Vol 16 No 10, October 2007
1009-1963/2007/16(10)/2903-05
Chinese Physics
© 2007 Chin. Phys. Soc.
and IOP Publishing Ltd

Realization of generalized synchronization between different chaotic systems via scalar controller

Zhou Ping(周 平)[a] and Cao Yu-Xia(曹玉霞)[b]

[a] *Institute for Nonlinear Systems, Chongqing University of Posts and Telecommunications, Chongqing* 400065, *China*

[b] *Library of Chongqing University of Posts and Telecommunications, Chongqing* 400065, *China*

(Received 24 February 2007; revised manuscript received 14 March 2007)

In this paper, a very simple generalized synchronization method between different chaotic systems is presented. Only a scalar controller is used in this method. The method of obtaining the scalar controller from chaotic systems is established. The sufficient and necessary condition of generalized synchronization is obtained from a rigorous theory, and the sufficient and necessary condition of generalized synchronization is irrelative to chaotic system itself. Theoretical analyses and simulation results show that the method established in this paper is effective.

Keywords: generalized synchronization, different chaotic systems, scalar controller
PACC: 0545

1. Introduction

Chaotic synchronization and control have aroused much interest of many authors in the past few decades.[1−12] Much work has been done on drive-response-type synchronization of chaos in dynamical systems[1,2] and synchronization in two systems coupled via feedback[3−5] from the physical viewpoint.

There exist two types of current concepts of synchronization, and they are the identical synchronization (IS) and generalized synchronization (GS). The identical synchronization means the case where states of two systems are equal or asymptotically become equal as time goes to infinity. The generalized synchronization refers to the case where states of two systems satisfy a functional relation or asymptotically satisfy a functional relation as time goes to infinity. Recently, the idea of generalized synchronization has been developed. Chaotic generalized synchronization has attracted great interest due to its theoretical challenge and its great potential applications in secure communications, chemical reactions, and modelling brain activity.[13]

In this paper, we realize a generalized synchronization between different chaotic systems via a scalar controller. We can obtain the scalar controller easily, and the very important result is the sufficient and necessary condition of generalized synchronization irrelative to chaotic system itself. Theoretical analyses and simulation results show that the method established in this paper is effective.

2. Generalized synchronization between different chaotic systems

Now, we consider the following three-dimensional chaotic system:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, x_3),\\ \dot{x}_2 &= f_2(x_1, x_2, x_3),\\ \dot{x}_3 &= f_3(x_1, x_2, x_3),\end{aligned} \tag{1}$$

where $x = (x_1, x_2, x_3)^{\mathrm{T}}$ are state variables, and they are all real numbers. $f_i (i = 1, 2, 3)$ are continuous functions, and of them only one is a nonlinear function. Many practical chaotic systems, for example, Chua's chaotic circuit,[14] chaotic K model[15] and Rössler chaotic system,[16] etc, are the same as system (1).

Without loss of generality, one can take $f_3(x_1, x_2, x_3)$ as the nonlinear function. System (1) can be changed to another form as follows:

$$\begin{aligned}\dot{x}_1 &= \sum_{i=1}^{3} a_i x_i,\\ \dot{x}_2 &= \sum_{i=1}^{3} b_i x_i,\\ \dot{x}_3 &= f_3(x_1, x_2, x_3),\end{aligned} \tag{2}$$

where a_i and $b_i\,(i = 1, 2, 3)$ are real numbers, and $a_3 \neq 0$, $b_3 \neq 0$ or $a_3 = 0$, $a_2 \neq 0$, $b_3 \neq 0$ or $b_3 = 0$, $b_1 \neq 0$, $a_3 \neq 0$.

Now, we consider the other chaotic system as follows in Eq.(3) that is different from chaotic system (2),

$$\begin{aligned}\dot{x}'_1 &= \sum_{i=1}^{3} a'_i x'_i,\\ \dot{x}'_2 &= \sum_{i=1}^{3} b'_i x'_i,\\ \dot{x}'_3 &= f'_3(x'_1, x'_2, x'_3),\end{aligned} \tag{3}$$

where a'_i, $b'_i\,(i = 1, 2, 3)$ are real numbers, and $a'_3 \neq 0$, $b'_3 \neq 0$ or $a'_3 = 0$, $a'_2 \neq 0$, $b'_3 \neq 0$ or $b'_3 = 0$, $b'_1 \neq 0$, $a'_3 \neq 0$.

Now, determine the reversible matrix A for system (2),

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{4}$$

where $A_{33} \neq 0$. If $a_3 \neq 0$, $b_3 \neq 0$,

$$\begin{aligned} A_{11} &= (-b_3/a_3)A_{12}, \quad A_{12} \neq 0,\\ A_{21} &= (-a_1 b_3/a_3 + b_1)A_{12},\\ A_{22} &= (-a_2 b_3/a_3 + b_2)A_{12},\\ A_{31} &= a_1 A_{21} + b_1 A_{22},\\ A_{32} &= a_2 A_{21} + b_2 A_{22},\\ A_{33} &= a_3 A_{21} + b_3 A_{22}. \end{aligned} \tag{5}$$

If $a_3 = 0$, $a_2 \neq 0$, $b_3 \neq 0$,

$$\begin{aligned} &A_{11} \neq 0, \quad A_{12} = 0, \quad A_{21} = a_1 A_{11},\\ &A_{22} = a_2 A_{11}, \quad A_{31} = a_1 A_{21} + b_1 A_{22},\\ &A_{32} = a_2 A_{21} + b_2 A_{22},\\ &A_{33} = a_3 A_{21} + b_3 A_{22}. \end{aligned} \tag{6}$$

If $b_3 = 0$, $b_1 \neq 0$, $a_3 \neq 0$,

$$\begin{aligned} &A_{11} = 0, \quad A_{12} \neq 0, \quad A_{21} = b_1 A_{12},\\ &A_{22} = b_2 A_{12}, \quad A_{31} = a_1 A_{21} + b_1 A_{22},\\ &A_{32} = a_2 A_{21} + b_2 A_{22},\\ &A_{33} = a_3 A_{21} + b_3 A_{22}. \end{aligned} \tag{7}$$

In the same way as dealing with chaotic system (2), we can determine the reversible matrix A' for system (3).

Now, we discuss the generalized synchronization between chaotic system (2) and chaotic system (3). Take system (2) as a drive system, system (3) as a response system.

Theorem For the following controlled response system:

$$\begin{aligned}\dot{x}'_1 &= \sum_{i=1}^{3} a'_i x'_i,\\ \dot{x}'_2 &= \sum_{i=1}^{3} b'_i x'_i,\\ \dot{x}'_3 &= \phi + f'_3(x'_1, x'_2, x'_3),\end{aligned} \tag{8}$$

where ϕ is a scalar controller given by

$$\begin{aligned}\phi = &-f'_3(x'_1, x'_2, x'_3) + A_{33} f(x_1, x_2, x_3)/A'_{33}\\ &+\Big[\sum_{i=1}^{3}(A_{31}a_i + A_{32}b_i)x_i - (A'_{31}a'_i + A'_{32}b'_i)x'_i\Big]/A'_{33}\\ &-\sum_{i=1}^{3} k_i(y_i - z_i)/A'_{33},\end{aligned}$$

with k_1, k_2 and k_3 being control parameters and real numbers, $(z_1, z_2, z_3)^{\mathrm{T}} = A(x_1, x_2, x_3)^{\mathrm{T}}$, and $(y_1, y_2, y_3)^{\mathrm{T}} = A'(x'_1, x'_2, x'_3)^{\mathrm{T}}$, the sufficient and necessary condition of generalized synchronization between drive chaotic system (2) and controlled response chaotic system (8) with respect to $(A_{11}x_1 + A_{12}x_2$, $A_{21}x_1 + A_{22}x_2$, $A_{31}x_1 + A_{32}x_2 + A_{33}x_3)$ and $(A'_{11}x'_1 + A'_{12}x'_2$, $A'_{21}x'_1 + A'_{22}x'_2$, $A'_{31}x'_1 + A'_{32}x'_2 + A'_{33}x'_3)$ is $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, and $k_2 k_3 - k_1 > 0$.

According to this theorem, system (2) and system (3) can achieve the generalized synchronization via scalar controller ϕ. The scalar controller ϕ depends on the system (2) and system (3), but the sufficient and necessary condition of generalized synchronization is not related to chaotic system (2) and system (3) themselves.

Proof

Define reversible linear transform $(z_1, z_2, z_3)^{\mathrm{T}} = A(x_1, x_2, x_3)^{\mathrm{T}}$ for system (2), and according to expressions (5) or (6) or (7), we can obtain the following equation system:

$$\begin{aligned}\dot{z}_1 &= z_2,\\ \dot{z}_2 &= z_3,\\ \dot{z}_3 &= l(z_1, z_2, z_3),\end{aligned} \tag{9}$$

where

$$\begin{aligned} l(z_1, z_2, z_3) = &A_{31}\sum_{i=1}^{3} a_i x_i\\ &+ A_{32}\sum_{i=1}^{3} b_i x_i + A_{33} f_3(x_1, x_{2,} x_3).\end{aligned}$$

Define reversible linear transform $(y_1, y_2, y_3)^{\mathrm{T}} = A'(x'_1, x'_2, x'_3)^{\mathrm{T}}$ for system (8), in the same as showing above, we can obtain the following equation system:

$$\begin{aligned}\dot{y}_1 &= y_2,\\ \dot{y}_2 &= y_3,\\ \dot{y}_3 &= l(z_1, z_2, z_3) - \sum_{i=1}^{3} k_i(y_i - z_i).\end{aligned} \tag{10a}$$

Defining synchronization error between system (9) and system (10) as $e_i = y_i - z_i\,(i = 1, 2, 3)$, we can obtain the error dynamical system as follows:

$$\begin{aligned}\dot{e}_1 &= e_2,\\ \dot{e}_2 &= e_3,\\ \dot{e}_3 &= -k_1e_1 - k_2e_2 - k_3e_3,\end{aligned} \tag{10b}$$

namely,

$$\begin{bmatrix} \dot{e}_1, \\ \dot{e}_2, \\ \dot{e}_3, \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0, \\ 0 & 0 & 1, \\ -k_1 & -k_2 & -k_3, \end{bmatrix} \begin{bmatrix} e_1, \\ e_2, \\ e_3, \end{bmatrix}.$$

According to the condition of Hurwitz,[17] we can obtain the sufficient and necessary condition for $\lim\limits_{t\to+\infty} e_i = 0\,(i = 1, 2, 3)$ as $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, and $k_2k_3 - k_1 > 0$.

Furthermore, $\lim\limits_{t\to+\infty} e_i = 0\,(i = 1, 2, 3)$ indicates

$$\begin{aligned}&\lim_{t\to+\infty}(A'_{11}x'_1 + A'_{12}x'_2) - (A_{11}x_1 + A_{12}x_2) = 0,\\ &\lim_{t\to+\infty}(A'_{21}x'_1 + A'_{22}x'_2) - (A_{21}x_1 + A_{22}x_2) = 0,\\ &\lim_{t\to+\infty}[(A'_{31}x'_1 + A'_{32}x'_2 + A'_{33}x'_3)\\ &- (A_{31}x_1 + A_{32}x_2 + A_{33}x_3)] = 0.\end{aligned}$$

Namely, controlled response chaotic system (8) and drive chaotic system (2) can achieve generalized synchronization with respect to $(A_{11}x_1 + A_{12}x_2,\ A_{21}x_1 + A_{22}x_2,\ A_{31}x_1 + A_{32}x_2 + A_{33}x_3)$ and $(A'_{11}x'_1 + A'_{12}x'_2,\ A'_{21}x'_1 + A'_{22}x'_2,\ A'_{31}x'_1 + A'_{32}x'_2 + A'_{33}x'_3)$, the sufficient and necessary condition of generalized synchronization is $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, and $k_2k_3 - k_1 > 0$.

3. Applications

Now, we take some chaotic systems for example.

3.1. Generalized synchronization between Chua's chaotic circuit system and Chaotic K model system

Chua's chaotic circuit[14] is shown as follows:

$$\begin{aligned}\dot{x}_1 &= a[-x_1 + x_2 - \phi(x_1)],\\ \dot{x}_2 &= x_1 - x_2 + x_3,\\ \dot{x}_3 &= -bx_2,\end{aligned} \tag{11}$$

where

$$\phi(x_1) = m_1x_1 + (m_0 - m_1)[|x_1 + 1| - |x_1 - 1|]/2,$$

$a = 10$, $b = 14.87$, $m_0 = -1.27$, and $m_1 = -0.68$.

Chaotic system (11) can be changed into the following form:

$$\begin{aligned}\dot{x}_1 &= -bx_2,\\ \dot{x}_2 &= x_1 - x_2 + x_3,\\ \dot{x}_3 &= a[-x_3 + x_2 - \phi(x_3)],\end{aligned} \tag{12}$$

where

$$\phi(x_3) = m_1x_3 + (m_0 - m_1)[|x_3 + 1| - |x_3 - 1|]/2.$$

Certainly, $a_3 = 0$, $a_2 \neq 0$, and $b_3 \neq 0$. According to expressions (6), we can obtain matrix A,

$$A = A_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -b & 0 \\ -b & b & -b \end{bmatrix}, \quad \text{and} \quad A_{11} \neq 0.$$

Chaotic K model[15] is given by

$$\begin{aligned}\dot{x}'_1 &= x'_1x'_2 - x'_3,\\ \dot{x}'_2 &= x'_1 - x'_2,\\ \dot{x}'_3 &= x'_1 + dx'_3,\end{aligned} \tag{13}$$

$d = 0.3$. Chaotic system (13) can be changed into the following form:

$$\begin{aligned}\dot{x}'_1 &= dx'_1 + x'_3,\\ \dot{x}'_2 &= x'_3 - x'_2,\\ \dot{x}'_3 &= x'_2x'_3 - x'_1.\end{aligned} \tag{14}$$

Certainly, $a_3 \neq 0$, and $b_3 \neq 0$. According to expressions (5), we can obtain matrix A',

$$A' = A'_{12}\begin{bmatrix} -1 & 1 & 0 \\ -d & -1 & 0 \\ -d^2 & 1 & -(1+d) \end{bmatrix}, \quad \text{and} \quad A'_{12} \neq 0.$$

Take Chua's chaotic circuit system (12) as a drive system, and chaotic K model system (14) as

a response system. According to the above theorem, we can obtain the scalar controller ϕ. Therefore, system (12) and system (14) can achieve the generalized synchronization with respect to $(A_{11}x_1, -bA_{11}x_2, A_{11}b(-x_1+x_2-x_3))$ and $(A'_{12}(-x'_1+x'_2), -A'_{12}(dx'_1+x'_2), A'_{12}(-d^2x'_1+x'_2-(1+d)x'_3))$. Some synchronization simulation results are obtained as follow. Choosing $k_1=1$, $k_2=1$ and $k_3=3$, and $A_{11}=1$, $A'_{12}=1$ leads to the simulation result shown in Fig.1. Choosing $k_1=10$, $k_2=8$ and $k_3=6$, and $A_{11}=1$, $A'_{12}=10$ yields the simulation result shown in Fig.2. In Figs.1 and 2 ε denotes the generalized synchronization error.

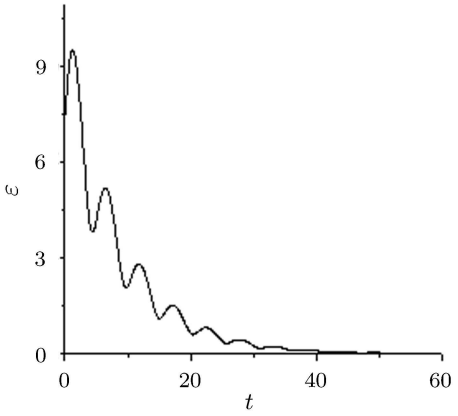


Fig.1. The generalized synchronization result.

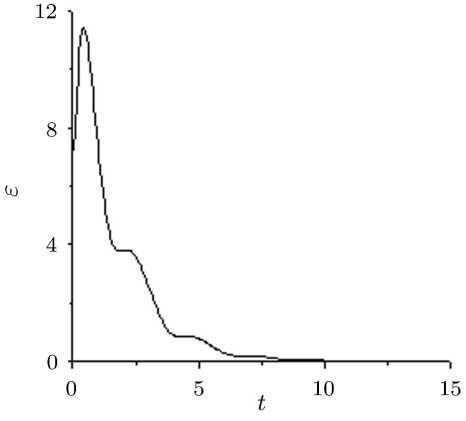


Fig.2. The generalized synchronization result.

3.2. Generalized synchronization between Chua's chaotic circuit system and Rössler chaotic system

The Rössler chaotic system[16] is given as

$$\dot{x}''_1 = -x''_2 - x''_3,$$
$$\dot{x}''_2 = x''_1 + ax''_2,$$
$$\dot{x}''_3 = x''_1x''_3 - bx''_3 + c, \tag{15}$$

where $a=0.2$, $b=8$, and $c=0.2$ are system parameters.

Certainly, $b_3=0$, $b_1\neq 0$, and $a_3\neq 0$. According to expressions (6), we can obtain matrix A'',

$$A'' = A''_{12}\begin{bmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & a^2-1 & -1 \end{bmatrix}, \quad \text{and} \quad A''_{12}\neq 0.$$

Take Chua's chaotic circuit system (12) as a drive system, and the Rössler chaotic system (15) as a response system. According to the above theorem, system (12) and system (15) can achieve the generalized synchronization with respect to $(A_{11}x_1, -bA_{11}x_2, A_{11}b(-x_1+x_2-x_3))$ and $(A''_{12}x''_2, A''_{12}(x''_1+ax''_2), A''_{12}(ax''_1+(a^2-1)x''_2-x''_3))$. Some synchronization simulation results are obtained as follows. Choosing $k_1=5$, $k_2=2$ and $k_3=4$, and $A_{11}=1$, $A''_{12}=1$ gives the simulation result in Fig.3. Choosing $k_1=10$, $k_2=5$ and $k_3=6$, and $A_{11}=1$, $A''_{12}=1$ yields the simulation result in Fig.4.

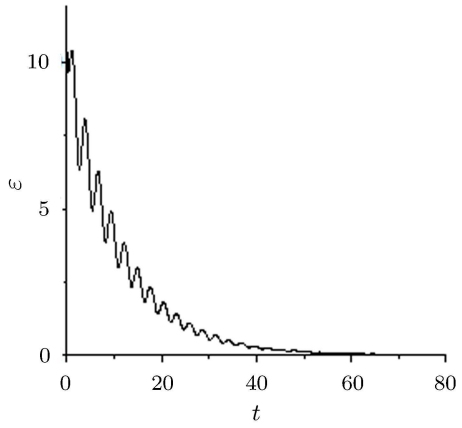


Fig.3. The generalized synchronization result.

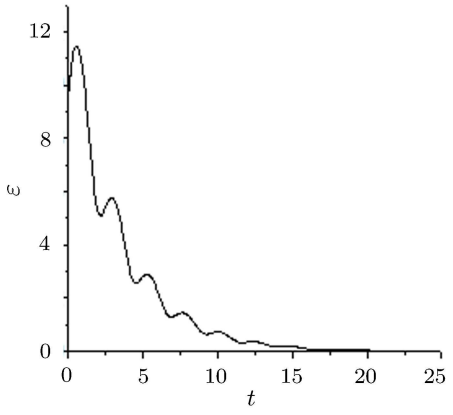


Fig.4. The generalized synchronization result.

3.3. Generalized synchronization between Chaotic K model system and Rössler chaotic system

Now, take chaotic K model system (14) as a drive system, and Rössler chaotic system (15) as a response system. According to the above theorem, system (14) and system (15) can achieve the generalized synchronization with respect to $(A'_{12}(-x'_1+x'_2)$, $-A'_{12}(dx'_1+x'_2)$, $A'_{12}(-d^2x'_1+x'_2-(1+d)x'_3))$ and $(A''_{12}x''_2$, $A''_{12}(x''_1+ax''_2)$, $A''_{12}(ax''_1+(a^2-1)x''_2-x''_3))$. Some synchronization simulation results are obtained as follow. Choosing $k_1=5$, $k_2=2$ and $k_3=4$, and $A'_{12}=1$, $A''_{12}=1$ gives the simulation result in Fig.5. Choosing $k_1=10$, $k_2=5$ and $k_3=6$, and $A'_{12}=1$, $A''_{12}=1$ leads to the simulation result in Fig.6.

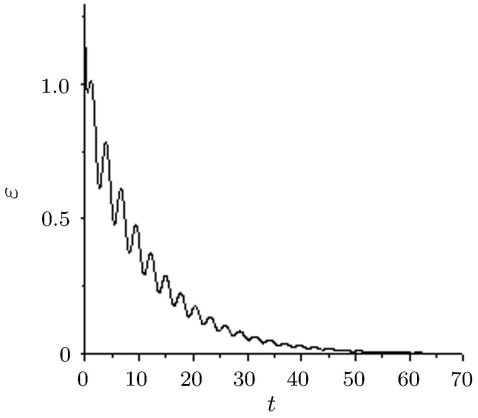


Fig.5. The generalized synchronization result.

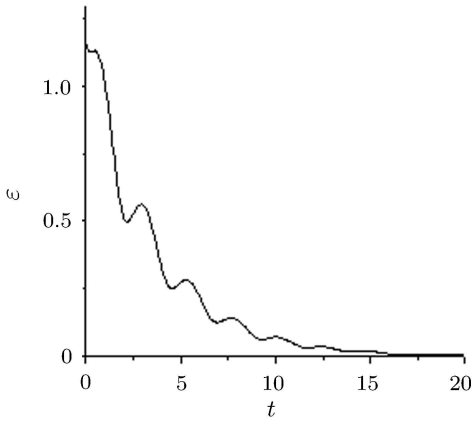


Fig.6. The generalized synchronization result.

4. Conclusion

A scalar controller is established to realize generalized synchronization between different chaotic systems. The method of obtaining the scalar controller from chaotic systems is very easy. The sufficient and necessary condition of generalized synchronization is obtained from a rigorous theory. We take some chaotic systems for example. Theoretical analyses and simulation results show that the method given in this paper is effective.

References

[1] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821

[2] Ott E, Grebogi C and Yorke J A 1990 *Phys. Rev. Lett.* **64** 1196

[3] Jiang G P, Tang K S and Chen G R 2003 *Chaos, Solitons & Fractals* **15** 925

[4] Chen S, Wang F and Wang C 2004 *Chaos, Solitons & Fractals* **20** 235

[5] Liu Y Z and Fei S M 2005 *Acta Phys. Sin.* **55** 3486 (in Chinese)

[6] Song Y Z, Zhao G Z and Qi D L 2006 *Chin. Phys.* **15** 2266

[7] Zhou P, Luo X H and Chen H Y 2005 *Acta Phys. Sin.* **54** 5048 (in Chinese)

[8] Sun F Y 2006 *Chin. Phys. Lett.* **23** 32

[9] Liu Y Z and Fei S M 2005 *Acta Phys. Sin.* **54** 3486 (in Chinese)

[10] Liu L, Su Y C and Liu C X 2006 *Acta Phys. Sin.* **55** 3933 (in Chinese)

[11] Wang F Q and Liu C X 2006 *Chin. Phys.* **15** 1971

[12] Wang J Z, Chen Z Q and Yuan Z Z 2006 *Chin. Phys.* **15** 1216

[13] Zhang X D and Li Z P 2006 *Commun. Theor. Phys.* **45** 461

[14] Yan S L, Wu S B and Pang H G 2001 *Acta Phys. Sin.* **50** 428 (in Chinese)

[15] Chen Y Y, Peng J H and Shen Q H 2001 *Acta Phys. Sin.* **50** 1871 (in Chinese)

[16] Zhou P 2003 *Acta Phys. Sin.* **52** 1871 (in Chinese)

[17] Lin H 1992 *Stability Theory* (Beijing: Beijing University Press)