Vol 16 No 10, October 2007
1009-1963/2007/16(10)/3067-05

Chinese Physics

© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

The choosing of reproducing kernel particle shape function with mathematic proof*

Xia Mao-Hui(夏茂辉)† and Li Jin(李 金)

College of Science, Yanshan University, Qinhuangdao 066004, *China*

(Received 8 October 2006; revised manuscript received 6 June 2007)

Many mechanical problems can be induced from differential equations with boundary conditions; there exist analytic and numerical methods for solving the differential equations. Usually it is not so easy to obtain analytic solutions. So it is necessary to give numerical solutions. The reproducing kernel particle (RKP) method is based on the Garlerkin Meshless method. According to the Sobolev space and Fourier transform, the RKP shape function is mathematically proved in this paper.

Keywords: point of interpolation, particle, reproducing kernel particle, shape function
PACC: 7155P, 0320

1. Introduction

The meshless method[1,4,6,8] is a new numerical method which has been developed rapidly in recent years. How to choose the shape function flexibly is very important in the meshless method. Based on the Fourier transform[2] and Sobolev spaces,[5,7] this paper gives the proof of reproducing kernel particle (RKP) shape function. The error of different shape functions is determined by the size of the neighbourhood of the R. When the neighbourhood size is big enough the error will be smaller, but the computing time is longer; otherwise the computing time is less.

2. Basic theory

We consider uniformly distributed particles $\left\{x_j^h\right\}$ in R^n and the associated particle shape function $\left\{\phi_j^h\right\}$ which is defined as

$$\begin{aligned}\phi_j^h(x) &= \phi_j^h(x_1,\ldots,x_n)\\ &= \phi\left(\frac{x-jh}{h}\right)\\ &= \phi\left(\frac{x_1-j_1h}{h},\ldots,\frac{x_n-j_nh}{h}\right),\end{aligned}$$

where $\phi \subset H^q(R^n)$ with $q \geq 1$ has a compact support: $\sup p\phi \subset B_R(0)$.

$$||\phi_j^h||_{H^1(R^n)} \leq h^{\frac{n}{2}-1}||\phi||_{H^1(R^n)}. \tag{1}$$

For all $x \in R^n$, $p \in P^k(R^n)$, with $\left\{\phi_j^h\right\}$ assumed to be reproducing of order k.

Let Ω be a bounded domain in R^n, consider a smooth function $u(x)$ in the domain Ω and study the error $u - \tilde{I}_h u$, where $\tilde{I}_h u$ is the interpolation of u in terms of ϕ_j^h, then $\left(\tilde{I}_h u\right)(x)$ will be defined as

$$\left(\tilde{I}_h u\right)(x) = \sum_{j\in A_x^h} u\left(x_j^h\right)\phi_j^h(x) \tag{2}$$

where $j \in A_x^h$ is the influence set for point x.

In Eq.(2), we assume that $u\left(x_j^h\right)$ is defined for all $j \in A_x^h$. If $p \in P^k$, for all $x \in R^n$, from the function $\sum\limits_{j\in Z} p\left(x_j^h\right)\phi_j^h(x) = p(x)$ we have

$$\sum_{j\in A_x^h} u\left(x_j^h\right)\phi_j^h(x) = \sum_{j\in Z^n} u\left(x_j^h\right)\phi_j^h(x) = p(x), \tag{3}$$

which simply is $\tilde{I}_h p = p$. Now let $s > n/2$, for some $x \in \Omega$, the particle x_j^h for all $j \in A_x^h$ may be outside Ω, and $u\left(x_j^h\right)$ may not be defined. To define them for all $u \in H^s(\Omega)$ and for all $x \in R^n$, we need an extension $\bar{u}$ of u in a sphere B_{R_0} containing Ω such that $\mathrm{dist}(\partial\Omega, \partial B_{R_0}) > \rho h$, and $\bar{u} \in H^s(B_{R_0})$. Then

$$\left(\tilde{I}_h u\right)(x) \equiv \left(\tilde{I}_h \bar{u}\right)(x) = \sum_{j\in A_x^h} \bar{u}\left(x_j^h\right)\phi_j^h(x). \tag{4}$$

For an extension of $\bar{u}$, let $\bar{u} = Eu$, where Eu is defined as $E : L_2(\Omega) \to L_2(R^n)$, $Eu|_\Omega = u$. Thus $\left(\tilde{I}_h u\right)(x)$ for all $x \in \Omega$, depends on a few values of

*Project supported by the Doctoral Scientists of Yanshan University (Grant No B272).
†E-mail: zhizihua666@126.com

$\bar{u}\left(x_j^h\right)$, where the particle x_j^h is just outside Ω, generally, $\phi_j^h\left(x_j^h\right) \neq \delta_{ij}$, hence $\left(\tilde{I}_h u\right)\left(x_j^h\right) \neq u\left(x_j^h\right)$. For $x \in R^n$ and $|\alpha| = k+1$, we define a function

$$\xi_\alpha^h(x) = x^\alpha - \sum_{i \in A_x^h} \left(x_i^h\right)^\alpha \phi_i^h(x). \tag{5}$$

Similarly, for $x \in R^n$ and $|\alpha| = k+1$, we also have

$$\xi_\alpha(x) = \xi_\alpha^1(x) = x^\alpha - \sum_{i \in A_x^h} (\mathrm{i})^\alpha \phi_i^h(x-i), \tag{6}$$

where A_x^1 is A_x^h with $h=1$. Let I_j^h be the cell centred at x_j^h and defined as

$$I_j^h = \left\{ x : ||x - x_j^h||_\infty \equiv \max_{i=1,2,\ldots,n} |x_i - x_j| \leq h/2 \right\}.$$

For each I_j^h, we define $A_j^h = \left\{k \in Z^n : \dot{\eta}_k^h \cap I_j^h \neq 0\right\}$ and $B_j^h = \left\{\cup_{k \in A_j^h} B_{Rh}\left(x_k^h\right)\right\} \cup I_j^h$.

Lemma 1 $\xi_\alpha^h(x)$, with $|\alpha| = k+1$, is periodic, i.e.

$$\xi_\alpha^h\left(x + x_j^h\right) = \xi_\alpha^h(x). \tag{7}$$

Lemma 2 Let $\alpha = \alpha(i), i = 1, 2, \ldots, M_k$ be an enumeration of multi-indices α with $|\alpha| = k+1$, and I_j^h be the cell centred at particle x_j^h, then, the following equalities will hold for $d_\alpha \in R$:

$$\begin{aligned} &|| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \xi_\alpha^h(x) ||_{H^1\left(I_j^h\right)}^2 \\ =\ & h^{2k+n} V^{\mathrm{T}} \left(A + h^2 B\right) V, \end{aligned} \tag{8}$$

where $V = \left[d_{\alpha(1)}, d_{\alpha(2)}, \ldots, d_{\alpha(M_k)}\right]^{\mathrm{T}}$, A and B are $M_k \times M_k$ matrices given by respectively

$$A_{lm} = \int_I \frac{1}{\alpha(l)!\alpha(m)!} \nabla \xi_{\alpha(l)} \cdot \nabla \xi_{\alpha(m)} \mathrm{d}x, \tag{9}$$

$$B_{lm} = \int_I \frac{1}{\alpha(l)!\alpha(m)!} \xi_{\alpha(l)} \cdot \xi_{\alpha(m)} \mathrm{d}x. \tag{10}$$

and $I = [-1/2, 1/2]^n$.

Proof According to functions (5) and (6), we can have $\xi_\alpha^h\left(\frac{x}{h}\right) = h^{-(k+1)} \xi_\alpha^h(x)$.

Now using the periodicity of $\xi_\alpha^h(x)$, we have

$$\begin{aligned} &|| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \nabla \xi_\alpha^h(x) ||_{H^0\left(I_j^h\right)}^2 \\ =\ & || \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \nabla \xi_\alpha^h(x) ||_{H^0\left(I_0^h\right)}^2 \\ =\ & h^{2(k+1)} || \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \nabla \left[\xi_\alpha\left(\frac{x}{h}\right)\right] ||_{H^0\left(I_0^h\right)}^2 \\ =\ & h^{2(k+1)} h^{n-2} || \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \nabla \xi_\alpha(y) ||_{H^0\left(I_0^h\right)}^2 \\ =\ & h^{2k+n} V^{\mathrm{T}} A V. \end{aligned} \tag{11}$$

Using a similar argument, we have

$$|| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \xi_\alpha^h(x) ||_{H^0\left(I_j^h\right)}^2 = h^{2k+2+n} V^{\mathrm{T}} B V.$$

Combining this function with equality (11), we have

$$|| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} d_\alpha \xi_\alpha^h(x) ||_{H^1\left(I_j^h\right)}^2 = h^{2k+n} V^{\mathrm{T}} \left(A + h^2 B\right) V.$$

This is the expected result.

Lemma 3 Let I_j^h be the cell centred at x_j^h, consider the corresponding set $\tilde{B}_j^h$, and suppose $u \in H^{k+2+q}\left(\tilde{B}_j^h\right)$ with $q > n/2$ when $n \geq 2$, and $q = 0$ when $n = 1$, t hen the following inequalities will hold:

1) for any $\delta > 0$,

$$\begin{aligned} &||u - \tilde{I}_h u||_{H^1\left(I_j^h\right)}^2 \\ \leq\ & \left(1 + \delta^2\right) || \sum_{|\alpha|=k+1} \frac{1}{\alpha!} \left(D^\alpha u\right)\left(x_j^h\right) \xi_\alpha^h(x) ||_{H^1\left(I_j^h\right)}^2 \\ &+ \left(1 + \frac{1}{\delta^2}\right) C h^{2k+2} \sum_{|\alpha|=k+2} ||D^\alpha u||_{H^q\left(\tilde{B}_j^h\right)}^2; \end{aligned} \tag{12}$$

2) for any $\delta > 0$,

$$\begin{aligned} &|| \sum_{|\alpha|=k+1} \frac{1}{\alpha!} \left(D^\alpha u\right)\left(x_j^h\right) \xi_\alpha^h(x) ||_{H^1\left(I_j^h\right)}^2 \\ \leq\ & \left(1 + \delta^2\right) ||u - \tilde{I}_h u||_{H^1\left(I_j^h\right)}^2 + \left(1 + \frac{1}{\delta^2}\right) \\ &\times C h^{2k+2} \sum_{|\alpha|=k+2} ||D^\alpha u||_{H^q\left(\tilde{B}_j^h\right)}^2. \end{aligned} \tag{13}$$

We can find the proof in Ref.[3].

Theorem 4 Let $\bar{\lambda}$ given in (9) be the maximal eigenvalue of matrix A and suppose $q > n/2$ when $n \geq 2$, and $q = 0$ when $n = 1$ then the following relationship will hold:

$$\sup_{u \in H^{k+2+q}(\Omega)} \lim_{h \to 0} \frac{||u - \tilde{I}_h u||_{H^1(\Omega)}^2}{h^{2k} Q_h(u)} = \bar{\lambda}, \tag{14}$$

where

$$Q_h(u) = |u|_{H^{k+1}(\Omega)}^2 + h \sum_{|\alpha|=k+2} ||D^\alpha u||_{H^q(\Omega)}^2. \tag{15}$$

Proof Letting $u \in H^{k+2+q}(\Omega)$, we first prove the following function:

$$\lim_{h \to 0} \frac{||u - \tilde{I}_h u||_{H^1(\Omega)}^2}{h^{2k} Q_h(u)} = \frac{\int_\Omega V^{\mathrm{T}}(x) A V(x) \mathrm{d}x}{|u|_{H^{k+1}(\Omega)}^2} \tag{16}$$

for $V^{\mathrm{T}}(x)=[D^{\alpha(1)}u(x),D^{\alpha(2)}u(x),\ldots,D^{\alpha(M_k)}u(x)]$, and $\alpha(i),1\leq i\leq M_k$ are the multi-indices with $|\alpha(i)|=k+1$. Let $u\in H^{k+2+q}(\Omega)$ and assume $\bar{u}$ to be an extension of u, then the following equality will hold since $\Omega\subset\bar{\Omega}_h$:

$$\begin{aligned}||u-\tilde{I}_hu||^2_{H^1(\Omega)}&\leq||\bar{u}-\tilde{I}_hu||^2_{H^1(\Omega)}\\&=\sum_{j\in\bar{A}^h}||\bar{u}-\tilde{I}_hu||^2_{H^1(I_j^h)}.\end{aligned}$$

Using equality (8) and $\bar{B}^h=\cup_{j\in\bar{A}^h}\tilde{B}_j^h$, for any $\delta>0$, we have

$$\begin{aligned}&||\bar{u}-\tilde{I}_hu||^2_{H^1(I_j^h)}\\&\leq(1+\delta^2)\sum_{j\in\bar{A}^h}||\sum_{|\alpha|=k+1}\frac{1}{\alpha!}(D^\alpha u)(x_j^h)\xi_\alpha^h(x)||^2_{H^1(I_j^h)}\\&\quad+\left(1+\frac{1}{\delta^2}\right)Ch^{2k+2}\sum_{j\in\bar{A}^h}\sum_{|\alpha|=k+2}||D^\alpha u||^2_{H^q(\tilde{B}_j^h)}\\&\leq(1+\delta^2)h^{2k}\sum_{j\in\bar{A}^h}h^nV_j^{\mathrm{T}}(A+h^2B)V_j\\&\quad+\left(1+\frac{1}{\delta^2}\right)Ch^{2k+2}\sum_{|\alpha|=k+2}||D^\alpha u||^2_{H^q(\tilde{B}_j^h)},\end{aligned}\tag{17}$$

where $V_j^{\mathrm{T}}=[D^{\alpha(1)}\bar{u}(x_j^h),\ D^{\alpha(2)}\bar{u}(x_j^h),\ \ldots,\ D^{\alpha(M_k)}\bar{u}(x_j^h)]$. Therefore, dividing both sides of inequality (17) by $h^{2k}Q_h(u)$, where $Q_h(u)$ is defined in equality (15), we have

$$\begin{aligned}&\frac{||\bar{u}-\tilde{I}_hu||^2_{H^1(I_j^h)}}{h^{2k}Q_h(u)}\\&\leq(1+\delta^2)\frac{\sum_{j\in\bar{A}^h}h^nV_j^{\mathrm{T}}(A+h^2B)V_j}{Q_h(u)}\\&\quad+\left(1+\frac{1}{\delta^2}\right)Ch^2\frac{\sum_{|\alpha|=k+2}||D^\alpha u||^2_{H^q(\tilde{B}_j^h)}}{Q_h(u)}.\end{aligned}\tag{18}$$

The quadratic form $V_j^{\mathrm{T}}(A+h^2B)V_j$ is

$$D^{\alpha(i)}\bar{u}(x_j^h)(A_{il}+h^2B_{il})D^{\alpha(l)}\bar{u}(x_j^h).$$

Since

$$\begin{aligned}&\lim_{h\to0}\sum_{j\in\bar{A}^h}h^nD^{\alpha(i)}\bar{u}(x_j^h)A_{il}D^{\alpha(l)}\bar{u}(x_j^h)\\&=\int_\Omega D^{\alpha(i)}u(x)A_{il}D^{\alpha(l)}u(x)\,\mathrm{d}x\end{aligned}$$

and $\lim_{h\to0}h^2\sum_{j\in\bar{A}^h}h^nD^{\alpha(i)}\bar{u}(x_j^h)B_{il}D^{\alpha(l)}\bar{u}(x_j^h)=0$. we have

$$\begin{aligned}&\lim_{h\to0}\sum_{j\in\bar{A}^h}h^nV_j^{\mathrm{T}}(A+h^2B)V_j\\&=\int_\Omega V^{\mathrm{T}}(x)AV(x)\,\mathrm{d}x.\end{aligned}\tag{19}$$

Since $|\bar{B}^h-\Omega|\to0$ as $h\to0$, we have

$$\begin{aligned}&\lim_{h\to0}\sum_{|\alpha|=k+2}||D^\alpha\bar{u}||^2_{H^q(\bar{B}^h)}\\&=\sum_{|\alpha|=k+2}||D^\alpha\bar{u}||^2_{H^q(\Omega)}.\end{aligned}\tag{20}$$

Also $\lim_{h\to0}Q_h(u)=|u|^2_{H^{k+1}(\Omega)}$. For any $\delta>0$, using equalities (19) and (20) in (18), we obtain

$$\begin{aligned}&\limsup_{h\to0}\frac{||u-\tilde{I}_hu||^2_{H^1(\Omega)}}{h^{2k}Q_h(u)}\\&\leq(1+\delta^2)\frac{\int_\Omega V^{\mathrm{T}}(x)AV(x)\,\mathrm{d}x}{|u|^2_{H^{k+1}(\Omega)}}.\end{aligned}$$

As $\delta>0$ is arbitrary, we have

$$\begin{aligned}&\limsup_{h\to0}\frac{||u-\tilde{I}_hu||^2_{H^1(\Omega)}}{h^{2k}Q_h(u)}\\&\leq\frac{\int_\Omega V^{\mathrm{T}}(x)AV(x)\,\mathrm{d}x}{|u|^2_{H^{k+1}(\Omega)}}.\end{aligned}\tag{21}$$

From equality (21), we also have

$$\frac{\int_\Omega V^{\mathrm{T}}(x)AV(x)\,\mathrm{d}x}{|u|^2_{H^{k+1}(\Omega)}}\leq\liminf_{h\to0}\frac{||u-\tilde{I}_hu||^2_{H^1(\Omega)}}{h^{2k}Q_h(u)}.\tag{22}$$

Combining inequalities (21) and (22) shows that $\lim_{h\to0}\frac{||u-\tilde{I}_hu||^2_{H^1(\Omega)}}{h^{2k}Q_h(u)}$ exists, and

$$\lim_{h\to0}\frac{||u-\tilde{I}_hu||^2_{H^1(\Omega)}}{h^{2k}Q_h(u)}=\frac{\int_\Omega V^{\mathrm{T}}(x)AV(x)\,\mathrm{d}x}{|u|^2_{H^{k+1}(\Omega)}}.$$

Because $\bar{\lambda}$ is the maximal eigenvalue of matrix A, according to $AV(x)=\lambda V(x)$ we have

$$\begin{aligned}&\int_\Omega V^{\mathrm{T}}(x)AV(x)\,\mathrm{d}x\\&\leq\bar{\lambda}\int_\Omega\sum_{i=1}^{M_k}|D^{\alpha(i)}u(x)|^2\mathrm{d}x\\&=\bar{\lambda}|u|^2_{H^{k+1}(\Omega)}.\end{aligned}$$

For any $\delta>0$, we have

$$\lim_{h\to0}\frac{||u-\tilde{I}_hu||^2_{H^1(\Omega)}}{h^{2k}Q_h(u)}\leq\bar{\lambda}.$$

Hence

$$\sup_{u\in H^{k+2+q}(\Omega)} \lim_{h\to 0} \frac{||u-\tilde{I}_h u||^2_{H^1(\Omega)}}{h^{2k} Q_h(u)} \leq \bar{\lambda}. \tag{23}$$

let $\bar{v} = [v_1, v_2, \ldots, v_{M_k}]^{\mathrm{T}}$ be an eigenvector of A corresponding to $\bar{\lambda}$, then there will exist $u \in P^{k+1}$ such that $V(x) = \bar{v}$. For a particular u, we have

$$\frac{\int_\Omega V^{\mathrm{T}}(x) A V(x)\,\mathrm{d}x}{|u|^2_{H^{k+1}(\Omega)}} = \bar{\lambda}.$$

From inequality (23) we conclude that

$$\sup_{u\in H^{k+2+q}(\Omega)} \lim_{h\to 0} \frac{||u-\tilde{I}_h u||^2_{H^1(\Omega)}}{h^{2k} Q_h(u)} = \bar{\lambda}.$$

This is the expected result.

3. Model of the algorithm

Now we assess the approximate value of shape function $\{\phi_j^h\}$ by the maximal eigenvalue of the matrix A, which is defined by Eq.(2), we know that $\bar{\lambda}$ is computable and it depends only on the shape function $\phi(x)$. We also know that $\bar{\lambda}$ does not depend on u and h. Based on equality (14) we have

$$||u-\tilde{I}_h u||_{H^1(\Omega)} \Big/ h^k \sqrt{Q_h(u)} \leq \sqrt{\bar{\lambda}},$$

where h is small enough. So we know that $\bar{\lambda}$ is determined only by shape function $\phi(x)$. Now we show the choice of shape function in one dimension. In one dimension, $\bar{\lambda}$ is given by

$$\bar{\lambda} = \left(|\xi_{k+1}|_{H^1(0,1)} \big/ (k+1)!\right)^2.$$

We consider four different classes of RKP shape function, reproducing of order 1, corresponding to four different weight functions $w(x)$ given as follows:

(a) Gaussian

$$w(x) = \begin{cases} \dfrac{\mathrm{e}^{\delta(x/R)^2} - \mathrm{e}^{\delta}}{1-\mathrm{e}^{\delta}}, & |x| \leq R, \\ 0, & |x| > R; \end{cases} \tag{24}$$

(b) Cubic spline

$$w(x) = \begin{cases} \dfrac{2}{3} - 4(x/R) + 4(x/R)^3, & |x| \leq R/2, \\ \dfrac{4}{3} - 4(|x|/R) + 4(x/R)^2 - \dfrac{4}{3}(|x|/R)^3, & R/2 \leq |x| \leq R, \\ 0, & |x| > R; \end{cases} \tag{25}$$

(c) Conical

$$w(x) = \begin{cases} \left[1-(x/R)^2\right]^l, & |x| \leq R, \\ 0, & |x| > R. \end{cases} \tag{26}$$

Computing $|\xi_{k+1}|_1$ for each of the weight functions with $R = 1.6$, we have

$$|\xi_{k+1}|_1 = \begin{cases} 0.225 & \text{from } w(x) \text{ in expression (26)}, & l = 2.5, \\ 0.211 & \text{from } w(x) \text{ in expression (24)}, & \delta = 2.5, \\ 0.097 & \text{from } w(x) \text{ in expression (25)}, & \\ 0.031 & \text{from } w(x) \text{ in expression (26)}, & l = 4.5. \end{cases}$$

Now we choose $u(x) = x^2$ to evaluate the choices of the shape function which is in the interval $\Omega = (0,1)$ and compute the error $||u-\tilde{I}_h u||_{H^1(\Omega)}$, where $\tilde{I}_h u$ is the interpolation of u with $h = 1/n$, $n = 50, 60, \ldots, 110$, $R = 1.6$. The result is summarized in Table 1.

Table 1. The error $||u-\tilde{I}_h u||_{H^1(\Omega)}$ of the shape function of $u(x)=x^2$ with four different weight functions.

n	$\|\|u-\tilde{I}_h u\|\|_{H^1(\Omega)}$			
	Conical $l=2.5$	Gaussian $\delta=2.5$	Cubic spline	Conical $l=4.5$
50	1.087×10^{-2}	1.397×10^{-2}	5.984×10^{-3}	2.186×10^{-3}
60	1.291×10^{-2}	1.135×10^{-2}	5.253×10^{-3}	1.735×10^{-3}
70	1.266×10^{-2}	8.849×10^{-3}	4.516×10^{-3}	1.407×10^{-3}
80	9.326×10^{-3}	7.916×10^{-3}	3.726×10^{-3}	1.235×10^{-3}
90	8.780×10^{-3}	6.763×10^{-3}	3.218×10^{-3}	1.152×10^{-3}
100	7.280×10^{-3}	5.980×10^{-3}	2.732×10^{-3}	8.879×10^{-4}
110	6.630×10^{-3}	5.684×10^{-3}	2.436×10^{-3}	8.017×10^{-4}

4. Conclusion

We know that the error is determined by the size of the neighbourhood of the R, and the bigger the neighbourhood, the smaller the error is, vise versa. This paper just studies uniformly distributed particles; non-uniformly distributed particles will be discussed in further study.

References

[1] Melenk J M and Babuska L l996 *Comput. Methods Appl. Mech. Engrg* **139** 289

[2] Pan W J 2000 *Fourier Transform* (The Press of Beijing University) (in Chinese)

[3] Babuska U Banerjee and Osborn J 2003 *Numer. Math.* **10** 1007

[4] Strousboulis T, Kopps K and Babuska I 2001 *Comput Methods Appl. Mech. Engrg* **l90** 4081

[5] Adam R A 1981 *Sobolev Spaces*(New York: Academic) (in Chinese)

[6] Carlos Z 2003 *Appl. Numer. Math.* **47** 575

[7] Wu H B 2006 *Acta Phys. Sin.* **55** 4987 (in Chinese)

[8] Zhang H, Liu L and Zhang J J 2007 *Acta Phys. Sin.* **56** 487 (in Chinese)