by a symmetry operator (12) is a symmetry transformation of stochastic dynamical system (6).

In particular, we further define an infinitesimal transformation generated by Y as follows:

$$\varphi\left(\boldsymbol{x},t;\varepsilon\right)=\boldsymbol{x}+\varepsilon\boldsymbol{\xi}\left(\boldsymbol{x},t\right), \tag{15}$$

where ε is an infinitesimal parameter. We also say that the stochastic dynamical system (6) is invariant up to ε-terms under the infinitesimal transformation generated by operator Y if and only if Y is a symmetry operator for (6). Through Eqs.(7) and (13), we can straightforwardly prove the following theorem:

Theorem 1[2] Suppose that $I=I\left(\boldsymbol{x},t\right)$ is a conserved quantity and Y is a symmetry operator for a stochastic system described by Eqs.(6). Then, $YI\left(\boldsymbol{x},t\right)$ is also a conserved quantity for the system.

A very similar theorem to that of Hojman within the framework of stochastic dynamical systems is formulated as follows:

Theorem 2[3] For given stochastic dynamical system (6), assume that there exists a function $\rho=\rho\left(\boldsymbol{x},t\right)$ satisfying

$$\begin{aligned}&\mathrm{d}\,\mathrm{iv}\boldsymbol{b}+\left(\partial_{t}+X_{0}\right)\rho=0,\quad \mathrm{div}\boldsymbol{g}_{r}+X_{r}\rho=0,\\&\left(r=1,\cdots,m\right),\end{aligned} \tag{16}$$

where X_{0} and X_{r} are the operators given by Eq.(8). Then the function $\boldsymbol{\xi}=\left(\xi^{i}\right)_{i=1}^{n}$ in a symmetry operator $Y=\sum\limits_{i=1}^{n}\xi^{i}\left(\boldsymbol{x},t\right)\dfrac{\partial}{\partial x^{i}}$ for stochastic system (6) yields the following conserved quantity for the system

$$I=\mathrm{div}\boldsymbol{\xi}+Y\rho. \tag{17}$$

3. The motion equation of stochastic Hamiltonian systems

Assume that the configuration of the dynamical system is specified by n generalized coordinates $q_{s}\,(s=1,\cdots,n)$ and the system is subjected to g ideal, nonholonomic constraints of Chetaev's form:[26]

$$f_{\beta}\left(\dot{\boldsymbol{q}},\boldsymbol{q}\right)=0,\qquad\left(\beta=1,\cdots,g\right). \tag{18}$$

The generalized forces imposed on the system are

$$\bar{Q}_{s}=Q_{s}+\sum_{i=1}^{m}\sigma_{si}\left(\boldsymbol{q}\right)\xi_{i}\left(t\right)+F_{s},\ \ \left(s=1,\cdots,n\right), \tag{19}$$

where Q_{s} include generalized forces of the usual meaning, and $\xi_{i}\left(t\right)$ are Gaussian white noises in the sense of Stratonovich with mean of zeros and correlation functions $E\left[\xi_{i}\left(t\right)\xi_{j}\left(t+\tau\right)\right]=2D_{ij}\delta\left(\tau\right)$, that is

$$\begin{aligned}&E\left\{\xi_{i}\left(t\right)\right\}=0,\\&E\left\{\xi_{i}\left(t\right)\xi_{j}\left(t+\tau\right)\right\}=2D_{ij}\delta\left(\tau\right),\\&\left(i,j=1,\cdots,m\right).\end{aligned} \tag{20}$$

F_{s}, the third term on the right-hand side of Eqs.(19), may represent the external forces applied to the system by the feedback control. $\boldsymbol{F}=\left(F_{1},\cdots,F_{n}\right)^{\mathrm{T}}$ can be written as

$$\boldsymbol{F}=\boldsymbol{B}\left(\boldsymbol{q}\right)\boldsymbol{u}, \tag{21}$$

while $\boldsymbol{B}$ is a $n\times k$ matrix, $\boldsymbol{u}$ is a k-dimensional feedback vector.

The motion equation of the system may be modeled as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot{q}_{s}}-\frac{\partial T}{\partial q_{s}}=&\ Q_{s}+Q_{s}'+\sum_{i=1}^{m}\sigma_{si}\left(\boldsymbol{q}\right)\xi_{i}\left(t\right)+F_{s}\\&+\sum_{\beta=1}^{g}\lambda_{\beta}\frac{\partial f_{\beta}}{\partial\dot{q}_{s}},\ \left(s=1,\cdots,n\right),\end{aligned} \tag{22}$$

$\sum\limits_{\beta=1}^{g}\lambda_{\beta}\dfrac{\partial f_{\beta}}{\partial\dot{q}_{s}}$, the fifth term on the right-hand side of Eqs.(22), may stand for the generalized constraint forces, and factors $\lambda_{\beta}\ (\beta=1,\cdots,g)$ are known as Lagrange multipliers. These Lagrange multipliers can be derived from the equations obtained by means of differentiating Eqs.(18) with respect to time together with Eqs.(22) as follows:

$$\lambda_{\beta}=\lambda_{\beta}\left(\dot{\boldsymbol{q}},\boldsymbol{q}\right),\quad\left(\beta=1,\cdots,g\right).$$

So, the generalized constraint forces have the form of $\Lambda_s = \sum_{\beta=1}^{g} \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}$.

Furthermore, if generalized forces Q_s are potential and Q'_s are non-potential, that is $Q_s = -\frac{\partial V}{\partial q_s}$, $(s = 1, \cdots, n)$, then Eqs.(22) can be rewritten as:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} &= Q'_s + \sum_{r=1}^{k} B_{sr}(\boldsymbol{q})\, u_r + \Lambda_s \\ &\quad + \sum_{i=1}^{m} \sigma_{si}(\boldsymbol{q}) \xi_i(t), \\ &\quad (s = 1, \cdots, n), \end{aligned} \tag{23}$$

where $L = T - V$ is the Lagrange function of the system. The generalized momentum conjugate to q_i is given by

$$p_s = \frac{\partial L}{\partial \dot{q}_s}. \tag{24}$$

The Hamiltonian function for the system is defined as

$$H = \sum_{s=1}^{n} p_s \dot{q}_s - L. \tag{25}$$

We can derive Eqs.(26) from Eq.(25) as follows:

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \frac{\partial H}{\partial q_s} = -\frac{\partial L}{\partial q_s}, \quad (s = 1, \cdots, n). \tag{26}$$

The Hamilton's equations equivalent to Eqs.(23) are[27]

$$\begin{aligned} \dot{q}_s &= \frac{\partial H}{\partial p_s}, \\ \dot{p}_s &= -\frac{\partial H}{\partial q_s} + Q'_s + \sum_{r=1}^{k} B_{sr}(\boldsymbol{q})\, u_r + \Lambda_s \\ &\quad + \sum_{i=1}^{m} \sigma_{si}(\boldsymbol{q}) \xi_i(t), \quad (s = 1, \cdots, n). \end{aligned} \tag{27}$$

Equations (27) are modelled as the motion equations of controlled Hamiltonian system with non-holonomic constraints excited by Gaussian white noise.

Furthermore, we can shift Eqs.(27) into Ito form

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})\,\mathrm{d}t + \boldsymbol{g}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{W}, \tag{28}$$

where

$$\boldsymbol{x} = (q_1, \cdots, q_n, p_1, \cdots, p_n)^{\mathrm{T}},$$

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x}) = \Big(&\frac{\partial H}{\partial p_1}, \cdots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1} + Q'_1 + F_1 + \Lambda_1, \cdots, \\ &-\frac{\partial H}{\partial q_n} + Q'_n + F_n + \Lambda_n \Big)^{\mathrm{T}}, \end{aligned}$$

$$\boldsymbol{g}(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nm} \end{bmatrix},$$

and $\boldsymbol{W}(t)$ is m-dimensional mutually independent Wiener process, and with

$$\begin{aligned} &E\{\Delta W_i(t)\} = E\{W_i(t + \Delta t) - W_i(t)\} = 0, \\ &E\{\Delta W_i(t)\, \Delta W_j(t)\} = 2D_{ij}\Delta t, \quad (i, j = 1, \cdots, m). \end{aligned}$$

Because of the appearance of elements $\mathbf{0}_{n\times m}$ in the matrix $\boldsymbol{g}(\boldsymbol{x})$ and the elements in the matrix $\sigma_{n\times m}$ only as functions of q_s, Wong–Zakai corrections $\frac{1}{2}\sum_{s=1}^{2n}\sum_{i,j=1}^{m} D_{ij} g_{sj} \frac{\partial g_{ki}}{\partial x_s}, (k = 1, \cdots, 2n)$ are equal to zero under this circumstance. We can rewrite Eqs.(27) as follows:

$$\begin{aligned} \mathrm{d}q_s &= \frac{\partial H}{\partial p_s}\mathrm{d}t, \\ \mathrm{d}p_s &= \left[-\frac{\partial H}{\partial q_s} + Q'_s + \sum_{r=1}^{k} B_{sr}(\boldsymbol{q})\, u_r + \Lambda_s \right] \mathrm{d}t \\ &\quad + \sum_{i=1}^{m} \sigma_{si}(\boldsymbol{q})\, \mathrm{d}W_i. \end{aligned} \tag{29}$$

Equations (27) and (29) are all called the motion equations of controlled stochastic Hamiltonian system with non-holonomic constraints. Equation (29) has the standard form of n-dimensional vector-valued stochastic differential equations described by Eqs.(6). Therefore, we can study the symmetries and conserved quantities for the controlled stochastic Hamiltonian system with constraints by the theory presented in Section 2.

4. Examples of symmetries and conservation laws for stochastic Hamiltonian systems

The formalism delineated in Section 2 will be used to find conserved quantities for stochastic Hamiltonian systems. We stress that the techniques related to non-Noetherian symmetries show their ability to produce conserved quantities of stochastic dynamical systems.

Example 1 Discuss the symmetry operator for Hamiltonian system.

Symmetry operator for Hamiltonian system must satisfy

$$[\partial_t + X_0,\ Y] = 0, \quad [X_r,\ Y] = 0, \quad (r = 1, \cdots, m),$$

where $X_0 = \sum\limits_{s=1}^{n} \dfrac{\partial H}{\partial p_s} \cdot \dfrac{\partial}{\partial q_s} - \sum\limits_{s=1}^{n} \dfrac{\partial H}{\partial q_s} \cdot \dfrac{\partial}{\partial p_s} = X_0^1 + X_0^2$ and $X_r = \sum\limits_{s=1}^{n} \sigma_s^r(q) \dfrac{\partial}{\partial p_s}$. Suppose $Y = \sum\limits_{s=1}^{n} V_s(q,p,t) \dfrac{\partial}{\partial q_s} + \sum\limits_{t=1}^{n} W_t(q,p,t) \dfrac{\partial}{\partial p_t} = Y_1 + Y_2$. Equation $[X_r,\ Y] = 0$ is equivalent to

$$[X_r,\ Y] = \sum_{t=1}^{n} \left[\sum_{s=1}^{n} \sigma_s \frac{\partial V_t}{\partial p_s} \right] \frac{\partial}{\partial q_t} - \sum_{t=1}^{n} \left[\sum_{s=1}^{n} \sigma_s \frac{\partial W_t}{\partial p_s} - \sum_{s=1}^{n} V_s \frac{\partial \sigma_t}{\partial q_s} \right] \frac{\partial}{\partial p_t} = 0.$$

We can also write it more concisely

$$\sum_{s=1}^{n} \sigma_s \frac{\partial V_t}{\partial p_s} = 0, \qquad (t = 1, \cdots, n),$$

$$\sum_{s=1}^{n} \sigma_s \frac{\partial W_t}{\partial p_s} - \sum_{s=1}^{n} V_s \frac{\partial \sigma_t}{\partial q_s} = 0, \qquad (t = 1, \cdots, n).$$

On the other hand, the equation corresponding to $[\partial_t + X_0,\ Y] = 0$ is more complicated

$$[\partial_t + X_0,\ Y] = [\partial_t, Y_1] + [\partial_t, Y_2] + \left[X_0^1, Y_1\right] + \left[X_0^1, Y_2\right] + \left[X_0^2, Y_1\right] + \left[X_0^2, Y_1\right] = 0, \tag{30}$$

where

$$\begin{aligned}
[\partial_t\ ,\ Y_1] &= \sum_{s=1}^{n} \frac{\partial V_s}{\partial t} \cdot \frac{\partial}{\partial q_s}, \\
[\partial_t\ ,\ Y_2] &= \sum_{s=1}^{n} \frac{\partial W_s}{\partial t} \cdot \frac{\partial}{\partial p_s}, \\
\left[X_0^1\ ,\ Y_1\right] &= \sum_{t=1}^{n} \left[\sum_{s=1}^{n} \left(\frac{\partial H}{\partial p_s} \cdot \frac{\partial V_t}{\partial q_s} - V_s \frac{\partial^2 H}{\partial p_t \partial q_s} \right) \right] \frac{\partial}{\partial q_t}, \\
\left[X_0^1\ ,\ Y_2\right] &= \sum_{t=1}^{n} \left[\sum_{s=1}^{n} \left(\frac{\partial H}{\partial p_s} \cdot \frac{\partial W_t}{\partial q_s} \right) \right] \frac{\partial}{\partial p_t} - \sum_{t=1}^{n} \left[\sum_{s=1}^{n} W_s \frac{\partial^2 H}{\partial p_t \partial p_s} \right] \frac{\partial}{\partial q_t}, \\
\left[X_0^2\ ,\ Y_1\right] &= \sum_{t=1}^{n} \left[\sum_{s=1}^{n} \left(-\frac{\partial H}{\partial q_s} \cdot \frac{\partial V_t}{\partial p_s} \right) \right] \frac{\partial}{\partial q_t} + \sum_{t=1}^{n} \left[\sum_{s=1}^{n} V_s \frac{\partial^2 H}{\partial q_t \partial q_s} \right] \frac{\partial}{\partial p_t}, \\
\left[X_0^2\ ,\ Y_2\right] &= \sum_{t=1}^{n} \left[\sum_{s=1}^{n} \left(-\frac{\partial H}{\partial q_s} \cdot \frac{\partial W_t}{\partial p_s} + W_s \frac{\partial^2 H}{\partial q_t \partial p_s} \right) \right] \frac{\partial}{\partial p_t}.
\end{aligned}$$

Equation (30) is complex and formidable. So, it is a hard work to deal with these general expressions for the symmetry operator of stochastic Hamiltonian system. We can find their solutions under a number of special cases and need some techniques.

If Hamiltonian of stochastic Hamiltonian system satisfies

$$\begin{aligned}
&\sum_{s=1}^{n} \left(\frac{\partial^2 H}{\partial p_t\, \partial q_s} + \frac{\partial^2 H}{\partial p_t\, \partial p_s} \right) = 0, \\
&\sum_{s=1}^{n} \left(\frac{\partial^2 H}{\partial p_t\, \partial q_s} + \frac{\partial^2 H}{\partial p_t\, \partial p_s} \right) = 0, \quad (t = 1, \cdots, n),
\end{aligned}$$

$$\sum_{s=1}^{n} \sigma_s \frac{\partial H}{\partial p_s} = 0,$$

$$\sum_{s=1}^{n} \sigma_s \frac{\partial H}{\partial p_s} - \sum_{s=1}^{n} H \frac{\partial \sigma_t}{\partial q_s} = 0, \quad (t = 1, \cdots, n)$$

then $Y = H \sum\limits_{s=1}^{n} \left(\dfrac{\partial}{\partial q_s} + \dfrac{\partial}{\partial p_s} \right)$ must be the symmetry operator for the stochastic Hamilton system.

Furthermore, suppose that the Hamiltonian is $H(q,p) = G(p) + F(q)$, the motion equations of the system are

$$\begin{aligned}
\mathrm{d}q_s &= \frac{\partial G}{\partial p_s} \mathrm{d}t, \\
\mathrm{d}p_s &= -\frac{\partial F}{\partial q_s} \mathrm{d}t - \frac{\partial F}{\partial q_s} \mathrm{d}W, \quad (s = 1, \cdots, n),
\end{aligned}$$

where W is Wiener process. Then, $Y = \sum\limits_{s=1}^{n} \left(\dfrac{\partial G}{\partial p_s} \dfrac{\partial}{\partial q_s} - \dfrac{\partial F}{\partial q_s} \dfrac{\partial}{\partial p_s} \right)$ is a symmetry operator of

the system.

Example 2 Discuss the symmetry and conserved quantity for the stochastic Hamilton system. Consider the stochastic dynamical system with $H = \frac{1}{2}\left(p_1^2+p_2^2+p_3^2\right)+(q_1p_1-q_2p_2+q_3p_3)$. The motion equations are

$$\begin{aligned}
&\mathrm{d}q_1 = (p_1+q_1)\,\mathrm{d}t,\\
&\mathrm{d}q_2 = (p_2-q_2)\,\mathrm{d}t,\\
&\mathrm{d}q_3 = (p_3+q_3)\,\mathrm{d}t,\\
&\mathrm{d}p_1 = -p_1\mathrm{d}t+q_2\mathrm{d}W,\\
&\mathrm{d}p_2 = p_2\mathrm{d}t+q_1\mathrm{d}W,\\
&\mathrm{d}p_3 = -p_3\mathrm{d}t-\frac{q_2p_1+q_1p_2}{q_3+p_3}\mathrm{d}W,
\end{aligned}$$

where W is Wiener process. Then, the operators X_0 and X_r $(r=1)$ in Eq.(8) are set as $X_0 = \sum\limits_{s=1}^{3}\frac{\partial H}{\partial p_s}\cdot\frac{\partial}{\partial q_s}-\sum\limits_{s=1}^{3}\frac{\partial H}{\partial q_s}\cdot\frac{\partial}{\partial p_s}$ and $X_1 = q_2\frac{\partial}{\partial p_1}+q_1\frac{\partial}{\partial p_2}-\frac{q_2p_1+q_1p_2}{q_3+p_3}\frac{\partial}{\partial p_3}$. For these operators, $I=\frac{1}{2}\left(p_1^2+p_2^2+p_3^2\right)+(q_1p_1-q_2p_2+q_3p_3)$ satisfies Eqs.(7), therefore it is the conserved quantity of the system.

Example 3 Discuss the two-degree-of-freedom dynamical system depicted in Fig.1.

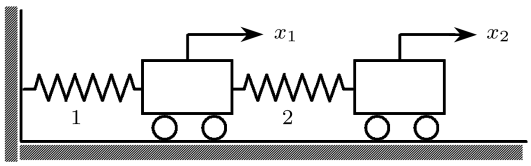


Fig.1. Two-degree-of-freedom dynamical system.

There are two masses m_1 and m_2 connected by two springs in the system. The spring 1 is a hard spring and it is characterized by $k_1x_1\left(1+\varepsilon x_1^2\right)$. The spring 2 is a linear spring, and its spring coefficient is k_2. The coordinates of the system are x_1 and x_2, the origins are the static equilibrium of their own respectively. The system is subjected to stochastic excitation. The equations of motion of the system are of the form

$$\begin{bmatrix} m_1 & 0\\ 0 & m_2\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2\\ -k_2 & k_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\varepsilon k_1x_1^3\\ 0\end{bmatrix}=\begin{bmatrix} m_1\left(x_2^2-\dfrac{m_2\dot{x}_2^2}{k_2(\mu-1)}\right)W\\ 0\end{bmatrix}.$$

W are Gaussian white noise in the sense of Stratonovich with correlation function

$$E\left[W(t)\,W(t+\tau)\right]=2D\delta(\tau).$$

Let $\omega_1^2=\frac{k_1}{m_1}$, $\omega_2^2=\frac{k_2}{m_2}$, $\mu=\frac{m_2}{m_1}$, then, we have

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix}+\begin{bmatrix}\omega_1^2+\mu\omega_2^2 & -\mu\omega_2^2\\ -\omega_2^2 & \omega_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\varepsilon\omega_1^2x_1^3\\ 0\end{bmatrix}=\begin{bmatrix}\left(x_2^2-\dfrac{\dot{x}_2^2}{(\mu-1)\omega_2^2}\right)W\\ 0\end{bmatrix}.$$

Let $x_1=y_1$, $x_2=y_2$, $\dot{x}_1=y_3$, $\dot{x}_2=y_4$, we have

$$\begin{aligned}
\dot{y}_1 &= y_3,\\
\dot{y}_2 &= y_4,\\
\dot{y}_3 &= -\left(\omega_1^2+\mu\omega_2^2\right)y_1+\mu\omega_2^2y_2-\varepsilon\omega_1^2y_1^3\\
&\quad+\left(x_2^2-\frac{\dot{x}_2^2}{(\mu-1)\omega_2^2}\right)W,\\
\dot{y}_4 &= \omega_2^2y_1-\omega_2^2y_2.
\end{aligned}$$

The stochastic differential equation of the dynamic system can be written as Ito form, (under this circumstance, Wong–Zakai amendment term is zero)

$$\mathrm{d}\boldsymbol{Y}=\boldsymbol{b}(\boldsymbol{Y},t)\,\mathrm{d}t+\boldsymbol{G}(\boldsymbol{Y},t)\,\mathrm{d}B,$$

where B is a standard Wiener process,

$$\boldsymbol{b}=\begin{bmatrix} y_3\\ y_4\\ -\left(\omega_1^2+\mu\omega_2^2\right)y_1+\mu\omega_2^2y_2-\varepsilon\omega_1^2y_1^3\\ \omega_2^2y_1-\omega_2^2y_2\end{bmatrix},$$

$$\boldsymbol{G}=\begin{bmatrix}0\\ 0\\ y_2^2-\dfrac{y_4^2}{(\mu-1)\omega_2^2}\\ 0\end{bmatrix}.$$

Then, we can easily check that $\mathrm{div}\boldsymbol{b}=0$ and $\mathrm{div}\boldsymbol{G}=0$ hold. We further find out that symmetry operator for this system does not exist, since no operator $Y=\sum\limits_{i=1}^{4}f^i(y,t)\,\partial_i$ satisfies Eqs.(13) (note that ∂_i stands for $\partial/\partial y_i$). Hence, it proves that conserved quantity does not exist for this system.

5. Conclusions

We have discussed symmetries and conserved quantities for stochastic Hamiltonian system. Because of the complexity of motion equations of stochastic dynamical systems, the expressions that symmetry operators must satisfy are lengthy and tedious. In order to obtain a conserved quantity for stochastic dynamical systems, one needs to solve a formidable set of partial differential equations. Generally speaking, determining symmetry and conserved quantities for stochastic dynamical systems is a tough work, though it is important. Usually, one can find stochastic conserved quantity only in some special cases.

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