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Quantization of the double-qubit structure and quantum computation*

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The quantization scheme of a double-qubit structure with superconducting quantum interference devices (SQUIDs) is given. By introducing unitary matrices and using spectral decompositions, the Hamiltonian operator of the system is exactly formulated in compact forms in spin-1/2 notation. A scheme of designing controlled-phase-shift (CPS) gates is also proposed by using this circuit system.

Keywords: double-qubit, quantization, spectral decomposition, controlled-phase-shift
PACC: 7335, 0367

1. Introduction

In recent years, the theory of quantum information and computation in quantum systems has aroused much interest.[1−4] A practical quantum computer, if built, would be comprised of a set of coupled two-state quantum systems[5] usually referred to as quantum bits (qubits), whose coherent time evolution must be controlled in a computation.[6] Experimentally, nuclear magnetic resonances in molecules,[7] trapped ions,[8] and quantum optical systems[9] have been investigated to achieve quantum computation. These systems have the advantages of high quantum coherence, but cannot be integrated easily to form large-scale circuits.

With the development of nanolithography technologies, it has been suggested that small solid-state devices can be promisingly used for quantum circuit integration to realize quantum computation. The superconducting Josephson junction can play an important role in realizing solid-state quantum computation. It has been recognized that suitable Josephson junction devices might serve as qubits in quantum information devices and that quantum logic operations can be performed by controlling gate voltages or magnetic fields.[6,10,11] The first challenge involved in the course of realizing quantum computer is to choose a suitable system with Josephson junctions serving as qubits. The simplest model of Josephson junction qubit was proposed earlier theoretically and experimentally.[12,13] Subsequently, Makhlin *et al*[6] presented an improved ideal design. This design avoided an undesired phase evolution effectively, by tuning the energy splitting between logical states during idle periods between operations. You *et al*[14] made a further study of this single charge qubit structure by considering the role of self-conductance, and proposed a scheme to couple single charge qubits by mutual inductance, and obtained some instructive conclusions.

In this paper, we introduce a new method to investigate the double-qubit structure. Firstly, we give the number-phase quantization scheme of the double-qubit structure with superconducting quantum interference devices (SQUIDs). Secondly, by introducing unitary matrices and using spectral decompositions,[15] we exactly formulate the Hamiltonian operator of the system in compact forms in spin-1/2 notation, which is important from the point of view of quantum computation. Finally, we also propose a scheme to design controlled-phase-shift(CPS) gates by using the system.

2. Number-phase quantization of the double-qubit structure

The double-qubit structure, where the interbit coupling is realized by mutual inductance between left and right SQUID loops, is shown in Fig.1. For

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each single-qubit structure, a superconducting island is coupled by two Josephson junctions (each with capacitance $C_{\mathrm{J}i}$ and Josephson coupling energy $E_{\mathrm{J}i}$) to a segment of a superconducting loop and through a gate capacitance $C_{\mathrm{g}i}$ to a voltage source $V_{\mathrm{g}i}$. For the

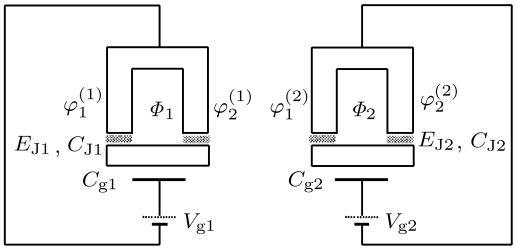


Fig.1. Double-qubit structure with SQUID.

SQUID loop, the following relationship can be deduced:

$$\varphi_1^{(i)} - \varphi_2^{(i)} = \frac{2e}{\hbar}\Phi_i, \tag{1}$$

where $\varphi^{(i)}$ $(i = 1, 2)$ is the difference in phase between the wave functions across the Josephson junction, and Φ_i is the total magnetic flux through the super-conducting loop. For convenience, expression (1) can be written as[16]

$$\varphi_1^{(i)} = \frac{e}{\hbar}\Phi_i, \quad \varphi_2^{(i)} = -\frac{e}{\hbar}\Phi_i, \quad (i = 1, 2), \tag{2}$$

where, we have supposed the initial phase difference $\varphi^{(i)}(0) = 0$.

In Ref.[16], Feynman *et al* gave

$$\dot{\varphi}^{(i)} = \frac{2eu_{\mathrm{J}i}}{\hbar}, \quad (i = 1, 2), \tag{3}$$

where $u_{\mathrm{J}i}$ is the electric potential difference across the Josephson junction. So when the tunnelling from one side to another takes place, the variation of Cooper-pair's potential energy is, respectively,

$$-\int_0^t u_{\mathrm{J}i} I_{ci} \sin\varphi_1^{(i)} \mathrm{d}t = E_{\mathrm{J}i}\left[1 - \cos\varphi^{(i)}\right], \quad \text{and} \quad -\int_0^t u_{\mathrm{J}i} I_{ci} \sin\varphi_2^{(i)} \mathrm{d}t = E_{\mathrm{J}i}\left[1 - \cos\varphi^{(i)}\right], \quad (i = 1, 2), \tag{4}$$

where $\varphi^{(i)} = \varphi_1^{(i)} = -\varphi_2^{(i)}$ and $I_{ci} = \dfrac{2e}{\hbar}E_{\mathrm{J}i}$ that is the critical electric current of the Josephson junction.

The magnetic flux Φ_i $(i = 1, 2)$ through each superconducting loop satisfies the following relationships:

$$\Phi_1 = \Phi_{x1} + \Phi_{L_1} + M\Phi_{L_2}/L_2, \quad \text{and} \quad \Phi_2 = \Phi_{x2} + \Phi_{L_2} + M\Phi_{L_1}/L_1, \tag{5}$$

where Φ_{xi} is the external flux, Φ_{L_i} the self-inductance flux, L_i the self-inductance of the loop and M the mutual inductance between the loops. From expression (5), we can obtain

$$\Phi_{L_1} = \frac{L_2\left(\Phi_1 - \Phi_{x1}\right)/M - \left(\Phi_2 - \Phi_{x2}\right)}{L_2/M - M/L_1}, \quad \text{and} \quad \Phi_{L_2} = \frac{L_1\left(\Phi_2 - \Phi_{x2}\right)/M - \left(\Phi_1 - \Phi_{x1}\right)}{L_1/M - M/L_2}. \tag{6}$$

Thus, if we consider $\varphi^{(i)}$ as the generalized coordinate, the potential energy related to $\varphi^{(i)}$ of the system is

$$\begin{aligned} V = & \sum_{i,j=1(i\neq j)}^{2} \left\{ \frac{1}{2L_i(L_j/M - M/L_i)^2} \left[\frac{L_j}{M}\left(\frac{\hbar}{e}\varphi^{(i)} - \Phi_{xi}\right) - \left(\frac{\hbar}{e}\varphi^{(j)} - \Phi_{xj}\right)\right]^2 + 2E_{\mathrm{J}i}\left[1 - \cos\varphi^{(i)}\right] \right\} \\ & + M \prod_{i,j=1(i\neq j)}^{2} \frac{1}{L_j(L_j/M - M/L_i)} \left\{ \frac{L_j}{M}\left[\frac{\hbar}{e}\varphi^{(i)} - \Phi_{xi}\right] - \left[\frac{\hbar}{e}\varphi^{(j)} - \Phi_{xj}\right] \right\}, \end{aligned} \tag{7}$$

and the kinetic energy related to $\dot{\varphi}^{(i)}$ is

$$T = \sum_{i=1}^{2} \left[\frac{1}{2}\left(2C_{\mathrm{J}i} + C_{\mathrm{g}i}\right)\left(\frac{\hbar}{2e}\right)^2 \dot{\varphi}^{(i)^2} + N_{\mathrm{g}i}\hbar\dot{\varphi}^{(i)} + \frac{2N_{\mathrm{g}i}^2 e^2}{C_{\mathrm{g}i}} \right], \tag{8}$$

where $N_{\mathrm{g}i} = -C_{\mathrm{g}i}V_{\mathrm{g}i}/(2e)$ is named the gate charge number. Then the Lagrangian of the system can be

obtained:

$$\begin{aligned}\mathcal{L} &= T - V \\ &= -\sum_{i,j=1(i\neq j)}^{2}\left\{\frac{1}{2L_i(L_j/M - M/L_i)^2}\left[\frac{L_j}{M}\left(\frac{\hbar}{e}\varphi^{(i)} - \Phi_{xi}\right) - \left(\frac{\hbar}{e}\varphi^{(j)} - \Phi_{xj}\right)\right]^2 + 2E_{Ji}\left[1-\cos\varphi^{(i)}\right]\right\} \\ &\quad -M\prod_{i,j=1(i\neq j)}^{2}\frac{1}{L_j(L_j/M - M/L_i)}\left\{\frac{L_j}{M}\left[\frac{\hbar}{e}\varphi^{(i)} - \Phi_{xi}\right] - \left[\frac{\hbar}{e}\varphi^{(j)} - \Phi_{xj}\right]\right\} \\ &\quad +\sum_{i=1}^{2}\left[\frac{1}{2}\left(2C_{\mathrm{J}i} + C_{\mathrm{g}i}\right)\left(\frac{\hbar}{2e}\right)^2\dot{\varphi}^{(i)^2} + N_{\mathrm{g}i}\hbar\dot{\varphi}^{(i)} + \frac{2N_{\mathrm{g}i}^2e^2}{C_{\mathrm{g}i}}\right]. \end{aligned} \tag{9}$$

Through the Josephson junction, Cooper-pairs can tunnel from or onto the island. The number n_i of excess Cooper-pairs on the island depends on the gate voltage. The number n_i satisfies the following relation:

$$2en_i = \frac{\hbar}{2e}\left(2C_{\mathrm{J}i} + C_{\mathrm{g}i}\right)\dot{\varphi}^{(i)} + 2N_{\mathrm{g}i}e, \quad (i=1,2), \tag{10}$$

where $2e$ is the charge of a Cooper-pair. The generalized momentum conjugate to $\varphi^{(i)}$ is

$$p_i = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}^{(i)}} = \frac{\hbar}{2e}\left[\frac{\hbar}{2e}\left(2C_{\mathrm{J}i} + C_{\mathrm{g}i}\right)\dot{\varphi}^{(i)} + 2N_{\mathrm{g}i}e\right] = n_i\hbar, \quad (i=1,2). \tag{11}$$

Note particularly that the generalized momentum p_i is proportional to n_i.

Thus, combining Eqs.(9)–(11) yields the classical Hamiltonian of the system

$$H = \sum_{i=1}^{2}\dot{\varphi}^{(i)}p_i - \mathcal{L} = H_1 + H_2 + H_3, \tag{12}$$

where

$$H_1 = 4E_{\mathrm{c}1}\left(n_{_1} - N_{\mathrm{g}1}\right)^2 + 2E_{\mathrm{J}1}\left[1-\cos\varphi^{(1)}\right] + \alpha_1\left[\frac{\hbar}{e}\varphi^{(1)} - \Phi_{x1}\right]^2 - \frac{2N_{\mathrm{g}1}^2e^2}{C_{\mathrm{g}1}}, \tag{13a}$$

$$H_2 = 4E_{\mathrm{c}2}\left(n_2 - N_{\mathrm{g}2}\right)^2 + 2E_{\mathrm{J}2}\left[1-\cos\varphi^{(2)}\right] + \alpha_2\left[\frac{\hbar}{e}\varphi^{(2)} - \Phi_{x2}\right]^2 - \frac{2N_{\mathrm{g}2}^2e^2}{C_{\mathrm{g}2}}, \tag{13b}$$

$$H_3 = \kappa\left(\frac{\hbar}{e}\varphi^{(1)} - \Phi_{x1}\right)\left(\frac{\hbar}{e}\varphi^{(2)} - \Phi_{x2}\right), \tag{13c}$$

$$E_{\mathrm{c}1} = \frac{e^2}{2\left(2C_{\mathrm{J}1} + C_{\mathrm{g}1}\right)}, \quad E_{\mathrm{c}2} = \frac{e^2}{2\left(2C_{\mathrm{J}2} + C_{\mathrm{g}2}\right)}, \tag{14}$$

$$\alpha_1 = \frac{L_2^2}{2L_1M^2\left(\frac{L_2}{M} - \frac{M}{L_1}\right)^2} + \frac{1}{2L_2\left(\frac{L_1}{M} - \frac{M}{L_2}\right)^2} - \frac{1}{L_1\left(\frac{L_2}{M} - \frac{M}{L_1}\right)\left(\frac{L_1}{M} - \frac{M}{L_2}\right)}, \tag{15a}$$

$$\alpha_2 = \frac{L_1^2}{2L_2M^2\left(\frac{L_1}{M} - \frac{M}{L_2}\right)^2} + \frac{1}{2L_1\left(\frac{L_2}{M} - \frac{M}{L_1}\right)^2} - \frac{1}{L_2\left(\frac{L_2}{M} - \frac{M}{L_1}\right)\left(\frac{L_1}{M} - \frac{M}{L_2}\right)}, \tag{15b}$$

$$\kappa = \frac{\left(\frac{M}{L_1L_2} + \frac{1}{M}\right)}{\left(\frac{L_2}{M} - \frac{M}{L_1}\right)\left(\frac{L_1}{M} - \frac{M}{L_2}\right)} - \frac{L_1}{L_2M\left(\frac{L_1}{M} - \frac{M}{L_2}\right)^2} - \frac{L_2}{L_1M\left(\frac{L_2}{M} - \frac{M}{L_1}\right)^2}, \tag{16}$$

with $E_{\mathrm{c}i} = \dfrac{e^2}{2\left(2C_{\mathrm{J}i} + C_{\mathrm{g}i}\right)}$ $(i=1,2)$ being the single-electron charging energy on the island. According to the standard canonical quantization principle, a pair of conjugate quantities n_i and φ_i are associated with a pair of

Hermitian operators $\hat{n}_i$ and $\hat{\varphi}_i$, and they satisfy the commutation relation $\left[\hat{\varphi}^{(i)}, \hat{n}_i\right] = -\,\mathrm{i}$. Thus, the quantized Hamiltonian of the system can be written as

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3, \tag{17}$$

where

$$\hat{H}_1 = 4E_{\mathrm{c}1}\left(\hat{n}_{_1} - N_{\mathrm{g}1}\right)^2 + 2E_{\mathrm{J}1}\left[1 - \cos\hat{\varphi}^{(1)}\right] + \alpha_1\left[\frac{\hbar}{e}\hat{\varphi}^{(1)} - \Phi_{x1}\right]^2 - \frac{2N_{\mathrm{g}1}^2 e^2}{C_{\mathrm{g}1}}, \tag{18a}$$

$$\hat{H}_2 = 4E_{\mathrm{c}2}\left(\hat{n}_{_2} - N_{\mathrm{g}2}\right)^2 + 2E_{\mathrm{J}2}\left[1 - \cos\hat{\varphi}^{(2)}\right] + \alpha_2\left[\frac{\hbar}{e}\hat{\varphi}^{(2)} - \Phi_{x2}\right]^2 - \frac{2N_{\mathrm{g}2}^2 e^2}{C_{\mathrm{g}2}}, \tag{18b}$$

$$\hat{H}_3 = \kappa\left[\frac{\hbar}{e}\hat{\varphi}^{(1)} - \Phi_{x1}\right]\left[\frac{\hbar}{e}\hat{\varphi}^{(2)} - \Phi_{x2}\right]. \tag{18c}$$

3. Hamiltonian operator of double-qubit structure in spin-1/2 notation

Now we discuss the various contributions to $\hat{H}$ given by expression (17) in detail. Here, we choose such a material that its superconducting energy gap Δ is the largest energy in the problem, even larger than the single-electron charging energy on the island. In this case quasiparticle tunnelling is suppressed at low temperatures, and such a situation can be reached that no quasiparticle excitation is found on the island.[17] Moreover, we consider a system in which the single-electron charging energy on the ith island is much larger than the Josephson coupling energy, i.e. $E_{ci} \gg E_{\mathrm{J}i}$. In this regime, a convenient basis is formed by the charge states, parameterized by the excess number $\hat{n}_i$ of Cooper-pairs on the island [17] as follows: $\hat{n}_i \left|n\right\rangle_i = n_i \left|n\right\rangle_i$.

Qubit-1 structure The commutation relation $\left[\hat{n}_1, \hat{\varphi}^{(1)}\right] = -\mathrm{i}$ implies

$$\left[\hat{n}_1, \exp\{\mathrm{i}\hat{\varphi}^{(1)}\}\right] = \exp\{\mathrm{i}\hat{\varphi}^{(1)}\}. \tag{19}$$

Using expression (19) and the basis state $\left|n\right\rangle_1$, we obtain

$$\exp\{\mathrm{i}\hat{\varphi}^{(1)}\}\left|n\right\rangle_1 = \left|n+1\right\rangle_1, \tag{20}$$

so, we can deduce

$$\cos\hat{\varphi}^{(1)} = \frac{1}{2}\sum_n \left(\left|n+1\right\rangle_{11}\left\langle n\right| + \left|n-1\right\rangle_{11}\left\langle n\right|\right). \tag{21}$$

At such a temperature that $k_{\mathrm{B}}T \ll E_{\mathrm{c}1}$,[12] we concentrate on a voltage near a degeneracy point where only two charge states, say $\left|n=0\right\rangle_1$ and $\left|n=1\right\rangle_1$, play a role, while all other charge states, having a much higher energy, can be ignored.[17] In this case, $\cos\hat{\varphi}^{(1)}$ can be presented as a matrix

$$\cos\hat{\varphi}^{(1)} = \frac{1}{2}\left(\left|1\right\rangle_{11}\left\langle 0\right| + \left|0\right\rangle_{11}\left\langle 1\right|\right) = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}. \tag{22}$$

By introducing the unitary operator

$$\hat{U}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{23}$$

we can diagonalize $\cos\hat{\varphi}^{(1)}$ as follows:

$$\begin{aligned}\hat{A}_1 &= \hat{U}_1^{-1}\cos\hat{\varphi}^{(1)}\hat{U}_1 \\ &= \cos\left[\hat{U}_1^{-1}\hat{\varphi}^{(1)}\hat{U}_1\right] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned} \tag{24}$$

Consequently, the spectral decomposition[15] of normal operator $\hat{A}_1$ is

$$\hat{A}_1 = \frac{1}{2}\left|0\right\rangle_{11}\left\langle 0\right| + \left(-\frac{1}{2}\right)\left|1\right\rangle_{11}\left\langle 1\right|. \tag{25}$$

Then we can obtain the operator function of normal operator $\hat{A}_1$

$$\hat{U}_1^{-1}\hat{\varphi}^{(1)}\hat{U}_1 = \arccos\hat{A}_1 = \arccos\frac{1}{2}\left|0\right\rangle_{11}\left\langle 0\right| + \arccos\left(-\frac{1}{2}\right)\left|1\right\rangle_{11}\left\langle 1\right| = \begin{pmatrix} \pi/3 & 0 \\ 0 & 2\pi/3 \end{pmatrix}. \tag{26}$$

By using the unitary operator $\hat{U}_1$ and expression (26), the Hamiltonian operator $\hat{H}_1$ given by expression (18a) can be presented as a matrix

$$\hat{H}_1^{'} = \hat{U}_1^{-1}\hat{H}_1\hat{U}_1 = \begin{pmatrix} H_{11}^{'} & H_{12}^{'} \\ H_{21}^{'} & H_{22}^{'} \end{pmatrix}, \tag{27}$$

where

$$H_{11}^{'} = 2E_{c1}N_{g1}^2 + 2E_{c1}\left(1 - N_{g1}\right)^2 - \frac{2N_{g1}^2e^2}{C_{g1}} + \alpha_1\Phi_{x1}^2 - \frac{2\alpha_1\pi\hbar}{3e}\Phi_{x1} + \frac{\alpha_1\pi^2\hbar^2}{9e^2} + E_{J1}, \tag{28}$$

$$H_{12}^{'} = H_{21}^{'} = 2E_{c1}\left(2N_{g1} - 1\right), \tag{29}$$

$$H_{22}^{'} = 2E_{c1}N_{g1}^2 + 2E_{c1}\left(1 - N_{g1}\right)^2 - \frac{2N_{g1}^2e^2}{C_{g1}} + \alpha_1\Phi_{x1}^2 - \frac{4\alpha_1\pi\hbar}{3e}\Phi_{x1} + \frac{4\alpha_1\pi^2\hbar^2}{9e^2} + 3E_{J1}, \tag{30}$$

The Hamiltonian operator $\hat{H}_1$ can be presented in spin-1/2 notation as

$$\hat{H}_1 = \Gamma_1\hat{\sigma}_x^{(1)} + \Lambda_1\hat{\sigma}_z^{(1)} + \Xi_1, \tag{31}$$

where

$$\Gamma_1 = \alpha_1\left(\frac{\pi\hbar}{3e}\Phi_{x1} - \frac{\pi^2\hbar^2}{6e^2}\right) - E_{J1}, \quad \Lambda_1 = H_{12}^{'}, \tag{32}$$

$$\Xi_1 = 2E_{c1}N_{g1}^2 + 2E_{c1}\left(1 - N_{g1}\right)^2 + \alpha_1\left(\Phi_{x1}^2 - \frac{\pi\hbar}{e}\Phi_{x1} + \frac{5\pi^2\hbar^2}{18e^2}\right) + 2E_{J1} - \frac{2N_{g1}^2e^2}{C_{g1}}. \tag{33}$$

For a particular case of $N_{g1} = 1/2$, near the degeneracy point, by a suitable manipulation of the gate voltage V_{g1}, the Hamiltonian operator $\hat{H}_1^{'}$ takes the simple form

$$\hat{H}_1^{'} = \begin{pmatrix} H_{11}^{'} & 0 \\ 0 & H_{22}^{'} \end{pmatrix}, \tag{34}$$

whose eigenvalues are $E_1 = H_{11}^{'}$ and $E_2 = H_{22}^{'}$, and the corresponding eigenstates are $\left|\chi_1^{'}\right\rangle_1 = |0\rangle_1$ and $\left|\chi_2^{'}\right\rangle_1 = |1\rangle_1$. Thus, the eigenstates of the system are, respectively,

$$|+\rangle_1 = \frac{|0\rangle_1 + |1\rangle_1}{\sqrt{2}} \quad \text{and} \quad |-\rangle_1 = \frac{|0\rangle_1 - |1\rangle_1}{\sqrt{2}}, \tag{35}$$

which are known as computational basis states[15] the same as $|0\rangle_1$ and $|1\rangle_1$, usually used in quantum computation and quantum information.

Qubit-2 structure Introducing the unitary operator

$$\hat{U}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{36}$$

and using similar methods as above turns the matrix form of the Hamiltonian operator $\hat{H}_2$, given by expression (18b) into

$$\hat{H}_2^{'} = \hat{U}_2^{-1}\hat{H}_2\hat{U}_2 = \begin{pmatrix} \tilde{H}_{11}^{'} & \tilde{H}_{12}^{'} \\ \tilde{H}_{21}^{'} & \tilde{H}_{22}^{'} \end{pmatrix}, \tag{37}$$

where

$$\tilde{H}_{11}^{'} = 2E_{c2}N_{g2}^2 + 2E_{c2}\left(1 - N_{g2}\right)^2 - \frac{2N_{g2}^2e^2}{C_{g2}} + \alpha_2\Phi_{x2}^2 - \frac{2\alpha_2\pi\hbar}{3e}\Phi_{x2} + \frac{\alpha_2\pi^2\hbar^2}{9e^2} + E_{J2}, \tag{38}$$

$$\tilde{H}_{12}^{'} = \tilde{H}_{21}^{'} = 2E_{c2}\left(2N_{g2} - 1\right), \tag{39}$$

$$\tilde{H}_{22}^{'} = 2E_{c2}N_{g2}^2 + 2E_{c2}\left(1 - N_{g2}\right)^2 - \frac{2N_{g2}^2e^2}{C_{g2}} + \alpha_2\Phi_{x2}^2 - \frac{4\alpha_2\pi\hbar}{3e}\Phi_{x2} + \frac{4\alpha_2\pi^2\hbar^2}{9e^2} + 3E_{J2}. \tag{40}$$

The Hamiltonian operator $\hat{H}_2$ can be presented in spin-1/2 notation as

$$\hat{H}_2 = \Gamma_2 \hat{\sigma}_x^{(2)} + \Lambda_2 \hat{\sigma}_z^{(2)} + \Xi_2, \tag{41}$$

where

$$\Gamma_2 = \alpha_2 \left(\frac{\pi\hbar}{3e} \Phi_{x2} - \frac{\pi^2\hbar^2}{6e^2} \right) - E_{\mathrm{J}2}, \quad \Lambda_2 = \tilde{H}_{12}^{'}, \tag{42}$$

$$\Xi_2 = 2E_{\mathrm{c}2} N_{\mathrm{g}2}^2 + 2E_{\mathrm{c}2} \left(1 - N_{\mathrm{g}2}\right)^2 + \alpha_2 \left(\Phi_{x2}^2 - \frac{\pi\hbar}{e} \Phi_{x2} + \frac{5\pi^2\hbar^2}{18e^2} \right) + 2E_{\mathrm{J}2} - \frac{2N_{\mathrm{g}2}^2 e^2}{C_{\mathrm{g}2}}. \tag{43}$$

For a particular case of $N_{\mathrm{g}2} = 1/2$, near the degeneracy point, by a suitable manipulation of the gate voltage $V_{\mathrm{g}2}$, the eigenvalues of $\hat{H}_2$ are $\tilde{E}_1 = \tilde{H}_{11}^{'}$ and $\tilde{E}_2 = \tilde{H}_{22}^{'}$, and the corresponding eigenstates are, respectively,

$$|+\rangle_2 = \frac{|0\rangle_2 + |1\rangle_2}{\sqrt{2}} \quad \text{and} \quad |-\rangle_2 = \frac{|0\rangle_2 - |1\rangle_2}{\sqrt{2}}. \tag{44}$$

Coupling term The coupling term $\hat{H}_3$ can be expressed in spin-1/2 notation as

$$\hat{H}_3 = \Gamma_3^{(1)} \hat{\sigma}_x^{(1)} + \Gamma_3^{(2)} \hat{\sigma}_x^{(2)} + \Gamma_3 \hat{\sigma}_x^{(1)} \hat{\sigma}_x^{(2)} + \Xi_3, \tag{45}$$

where

$$\Gamma_3^{(1)} = \frac{\kappa\pi\hbar}{6e} \left(\Phi_{x2} - \frac{\pi\hbar}{2e} \right), \quad \Gamma_3^{(2)} = \frac{\kappa\pi\hbar}{6e} \left(\Phi_{x1} - \frac{\pi\hbar}{2e} \right), \quad \Gamma_3 = \frac{\kappa\pi^2\hbar^2}{36e^2}, \tag{46}$$

$$\Xi_3 = \kappa \left(\Phi_{x1} - \frac{\pi\hbar}{2e} \right) \left(\Phi_{x2} - \frac{\pi\hbar}{2e} \right). \tag{47}$$

By substituting expressions (31), (41) and (45) into expression (17) and considering a particular case of $N_{\mathrm{g}i} = 1/2$, near the degeneracy point, the Hamiltonian operator $\hat{H}$ of the system can be written in spin-1/2 notation as

$$\hat{H} = \left(\Gamma_1 + \Gamma_3^{(1)} \right) \hat{\sigma}_x^{(1)} + \left(\Gamma_2 + \Gamma_3^{(2)} \right) \hat{\sigma}_x^{(2)} + \Gamma_3 \hat{\sigma}_x^{(1)} \hat{\sigma}_x^{(2)} + \Xi_1 + \Xi_2 + \Xi_3. \tag{48}$$

From expression (48), one can see that the quantum two-level system can be regarded as a spin-1/2 particle. Here, the effective 'magnetic field' is $B_x^{(i)} = \Gamma_i + \Gamma_3^{(i)}$ and the exchange energy between the two 'particles' is Γ_3. Note that we have included the Bohr magneton μ_{B} in the definition of the 'magnetic field' $B_x^{(i)}$. Using these compact forms, one can conveniently describe the evolution of the system in the quantum computing language. The two-qubit operations can be realized by evolving the system according to the following unitary transformation

$$\hat{U}(\tau) = \exp(-\mathrm{i}\hat{H}\tau/\hbar). \tag{49}$$

4. Quantum computation

The Hamiltonian given by expression (48) is the Ising-like[18] Hamiltonian with the homogeneous 'magnetic field' along the x-direction. In the two most notable cases, i.e. the Ising-like and the XY systems, it is well-known that universal quantum computation is possible.[18] So there exists a particularly simple realization of the controlled-phase-shift (CPS) gate,[15] namely

$$\hat{U}_{\mathrm{CPS}} = \exp\{\mathrm{i}\beta[1 - \hat{\sigma}_x^{(1)} - \hat{\sigma}_x^{(2)} + \hat{\sigma}_x^{(1)} \hat{\sigma}_x^{(2)}]\}. \tag{50a}$$

When the circuit components parameters and the external magnetic flux are chosen such that $\Gamma_1 + \Gamma_3^{(1)} = \Gamma_2 + \Gamma_3^{(2)} = -\Gamma_3 = \dfrac{\hbar\beta}{\tau}$ for the switching time τ, we can obtain the CPS gate,

$$\hat{U}_{\mathrm{CPS}} = \exp\{\mathrm{i}\beta[1 - (\Xi_1 + \Xi_2 + \Xi_3)/\Gamma_3]\} \hat{U}(\tau) = \exp\{\mathrm{i}\beta[1 - \hat{\sigma}_x^{(1)} - \hat{\sigma}_x^{(2)} + \hat{\sigma}_x^{(1)} \hat{\sigma}_x^{(2)}]\}. \tag{50b}$$

The phase factor $\exp\{\mathrm{i}\beta[1-(\Xi_1+\Xi_2+\Xi_3)/I_3]\}$ corresponds to an overall energy level shift of the system. In the space comprised of the basis vectors, $|+\rangle_1|+\rangle_2$, $|+\rangle_1|-\rangle_2$, $|-\rangle_1|+\rangle_2$ and $|-\rangle_1|-\rangle_2$, the matrix of the operator $\hat{U}_{\mathrm{CPS}}$ is

$$\hat{U}_{\mathrm{CPS}}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\mathrm{e}^{\mathrm{i}4\beta}\end{pmatrix}. \tag{51}$$

Thus, we can obtain

$$\hat{U}_{\mathrm{CPS}}=|+\rangle_1|+\rangle_{22}\langle+|_1\langle+|+|+\rangle_1|-\rangle_{22}\langle-|_1\langle+|+|-\rangle_1|+\rangle_{22}\langle+|_1\langle-|+\exp(\mathrm{i}4\beta)|-\rangle_1|-\rangle_{22}\langle-|_1\langle-|. \tag{52}$$

From expression (52) it is found that, if we regard the first qubit as control and the second qubit as target, when the control qubit is $|+\rangle_1$ the states of the target qubit must be left alone. But, when the control qubit is $|-\rangle_1$, a phase shift $\exp(\mathrm{i}4\beta)$ is applied to the target qubit that is $|-\rangle_2$, but not the one that is $|+\rangle_2$. By introducing the Hadamard gate $\hat{U}_{\mathrm{H}i}=\dfrac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$, $(i=1,\,2)$, one can obtain the CPS gate $\hat{U}'_{\mathrm{CPS}}$ for the basis vectors $|0\rangle_1|0\rangle_2$, $|0\rangle_1|1\rangle_2$, $|1\rangle_1|0\rangle_2$ and $|1\rangle_1|1\rangle_2$,

$$\begin{aligned}\hat{U}'_{\mathrm{CPS}}&=\hat{U}^{+}_{\mathrm{H1}}\hat{U}^{+}_{\mathrm{H2}}\hat{U}_{\mathrm{CPF}}\hat{U}_{\mathrm{H1}}\hat{U}_{\mathrm{H2}}\\&=|0\rangle_1|0\rangle_{22}\langle0|_1\langle0|+|0\rangle_1|1\rangle_{22}\langle1|_1\langle0|+|1\rangle_1|0\rangle_{22}\langle0|_1\langle1|+\exp(\mathrm{i}4\beta)|1\rangle_1|1\rangle_{22}\langle1|_1\langle1|,\end{aligned} \tag{53}$$

which leaves the basis vectors $|0\rangle_1|0\rangle_2$, $|0\rangle_1|1\rangle_2$, $|1\rangle_1|0\rangle_2$ alone, but applies a phase shift $\exp(\mathrm{i}4\beta)$ to $|1\rangle_1|1\rangle_2$.

5. Conclusions

The scheme of number-phase quantization of the double-qubit structure with SQUIDs is proposed. By introducing the unitary matrices and using the spectral decompositions, we exactly formulate the Hamiltonian operator of the double-qubit structure in compact forms in spin-1/2 notation. Using these compact forms, one can conveniently describe the evolution of the systems in the quantum computing language. We also propose a scheme to design controlled-phase-shift(CPS) gates by using this system. From the point of view of quantum computation and quantum information, the above conclusions will be helpful to the study of charge qubits.

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