Vol 16 No 11, November 2007
1009-1963/2007/16(11)/3154-07

Chinese Physics

© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

An algorithm and its application for obtaining some kind of infinite-dimensional Hamiltonian canonical formulation*

Ren Wen-Xiu(任文秀)[a)b)] and Alatancang(阿拉坦仓)[a)†]

[a)] *Department of Mathematics, Inner Mongolia University, Hohhot* 010021, *China*

[b)] *Department of Mathematics, Inner Mongolia University of Technology, Hohhot* 010051, *China*

(Received 9 May 2007; revised manuscript received 19 July 2007)

Using factorization viewpoint of differential operator, this paper discusses how to transform a nonlinear evolution equation to infinite-dimensional Hamiltonian linear canonical formulation. It proves a sufficient condition of canonical factorization of operator, and provides a kind of mechanical algebraic method to achieve canonical '$\partial/\partial x'$-type expression, correspondingly. Then three examples are given, which show the application of the obtained algorithm. Thus a novel idea for inverse problem can be derived feasibly.

Keywords: nonlinear evolution equation, infinite-dimensional Hamiltonian canonical system, factorization of differential operator, commutator

PACC: 0200, 0340

1. Introduction

The theory and methods of Hamiltonian system have received extensive attention.[1−6] Especially, research and development of the canonical system has become more and more necessary and significant due to its concise formula providing lots of new view and good approaches for physics and mechanics. For example, utilizing the canonical form of elliptic partial differential equation, Zhong[7] has successfully extended the traditional method of separation of variables to the symplectic-Fourier expansion method. The purpose of this paper is focused on transforming the original equation into canonical form. Commonly, such problem is named as an inverse problem (IP)[8] of an infinite-dimensional Hamiltonian canonical system. Despite the traditional method being powerful for constructing Hamilton framework,[9] there are still some defects, such as requiring variational principle and computing Legendre transformation, which have limited the applications of the Hamilton method. In particular, it seems considerably difficult for the non-self-adjoint differential equation. So far, no unified path has been discovered. Therefore, we must explore a novel channel to conquer these shortcomings.

We cite a definition: an evolution equation is called the infinite-dimensional linear Hamiltonian canonical formulation,[10,11] if it can be written in the form

$$\dot{\omega} = H\omega. \tag{1}$$

Here

$$H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$$

is a Hamiltonian canonical $2m \times 2m$ matrix differential operator, in which A, A^* are mutually adjoint $m \times m$ matrix operators, and B, C are self-adjoint, respectively. In addition to this definition, there is another equivalent description for H in some sense:[12] if and only if $H^* = JHJ$ holds, the matrix differential operator H is Hamiltonian canonical, where

$$J = \begin{bmatrix} 0 & I_{m\times m} \\ -I_{m\times m} & 0 \end{bmatrix},$$

and $I_{m\times m}$ is the matrix identity operator. Recalling Eqs.(1), we can take the left-hand term $\dot{\omega}$ as the first-order derivative of ω to the dependent variable t or x, in the case of appearing canonical '$\partial/\partial x'$-type: $\partial\omega/\partial x = H\omega$ and '$\partial/\partial t'$-type: $\partial\omega/\partial t = H\omega$, where Hamiltonian operators H have different meanings. Namely, the former does not explicitly contain the total derivative operator D_x, the latter does not

*Project supported by the National Natural Science Foundation of China (Grant No 10562002) and the Natural Science Foundation of Nei Mongol, China (Grant No 200508010103).

†Corresponding author. E-mail: alatanca@imu.edu.cn

contain D_t. According to the Vainberg theorem, we know that a given nonlinear evolution equation

$$u_t = F(x, u, u^{(1)}, \cdots, u^{(n)}) \tag{2}$$

cannot be replaced by form of '$\partial/\partial t'$. After referring to Ref.[13], it is very natural for us to think whether or not Eq.(2) can be converted into the canonical form '$\partial/\partial x'$, i.e. $\partial\omega/\partial x = H\omega$, which distinguishes it from the classical pattern in essence.

Unlike determining the Lagrange functional and dual coordinates in the traditional sense, the demonstrated approach in Refs.[8,12] exploited an algebraic relationship to establish canonical representation by taking advantage of Hamiltonian canonical definition directly. Enlightened with this thought, we make an effort to consider IP again in another viewpoint. Herein, we obtain an extendible method attributing to the algorithm in Ref.[8] via the decomposition of differential operator. Through determining the Hamiltonian solution of a special equation, we provide some related examples to illustrate the application of the method. Intuitively, our results are consistent with the main criteria from Zhang Hong-Qing about the IP of the Hamiltonian canonical system for linear self-adjoint differential operator.[14] This method is practicable.

2. An algebraic method of obtaining infinite-dimensional Hamiltonian canonical system

2.1. Symbol and idea

For preparation in advance, we firstly introduce the total derivative operator

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + u_{3x}\frac{\partial}{\partial u_{xx}} + \cdots$$

over space $X \times U : (x, t; u)$, among which there are the following fundamental multiplications:

$$D_x^i D_x^j = D_x^{i+j},\ (i, j \geq 0);$$
$$D_x D_t = D_t D_x; \quad D_x Q = Q D_x + D_x Q,$$

for differential function Q. Then we employ commutator of differential operators T, S as $[T, S] = TS - ST$, such as $[D_x, D_t] = 0$; more generally, we allow differential function to appear in commutator computation, i.e. here take function as a multiplication operator if there is only the right multiplication of the unit operator I, for instance, let $T = u$, $S = D_x$, then we obtain

$$[u, D_x] = [uI, D_x] = uID_x - D_x uI$$
$$= uD_x - uD_x - u_x = -u_x.$$

Secondly, according to the Wu-main variable idea, we think that system (2) must be a polynomial in the derivative u_x with matrix differential operators as coefficients. By the relation $u_x = D_x u$, and upgrading the dimension of system to even 2m (usually obey $2m \geq n$), we can rewrite Eq.(2) into

$$(a_0 D_x^n + a_1 D_x^{n-1} + \cdots + a_{n-1} D_x + a_n I)u = 0, \tag{3}$$

where, for $k = 0, 1, \cdots, n$, coefficient a_k denotes a $2m \times 1$ matrix differential operator, which depends on x, t, derivatives $u^{(j)}$ for $j \leq n - k$ and operators D_t^i up to a known number except D_x. No matter whether being changed or not, letter n all along is a sign for the order of the system. For convenience, we denote Eqs.(1) and Eqs.(3) as $M\omega = 0, \Delta u = 0$ respectively, where $M = ID_x - H, \Delta = \sum_{k=0}^{n} a_k D_x^{n-k}$, in this case, involved I is the 2m×2m matrix identical operator. Through taking the place of D_x, we define the characteristic polynomial of Δ as $f(x) = \sum_{k=0}^{n} a_k x^{n-k}$, and mark $f(H) = \sum_{k=0}^{n} H^{n-k} a_k$.

Finally, we shall discuss IP in factorization of differential operators. We think that the whole process should be the following: the first step is to factorize Δ in form of

$$\Delta = MQ, \tag{4}$$

then to convert Eqs.(3) into Eqs.(1) in terms of transformation: $\omega = Qu \neq 0$, in which the appropriate selection of M and Q is pivotal for building Hamiltonian canonical framework. Indeed, using the definition, we easily deduce

$$M^* = (ID_x)^* - H^* = -ID_x - JHJ$$
$$= J(ID_x - H)J = JMJ,$$

hence M is a first order Hamiltonian canonical operator in D_x, and formula (4) is called the canonical decomposition of Δ. Thus we state that the key for the IP is to determine the canonical decomposition. However, the existence of decomposition (4) is only a sufficient criterion of the IP, and the uniqueness of decomposition cannot be ensured, which means that all of M and Q are variable factors just as $\Delta = M_1 Q_1 = M_2 Q_2$. Particularly, for a fixed Hamiltonian operator M, if

$0 \in \rho(M)$ holds, then factorization (4) is existential and unique iff $Q = M^{-1}\Delta$. We will try to structurally reveal it in the next section.

2.2. Main results about canonical decomposition

Using the order of operator defined by the highest power of D_x, and the above-mentioned form of Q, we take Q as a $n-1$ order operator, which should be a linear combination of $I, D_x, D_x^2, \cdots, D_x^{n-1}$, i.e. has the form as

$$Q = \sum_{k=0}^{n-1} b_k D_x^{n-1-k}, \tag{5}$$

where for $k = 0, 1, \cdots, n-1$, undetermined coefficient b_k is complicated, but it necessarily satisfies $Qu \neq 0$ within any way. Next, we will take means of operator identities, and consult the idea in Ref.[15], to reach it.

Assume coefficients

$$b_k = b_k(x, u, u^{(1)}, \cdots, u^{(n-1-k)}, D_t, D_{tt}, \cdots),$$

for all of $k \leq n-1$. By operational rules of D_x and commutator symbol, substituting Eq.(5) into Eq.(4), we find

$$\begin{aligned}&(b_0 - a_0)D_x^n + \sum_{k=1}^{n-1}(b_k - Hb_{k-1} - [b_{k-1}, D_x] - a_k)D_x^{n-k}\\ &-(Hb_{n-1} + [b_{n-1}, D_x] + a_n)I = 0.\end{aligned} \tag{6}$$

If coefficients of $I, D_x, D_x^2, \cdots, D_x^n$ vanish, Eq.(6) clearly yields

$$\begin{aligned}&b_0 = a_0;\ \ b_k = a_k + Hb_{k-1} + [b_{k-1}, D_x],\\ &(k = 1, \cdots, n-1);\\ &a_n + Hb_{n-1} + [b_{n-1}, D_x] = 0.\end{aligned} \tag{7}$$

Note that it is sufficient to make Eq.(6) identically vanish while coefficients of the power of D_x are all zero. Therefore we have the following theorem.

Theorem Suppose that the operator equation related to a known system (3) is

$$f(H) + \sum_{k=0}^{n-1} H^{n-1-k} c_k = 0, \tag{8}$$

where $c_0 = [a_0, D_x]$ and

$$\begin{aligned}c_k = &\sum_{i=0}^{k}[H^{k-i}a_i, D_x]\\ &+ \sum_{i=0}^{k-1}[H^{k-1-i}c_i, D_x], \quad (k = 1, \cdots, n-1).\end{aligned}$$

Let solutions H of Eq.(8) exist. If at least there is one H be Hamiltonian canonical, which is called Hamiltonian canonical solution (HCS) in remains, then Δ determined by Eqs.(3) can generate canonical decomposition associated with H.

Proof By sequences $\{a_k\}_{k=0}^n$ and $\{c_k\}_{k=0}^{n-1}$, we define a new one with the expression:

$$\begin{aligned}&\{b_k\} : b_0 = a_0;\\ &b_k = \sum_{i=0}^{k} H^{k-i}a_i + \sum_{i=0}^{k-1} H^{k-1-i}c_i,\\ &(k = 1, 2, \cdots, n-1).\end{aligned}$$

Obviously, three sequences have the relations $c_k = [b_k, D_x], (0 \leq k \leq n-1)$ and two formulae ahead of Eq.(7). In terms of those relations and Eq.(8), and noting the basic identity

$$Hb_{k-1} + [b_{k-1}, D_x] = -Mb_{k-1} + b_{k-1}D_x,$$

we can obtain

$$a_n = -Hb_{n-1} - [b_{n-1}, D_x] = Mb_{n-1} - b_{n-1}D_x.$$

Furthermore, we have

$$\begin{aligned}\Delta &= \sum_{k=0}^{n} a_k D_x^n\\ &= b_0 D_x^n + \sum_{k=1}^{n-1}(b_k + Mb_{k-1} - b_{k-1}D_x)D_x^{n-k}\\ &\quad + Mb_{n-1} - b_{n-1}D_x\\ &= \sum_{k=0}^{n-1} b_k D_x^{n-k} + M\sum_{k=1}^{n} b_{k-1}D_x^{n-k}\\ &\quad - \sum_{k=1}^{n} b_{k-1}D_x^{n-k+1} = MQ.\end{aligned}$$

i.e. a required canonical decomposition is shown in terms of the existence of HCS.

From theorem, we can easily deduce the following results.

Corollary 1 For $k = 0, 1, \cdots, n-2$, let coefficient b_k of Q satisfies $[b_k, D_x] = 0$, but $[b_{n-1}, D_x] = 0$ holds unnecessarily. For operator equations $f(H) + [b_{n-1}, D_x] = 0$ if HCS exists, then Δ determined by Eqs.(3) can generate canonical decomposition.

Corollary 2 Let for $k = 0, 1, \cdots, n-1$, coefficient a_k of Δ determined by Eqs.(3) satisfies $[a_k, D_x] = 0$. For operator equations $f(H) = 0$ if HCS exists such that it is commuted with D_x, then Δ has canonical decomposition.

Proof Similar to the previous proof, firstly define new sequence

$$\Big\{b_k = \sum_{j=0}^{k} H^j a_{k-j}\Big\}_{k=0}^{n-1}$$

by $\{a_k\}_{k=0}^n$. Then using the simplified relationship

$$\begin{aligned} &b_0 = a_0;\ \ b_k = a_k + Hb_{k-1},\\ &(k = 1, 2, \cdots, n-1);\\ &a_n + Hb_{n-1} = 0, \end{aligned} \tag{9}$$

and known conditions $[a_k, D_x] = 0$, $[H, D_x] = 0$, we find

$$\begin{aligned} [b_k, D_x] &= \Big[\sum_{j=0}^{k} H^j a_{k-j}, D_x\Big]\\ &= \sum_{j=0}^{k} [H^j a_{k-j}, D_x]\\ &= \sum_{j=0}^{k} H^j [a_{k-j}, D_x] = 0,\\ &\qquad (0 \le k \le n-1). \end{aligned}$$

This is a special case when $[b_{n-1}, D_x] = 0$ in corollary 1.

As we predict, it is evident that a new method for IP can be obtained through canonical decomposition for Δ.

2.3. An arithmetic description of constructing canonical expression

In Section 2.2, theorem indicated that IP depended on HCS of an operator equation. It not only showed a sufficient condition for some known systems in Hamiltonian canonical structure, but also implied an approach to obtain this form stemmed from the process of proof. So, we shall concentrate attention on process of constructing canonical expression. For convenience, we picture main mechanical steps in details as the following.

First, we transform Eq.(2) into Eqs.(3), and denote corresponding operator Δ, coefficients $\{a_k\}_{k=0}^n$. In case of meeting an odd-dimensional system, we must change it into even by appending additional zero-equations at arbitrary position. Second, according to the commutativity of coefficients $\{a_k\}_{k=0}^n$ and D_x, meanwhile letting $H = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, we rewrite Eq.(8) or its simplified form with undetermined block operators A, B, C, D. Third, solve four parameters of this rewritten equations, and further verify $B^* = B$, $C^* = C$, $D = -A^*$, i.e. qualify $H = \left[\begin{smallmatrix} A & B \\ C & -A^* \end{smallmatrix}\right]$ as HCS, where A^* is the adjoint of A. Because solutions as candidate Hamiltonian operators are not single, we need to select and confirm them technically. But, sometimes no HCS exists. In this case, we cannot ensure the inexistence of a canonical representation of Eq.(2), namely, the effectiveness of our method is lost. Finally, through iterative computations to $\{b_i\}_{i=0}^{n-1}$ by expression (7) or expression (9), the Hamiltonian canonical system of Eq.(2) is devised in '$\partial/\partial x'$-type:$\dot{\omega} = H\omega$, where vector $\omega = (\omega_1, \omega_2, \cdots, \omega_{2m})^{\mathrm{T}}$ consists of components

$$\omega_i = \Big(\sum_{k=0}^{n-1} b_k D_x^{n-1-k} u\Big)_i,$$

for $i = 1, \cdots, 2m$.

Remark The corresponding way with corollary 2 is more tractable than the one in Ref.[8], which is always restricted to be an evolution equation with constant coefficients.

3. Examples

To illustrate the effectiveness of the obtained algorithm, three examples are presented.

Example 1 Consider the nonlinear equation

$$u_{3x} + f(u)u_x + u_t = 0, \tag{10}$$

which arises from Ref.[12]. In the above notation, set

$$\Delta = \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix} D_x^3 + \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix} D_x^2 + \begin{bmatrix} 0\\0\\0\\f(u) \end{bmatrix} D_x + \begin{bmatrix} 0\\0\\0\\D_t \end{bmatrix}, \quad a_0 = \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0\\0\\0\\f(u) \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0\\0\\0\\D_t \end{bmatrix}$$

such that this system is represented by form (3).

Let

$$H=\begin{bmatrix}A & B\\ C & D\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & b_{11} & b_{12}\\ a_{21} & a_{22} & b_{21} & b_{22}\\ c_{11} & c_{12} & d_{11} & d_{12}\\ c_{21} & c_{22} & d_{21} & d_{22}\end{bmatrix}.$$

By the relations in the theorem, we find

$$c_0=\begin{bmatrix}0\\0\\0\\0\end{bmatrix},\quad c_1=\begin{bmatrix}[B,D_x]\\ [D,D_x]\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix},\quad c_2=\begin{bmatrix}0\\0\\0\\-f'(u)u_x\end{bmatrix}+\begin{bmatrix}[AB+BD+[B,D_x],D_x]\\ [CB+D^2+[D,D_x],D_x]\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}.$$

From this, the operator equation (8) with respect to H can split into

$$(A^2B+ABD+A[B,D_x]+BCB+BD^2+B[D,D_x]+[AB+BD+[B,D_x],D_x])\begin{bmatrix}0\\1\end{bmatrix}+B\begin{bmatrix}0\\f(u)\end{bmatrix}=0,$$

$$(CAB+CBD+C[B,D_x]+DCB+D^3+D[D,D_x]+[CB+D^2+[D,D_x],D_x])\begin{bmatrix}0\\1\end{bmatrix}$$

$$+D\begin{bmatrix}0\\f(u)\end{bmatrix}+\begin{bmatrix}0\\D_t-f'(u)u_x\end{bmatrix}=0.$$

To derive the corresponding HCS, we first particularly fix block operators

$$B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\quad D=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\quad A=-D^*=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix},$$

then using the fact

$$D\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix},\quad B\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix},\quad [B,D_x]=0,\quad [D,D_x]=0,\quad [A,D_x]=0,$$

we can simplify the above equations into

$$c_{22}=-f(u);\ \ c_{11}=[c_{12},D_x];\ \ -c_{21}+c_{12}+D_t=0.$$

Thus we can determine a special solution in the form as

$$c_{11}=0,\quad c_{21}=(1/2)D_t+\alpha,\quad c_{12}=-(1/2)D_t+\alpha,$$

where α is any constant, namely, a HCS generated by Eq.(10) is

$$H=\begin{bmatrix}0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & -(1/2)D_t+\alpha & 0 & 0\\ (1/2)D_t+\alpha & -f(u) & 1 & 0\end{bmatrix}.$$

Further, according to formula (7), we have

$$b_0=\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\quad b_1=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad b_2=\begin{bmatrix}-1\\0\\-(1/2)D_t+\alpha\\0\end{bmatrix}.$$

Consequently, Eq.(10) is changed into the canonical system

$$\partial\omega/\partial x=H\omega,$$

where

$$\omega=b_0u_{xx}+b_1u_x+b_2u=\begin{bmatrix}-u\\ u_x\\ -(1/2)u_t+\alpha u\\ u_{xx}\end{bmatrix}.$$

In particular, for $f(u)=6u$ or $f(u)=6u^2$ in Eq.(10), the canonical expressions of Korteweg-de Vries equation and modified Korteweg-de Vries equation can be automatically obtained, which actually agree with those results in Ref.[12]. It means that our method is valid. Through comparison we see that the main advantage of this method is in shifting IP into directly solving HCS of a certain operator equation. If we can obtain more HCSs, then more Hamiltonian canonical patterns for the same equation will be showed. Of course, searching for satisfied HCS is complicated.

Example 2 Consider equation

$$u_{xx}-u_{tt}+f(u)=0,$$

Same as the step ahead, equivalently we take

$$\Delta = \begin{bmatrix} 0 \\ 1 \end{bmatrix} D_x^2 + \begin{bmatrix} 0 \\ -D_t^2 + f(u)u^{-1} \end{bmatrix},$$

$$a_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$a_2 = \begin{bmatrix} 0 \\ -D_t^2 + f(u)u^{-1} \end{bmatrix},$$

let undetermined Hamiltonian operator be $H = \left[\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right]$. Solving equations

$$x_1 x_2 + x_2 x_4 + [x_2, D_x] = 0$$

and

$$x_3 x_2 + x_4^2 + [x_4, D_x] = D_t^2 - f(u)u^{-1},$$

we choose a HCS as

$$H = \begin{bmatrix} \gamma D_t^{2k} & 1 \\ D_t^2 - f(u)/u - \gamma^2 D_t^{4k} & -\gamma D_t^{2k} \end{bmatrix},$$

where γ is arbitrary constant, and $k \in N$. After computing $b_0 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $b_1 = \left[\begin{smallmatrix} 1 \\ -\gamma D_t^{2k} \end{smallmatrix}\right]$, the state variables

$$\omega = \begin{bmatrix} u \\ u_x - \gamma \partial^{2k} u / \partial t^{2k} \end{bmatrix}$$

are determined which correspond to canonical form: $\partial\omega/\partial x = H\omega$. Moreover, if set $f(u)$ equal to $-\alpha u - \beta u^3$ or $mu - g^2 u^3$, then it means the canonical forms of Klein–Gordon equation and Landau–Ginsburg–Higgs equation.

Example 3 Discuss a kind of equation

$$u_{xx} - u_t + f(u) = 0,$$

which depends on term $f(u)$. For instance, when $f(u) = u(1-u)$, it indicates the Fisher equation; when $f(u) = u^2(1-u)$, it is the Huxley equation; $f(u) = \alpha u(1-u^2)$, to be the Chaffee–Infante equation; more, $f(u) = u(u-\alpha)(1-u)$ means the Fitzhugh–Nagumo equation; $f(u) = \mid u \mid^2 u$ represents the Schrödinger equation, and so on.

For briefness, we only show the last result that it can be equivalent to the canonical system

$$\frac{\partial}{\partial x} \begin{bmatrix} u \\ u_x + \sqrt{D_t} u \end{bmatrix} = \begin{bmatrix} -\sqrt{D_t} & 1 \\ -f(u)u^{-1} & \sqrt{D_t} \end{bmatrix} \begin{bmatrix} u \\ u_x + \sqrt{D_t} u \end{bmatrix}.$$

Here note that $\sqrt{D_t}\sqrt{D_t} = D_t$.

4. Feasibility of the algorithm

From the above examples, we conclude that the adopted algorithm is theoretically built on the decomposition (4). In other words, when operator $\Delta - MQ$ acts on arbitrary differential function, it always vanishes. Actually, for the IP, $(\Delta - MQ)u = 0$ holding is enough, where u is a solution of Eqs.(3). For example, consider the canonical description of the Bretherton equation

$$\alpha u_{4x} + u_{xx} + u_{tt} - \beta u^3 = 0.$$

We verify that the Hamiltonian operator

$$H = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1/\alpha \\ D_t^2 - 3\beta u^2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

is not a HCS of Eq.(8), hence corresponding canonical form cannot be determined by the given method in Section 2.3. However, we can obtain it while replacing Eq.(4) by $\Delta u = MQu$, i.e. formula (7) is replaced by

$$b_0 = a_0;$$

$$b_k = a_k + H b_{k-1} + [b_{k-1}, D_x], \ (k = 1, 2);$$

$$b_3 u = D_x^{-1} H b_2 u_x; \ \ a_4 + H b_3 = 0,$$

and with previous mentioned steps to obtain state variables

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{bmatrix} u_{3x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_{xx} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_x + \begin{bmatrix} 0 \\ 0 \\ -D_t^2 + \beta u^2 \\ 0 \end{bmatrix} u = \begin{bmatrix} -u_x \\ u_{xx} \\ -u_{tt} + \beta u^3 \\ \alpha u_{3x} \end{bmatrix}.$$

So, we will draw a more general criterion on IP as following: let u be a solution of system (3) defined by Δ, and H be a Hamiltonian operator. A sufficient condition for IP of Eq.(3) related to H is

$$u \in \mathrm{Ker}(\Delta - MQ),$$

where operator Q is in the form of Eq.(5).

Obviously, when Δ is the linear self-adjoint operator, the above criterion agrees with the idea of the theorem in Ref.[14]; therefore, result (11) should be feasible, and implies an ideal way to orderly compute H and ω in canonical expression. Though we do not prove this rigorously, it provides a novel view for IPs. This is a worthwhile approach.

References

[1] P J Olver 1993 *Applications of Lie Groups to Differential Equations* (New York: Springer) Second Edition pp389–463

[2] Ren W X and Alatancang 2007 *Commun. Theor. Phys.* **48** 211

[3] Masakazu Sueyoshi and Takahiro Iwayama 2007 *Fluid Dynamics Research* **39** 346

[4] Zheng Y and Zhang H Q 1996 *Acta Mech. Sin.* **28** 119 (in Chinese)

[5] Yao Y Q and Chen D Y 2007 *Chin. Phys.* **16** 611

[6] Zheng Y and Chen Y 2002 *Applied Mathematics A Journal of Chinese Universities* **17** 177

[7] Zhong W X 2004 *Journal of Dalian University of Technology* **44** 1 (in Chinese)

[8] Alatancang, Zhang H Q and Zhong W X 2000 *Appl. Math. Mech.* **21** 733

[9] Guo Y X, Yu Y and Huang H J 2001 *Chin. Phys.* **10** 1

[10] Zheng Y, Zhang H Q and Tang L M 1994 *Journal of Dalian University of Technology* **34** 621 (in Chinese)

[11] Chen Y, Zheng Y and Zhang H Q 2003 *Appl. Math. Mech.* **24** 22

[12] Alatancang, Zhang H Q and Zhong W X 1999 *Acta Mech. Sin.* **31** 347 (in Chinese)

[13] Maxim V and Pavlov 1998 *Phys. Lett.* A **243** 295

[14] Zhang H Q 1994 *Thirty Years for Nonholonomic Mechanics in China* (Kaifeng: Henan University Press) pp182–186 (in Chinese)

[15] Luis Verde-Star 2007 *Adv. in Appl. Math.* **39** 329