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The quantum Kirchhoff equation and quantum current and energy spectrum of a homogeneous mesoscopic dissipation transmission line*

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On the basis of quantization of charge, the loop equations of quantum circuits are investigated by using the Heisenberg motion equation for a mesoscopic dissipation transmission line. On the supposition that the system has a symmetry under translation in charge space, the quantum current and the quantum energy spectrum in the mesoscopic transmission line are given by solving their eigenvalue equations. Results show that the quantum current and the quantum energy spectrum are not only related to the parameters of the transmission line, but also dependent on the quantized character of the charge obviously.

Keywords: mesoscopic dissipation transmission line, quantum Kirchhoff's equation, quantum current, quantum energy spectrum

PACC: 7335, 0365

1. Introduction

With the development of nanotechnology and nanoelectronics, the miniaturization of integrated circuits and components becomes more and more important. When the integration density of circuits increases, and the sizes of elemental unit-circuit and device decrease, the power dissipation of unit area will rise and the traditional circuits and devices will approach 'physical limit' of the working principle. And in this case quantum mechanical effects should be considered.[1−3] Recently, the quantum fluctuations in mesoscopic RLC (or LC) circuits have been investigated widely, but these researches are mainly restricted to the lumped parameter circuit.[4−13] Louisell[14] has suggested an quantization method for a mesoscopic dissipationless transmission line, and Wang[15,16] studied the quantum fluctuations of charge and current in a transmission line in Louisell's way, but did not take into account the quantized property of charge. In this paper, within the frame of discrete charge,[17−25] the quantum loop voltage equations of homogeneous mesoscopic dissipation transmission line are studied by using the Heisenberg motion equation; and on the supposition that the system has a symmetry under translation in charge space, the quantum current and the quantum energy spectrum of the transmission line are given by solving the eigenvalue equations.

2. Quantum Kirchhoff's equation and quantum vibration of charge

We consider a homogeneous mesoscopic dissipation transmission line, each unit has a resistance R, inductance L and capacitance C. The circuit model is shown in Fig.1. For the charge in the lth unit ($l \in x$) of transmission line, its classical motion equation is

$$LC\ddot{q}_l + RC\dot{q}_l + 2q_l = q_{l+1} + q_{l-1}. \tag{1}$$

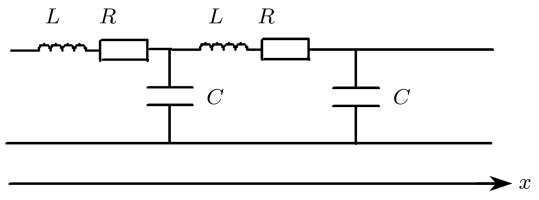


Fig.1. A homogeneous mesoscopic dissipation transmission line.

However, for the mesoscopic system, we are required to consider the quantum effect of charge.[17−20] Ac-

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cording to the quantum property of charge, we let the eigenvalue of charge operator $\hat{q}_l$ have a discrete value, i.e. $\hat{q}_l|n_l\rangle = n_l q_e|n_l\rangle$, where q_e is the elementary charge and $n_l \in z$, z is a set of integers. This allows us to introduce a minimum 'shift operator': $\hat{Q}_l = \exp(\mathrm{i}q_e\hat{p}_l/\hbar)$, where the canonical current $\hat{p}_l$ is Dirac conjugate of charge operator $\hat{q}_l$, i.e. $[\hat{q}_l, \hat{p}_l] = \mathrm{i}\hbar$. The discrete derivatives are defined by the minimum shift operators

$$D_l := (\hat{Q}_l - 1)/q_e, \quad \text{and} \quad \bar{D}_l := (1 - \hat{Q}_l^+)/q_e. \tag{2}$$

The quantum Hamilton operator then can be expressed as[18]

$$\hat{H} = \sum_l \left[-\frac{\hbar^2}{2L} \exp\left(-\frac{R}{L}t\right) D_l \bar{D}_l + \frac{1}{2C} \exp\left(\frac{R}{L}t\right) (q_{l+1} - q_l)^2 \right]. \tag{3}$$

With the relation[17] $\hat{Q}_l\hat{Q}_l^+ = \hat{Q}_l^+\hat{Q}_l = 1$, the time evolutions of charge and current in the lth unit of transmission line can be given by the Heisenberg motion equation, respectively, as follows:

$$\frac{\mathrm{d}\hat{q}_l}{\mathrm{d}t} = \frac{\hbar}{q_e L} \exp\left(-\frac{R}{L}t\right) \sin\left(\frac{q_e}{\hbar}\hat{p}_l\right), \quad \text{and} \quad \frac{\mathrm{d}\hat{p}_l}{\mathrm{d}t} = \frac{1}{CL} \exp\left(\frac{R}{L}t\right) (\hat{q}_{l+1} + \hat{q}_{l-1} - 2\hat{q}_l). \tag{4}$$

Further, we can obtain

$$LC\frac{\mathrm{d}^2\hat{q}_l}{\mathrm{d}t^2} + RC\frac{\mathrm{d}\hat{q}_l}{\mathrm{d}t} + 2\cos\left(\frac{q_e}{\hbar}\hat{p}_l\right)\hat{q}_l = \cos\left(\frac{q_e}{\hbar}\hat{p}_l\right)(\hat{q}_{l+1} + \hat{q}_{l-1}). \tag{5}$$

Equation (5) is known as quantum Kirchhoff equation of a homogeneous mesoscopic dissipation transmission line because the Planck constant and charge quantum are included. When $q_e \to 0$, taking first approximation yields $\cos(q_e\hat{p}_l/\hbar) \to 1$, and Eq.(5) is converted into the simple form of transmission line Eq.(1).

Taking the average of Eq.(5) and using $\left\langle \frac{\mathrm{d}\hat{A}}{\mathrm{d}t} \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \hat{A} \right\rangle$, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\langle\hat{q}_l\rangle + \frac{R}{L}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{q}_l\rangle + \omega^2\langle\hat{q}_l\rangle = f(t), \tag{6}$$

where

$$\omega^2 = \frac{2}{LC\langle q_l\rangle}\left\langle \cos\left(\frac{q_e}{\hbar}\hat{p}_l\right)\hat{q}_l \right\rangle,$$

and

$$f(t) = \frac{1}{LC}\left\langle \cos\left(\frac{q_e}{\hbar}\hat{p}_l\right)(\hat{q}_{l+1} + \hat{q}_{l-1}) \right\rangle. \tag{7}$$

Suppose the coupled interaction between neighbouring units to take the harmonic form $f(t) = F_0\cos(\omega' t)$, where F_0 and ω' denote the coupled strength and frequency respectively, then we will have the solution $\langle q_l\rangle = A_q \exp(\mathrm{i}\omega' t)$. From Eq.(6), we can obtain

$$A_q = F_0/[-\omega'^2 + \mathrm{i}\omega' R/L + \omega^2]. \tag{8}$$

The vibration amplitude of quantum charge is

$$|A_q| = F_0/\sqrt{\Delta^2 + (\omega' R/L)^2}, \tag{9}$$

where $\Delta = \omega^2 - \omega'^2$. The result shows that the difference between the frequency of the coupled action and the eigenfrequency of transmission line can affect the vibration amplitude of charge quantum.

3. Quantum current

With the minimum shift operators, the quantum current Eq.(4) can be rewritten as

$$\hat{I}_l = \frac{\mathrm{i}\hbar}{2q_e L} \exp\left(-\frac{R}{L}t\right)\left(\hat{Q}_l^+ - \hat{Q}_l\right), \tag{10}$$

with the minimum shift operators satisfying the following relations:[17]

$$\hat{Q}_l|n_l\rangle = |n_l - 1\rangle, \quad \text{and} \quad \hat{Q}_l^+|n_l\rangle = |n_l + 1\rangle. \tag{11}$$

Taking account of charge quantization $\hat{q}_l|n_l\rangle = n_l q_e|n_l\rangle$ and the relations of Eq.(11), we can obtain the quantum current of transmission line by solving eigenvalue equation $\hat{I}_l|\phi_l\rangle = I_l|\phi_l\rangle$ in charge representation.

Expanding the wave vector $|\phi_l\rangle$ according to the eigenstate $|n_l\rangle$ as $|\phi_l\rangle = \sum_{n_l} \phi_{n_l}|n_l\rangle$, where $\phi_{n_l} = \langle n_l|\phi_l\rangle$, from the eigenvalue equation of current and the completeness of $|n_l\rangle$, we have

$$\frac{\mathrm{i}\hbar}{2q_\mathrm{e}L}\exp\left(-\frac{R}{L}t\right)\sum_{n_l'}\langle n_l|\hat{Q}_l^+ - \hat{Q}_l|n_l'\rangle\phi_{n_l'} = I_l\phi_{n_l}. \tag{12}$$

The orthonormality of $|n_l\rangle$ leads to the non-zero matrix elements of $\hat{Q}_l$ and $\hat{Q}_l^+$ as follows:

$$\sum_{n_l'}\langle n_l|\hat{Q}_l^+|n_l'\rangle = \delta_{n_l,n_l'+1},$$

and

$$\sum_{n_l'}\langle n_l|\hat{Q}_l|n_l'\rangle = \delta_{n_l,n_l'-1}. \tag{13}$$

Inserting Eq.(13) into Eq.(12), we obtain the wave function equation

$$\frac{\mathrm{i}\hbar}{2q_\mathrm{e}L}\exp\left(-\frac{R}{L}t\right)(\phi_{n_l-1} - \phi_{n_l+1}) = I_l\phi_{n_l}. \tag{14}$$

In order to obtain the quantum current by solving Eq.(14), we suppose the system to have a symmetry under translation in charge space. Since $\hat{Q}_l$ and $\hat{Q}_l^+$ are both commutative with $\hat{I}_l$, and they have a common eigenfunction, we can have[18,19]

$$\hat{Q}_l|\phi_l\rangle = \exp(\mathrm{i}\theta)|\phi_l\rangle,$$

and

$$\hat{Q}_l^+|\phi_l\rangle = \exp(-\mathrm{i}\theta)|\phi_l\rangle, \tag{15}$$

where, $\theta(0 < \theta < 2\pi)$ denotes the phase angle in the lth unit, and it is related to the electronic parameter of transmission line. In a way similar to that in dealing with Eqs.(12)–(14), we have

$$\phi_{n_l} = \exp(-\mathrm{i}n_l\theta). \tag{16}$$

Inserting expression (16) into Eq.(14) yields

$$I_l = \frac{\hbar}{q_\mathrm{e}L}\exp\left(-\frac{R}{L}t\right)\sin\theta. \tag{17}$$

The result shows that the amplitude of per-unit quantum current is related to resistance and inductance, but unrelated to capacitance.

4. Quantum energy spectrum

We can rewrite Eq.(3) as follows:

$$\hat{H} = \hat{H}_0 + \hat{H}_\mathrm{I}, \tag{18}$$

where

$$\hat{H}_0 = \sum_l \hat{H}_l = \sum_l\left[\frac{2\hbar^2}{Lq_\mathrm{e}^2}\exp\left(-\frac{R}{L}t\right)\sin^2\left(\frac{q_\mathrm{e}}{2\hbar}\hat{p}_l\right) + \frac{1}{C}\exp\left(\frac{R}{L}t\right)\hat{q}_l^2\right], \tag{19}$$

$$\hat{H}_\mathrm{I} = -\frac{1}{C}\sum_l\left[\exp\left(\frac{R}{L}t\right)\hat{q}_{l+1}\hat{q}_l\right]. \tag{20}$$

From expression (19), we can find that the Hamiltonian of the system comprises a nonlinear magnetic energy part, which makes it difficult to investigate the energy spectrum property of transmission line. For simplicity, we calculate the energy by using the basic vector of noninteraction Hamiltonian $\hat{H}_l$.

In expression (19), the site Hamiltonian $\hat{H}_l$ corresponds to that for an RLC circuit with a capacitance $C/2$. For sub–section $\{|l\rangle, l \in z\}$, we have

$$\hat{H}_l|l\rangle = E_l|l\rangle, \tag{21}$$

where $\hat{q}_l$ acts only on the lth unit. Since the total Hilbert space is the direct product of sub-space, i.e. $|\{l\}> = |1\rangle \otimes |2\rangle \otimes \cdots |l\rangle \otimes \cdots$, the complete Hamiltonian is Hermitian, and its matrix elements become tri-diagonal in basis vector $|\{l\}\rangle$, namely

$$\langle l|\hat{H}|l\rangle = E_l, \tag{22}$$

$$\langle l|\hat{H}|l+1\rangle = \langle l-1|\hat{H}|l\rangle = -\frac{1}{C}\exp\left(\frac{R}{L}t\right)q_\mathrm{e}^2. \tag{23}$$

Consider that Hamiltonian $\hat{H}$ changes with time slowly, and let $\hat{H}|\psi\rangle = E|\psi\rangle$, then we will have

$$\sum_{\{l\}}\psi_{\{l\}}\langle\{l'\}|\hat{H}|\{l\}\rangle = E\sum_{\{l\}}\psi_{\{l\}}\langle\{l'\}|\{l\}\rangle, \tag{24}$$

namely

$$\psi_l\sum_l E_l - \frac{1}{C}\exp\left(\frac{R}{L}t\right)q_\mathrm{e}^2(\psi_{l+1} + \psi_{l-1}) = E\psi_l. \tag{25}$$

Similarly, by taking $\psi_l = \exp(\mathrm{i}l\theta)$, we can obtain the energy spectrum

$$E = \sum_l E_l - \frac{2\cos\theta}{C}\exp\left(\frac{R}{L}t\right)q_\mathrm{e}^2, \tag{26}$$

where θ is the continuous variable because the transmission line is infinite. It shows that the energy stored in LC of a unit is now spread into a band due to the interaction between neighbouring units. Equation (26) includes the energy E_l of a nonstandard harmonic oscillator, and its quantity is determined by the parameters of transmission line.

5. Conclusion

On the basis of the discrete charge, by quantizing a homogeneous mesoscopic dissipation transmission line, the quantum Kirchhoff equation is obtained, and then the quantum current and quantum energy spectrum are given. From Eq.(5), we can find that the coupling action in the lth unit between the front unit and the rear unit plays a driving source role. Though these sources are localized apparently, the quantum current in each unit is non-localized because the current is related to the coupling action of neighbouring units. Equation (17) indicates that the value of quantum current is within $[-\hbar/(q_{\rm e}L), \hbar/(q_{\rm e}L)]$, and the quantum current varies sinusoidally with the phase angle θ. In the presence of resistance, the vibration of quantum current decays with time exponentially, whose rate is determined by the time constant of the cell. The result of Eq.(26) shows that the quantum energy spectrum depends on not only the property of charge quantization, but also the electric parameters of the transmission line. We expect these results to be helpful for designing and applying quantum circuits in quantum computers.

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