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Stability problem in Rindler spacetime*

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The stability problem of the Rindler spacetime is carefully studies by using the scalar wave perturbation. Using two different coordinate systems, the scalar wave equation is investigated. The results are different in the two cases. They are analysed and compared with each other in detail. The following conclusions are obtained: (a) the Rindler spacetime as a whole is not stable; (b) the Rindler spacetime can exist stably only as part of the Minkowski spacetime, and the Minkowski spacetime can be a real entity independently; (c) there are some defects for the scalar wave equation written by the Rindler coordinates, and it is unsuitable for the investigation of the stability properties of the Rindler spacetime. All these results may shed some light on the stability properties of the Schwarzschild black hole. It is natural and reasonable for one to infer that: (a) perhaps the Regge–Wheeler equation is not sufficient to determine the stable properties; (b) the Schwarzschild black hole as a whole might be really unstable; (c) the Kruskal spacetime is stable and can exist as a real physical entity; whereas the Schwarzschild black hole can occur only as part of the Kruskal spacetime.

Keywords: stable problem, Rindler spacetime, Regge-Wheeler equation
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1. Introduction

The stability of the Schwarzschild black hole is a vitally important problem in general relativity. In 1957, Regge and Wheeler first treated the problem by using the Schwarzschild coordinates.[1] They obtained the perturbation equation, that is, the well-known Regge–Wheeler equation. The horizon and the spatial infinity consist of the boundaries for the perturbation problem. It is easy to treat the boundary of the spatial infinity; whereas the boundary conditions at the horizon are complicated. The background metric has an apparent singularity at the horizon $r = 2m$ in the Schwarzschild coordinates; this in turn may result in spurious divergence in the perturbation and cause confusion. A proposition was given by Vishveshwara in 1970. Because the metric is regular at $r = 2m$ in the Kruskal coordinates system, he transformed the perturbation fields into it. By using the Schwarzschild time $t = 0$ to define the initial time, it is obtained that the black hole is stable.[2]

In practice, there is still another problem concerning the Schwarzschild coordinates. Due to its metric component $g_{00} = 0$ at the horizon $r = 2m$, the Schwarzschild time coordinate t loses its meaning at the horizon. This is the drawback of the Schwarzschild time coordinate t. For example, a particle near the horizon $r = 2m$ falls into the black hole by a finite proper time; but it is described by the Schwarzschild time coordinate as an infinite process as $t \to +\infty$, and $r \to 2m$. This example shows that the Schwarzschild time coordinate t is a tortoise coordinate. It is well-known that $r_* = r + 2m\ln(2m/r - 1)$ is a tortoise coordinate too.

Actually, this tortoise property of t appears more easily to escape one's notice, and the researchers have hardly considered its influence on the stability problem: $t = 0$ as the initial time at the horizon is taken for granted.[1,2] One should doubt its validity. We have first taken notice of it and worked over it. For simplicity, one might select some 'good' time coordinate to define the initial time. Using the new 'good' coordinate time, the stability problem should be reinvestigated again. The Kruskal time T and the Painlevé time are all 'good' coordinate times. We have employed them to study the problem and obtained unusual results. By using the Kruskal time coordinate $T = C =$ const to define the initial time, we find that its stability properties controversially depend on

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the sign of the initial time $T = C =$ const: the Schwarzschild black hole is stable when $T = C \geq 0$, whereas the Schwarzschild black hole is unstable when $T = C < 0$.[3−6] The same is also true of the Painlevé time: when the initial time slice intersects the future horizon, it is stable; whereas it is unstable as the initial time slice intersects the past horizon. These unexpected results are in contrast with the conclusion that it is taken for granted that the Schwarzschild black hole is stable.

Which one is correct? Is the Schwarzschild black hole stable or not? Where do the contradictories come from? and why so? Do they result just from an effect caused by the tortoise property of the time coordinate t? Actually, we have already replaced it by 'good' time, which in turn excludes this possibility. Perhaps this complicated situation really shows that the Regge–Wheeler equation, which is obtained in the Schwarzschild coordinates, is not sufficient to determine the stability properties. As shown before, the Schwarzschild coordinate t is a tortoise coordinate like the well-known tortoise coordinate r_*; it does not cover the event horizon. So, it is possibly not qualified for the stability study.

The Kruskal spacetime is the extension of the Schwarzschild black hole. The Kruskal coordinates are qualified for the study of the perturbation equation and the stability problem. But the background metric is varying with time in the Kruskal coordinates,[2] subsequently, it is almost impossible to restudy the problem in the whole Kruskal coordinates. So there is still not a convincing answer to the stability problem of the Schwarzschild black hole.

On the ground of the difficulties, we want to study the stability problem of the Schwarzschild black hole indirectly, and in turn we are led to study that of the Rindler spacetime.

The Rindler spacetime is part of the Minkowski spacetime whose constant spatial coordinates describe accelerated observers with constant accelerations. The interest in Rindler spacetime lies in almost the same geometrical structure as that of the Schwarzschild black-hole. Actually, the Hawking radiation in the Schwarzschild black-hole is closely connected with the Unruh effect in the Rindler spacetime.[7,8]

The Rindler coordinates are the same as those of the Schwarzschild time coordinates; its time is a tortoise coordinate too. What is an effect on the stability study of the Rindler spacetime? In order to find it, we study the Klein–Gordon equation in the whole Rindler coordinates in Section 2 and discuss the stability properties of the Rindler spacetime. The same problem is treated in Section 3 by the Minkowski coordinates; some arguments are given and conclusion is obtained in Section 4.

2. The scalar field equation in the Rindler spacetime

The metric of the Rindler spacetime is

$$\mathrm{d}s^2 = -z^2\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{1}$$

with $0 < z < +\infty$, t, x, $y \in (-\infty, +\infty)$. The Minkowski spacetime's metric is

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}Z^2 + \mathrm{d}x^2 + \mathrm{d}y^2. \tag{2}$$

The transformation equations between them are

$$z = \sqrt{Z^2 - T^2} \tag{3}$$

and

$$\mathrm{e}^{2t} = \frac{Z+T}{Z-T}, \tag{4}$$

or

$$Z = z\cosh t \tag{5}$$

and

$$T = z\sinh t. \tag{6}$$

Of course, the Rindler spacetime's completion is the Minkowski spacetime, and corresponds to only the part $Z > 0$ and $Z^2 - T^2 > 0$ of the Minkowski spacetime. $Z^2 - T^2 = 0$ is the horizon of the Rindler spacetime. The horizon is denoted alternatively by $z = 0$ in the Rindler coordinates. The metric (1) is obviously singular at the horizon.

In the Rindler coordinates, the Klein–Gordon equation of the scalar field with mass μ is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}g^{\mu\nu}\frac{\partial \Psi}{\partial x^\nu}\right] - \mu^2\Psi = 0, \tag{7}$$

which turns out to be

$$-\frac{1}{z}\frac{\partial^2\Psi}{\partial t^2} + \frac{1}{z}\frac{\partial}{\partial z}\left[z\frac{\partial\Psi}{\partial z}\right] + \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - \mu^2\Psi = 0. \tag{8}$$

The coordinates t, x and y range from $-\infty$ to $+\infty$, therefore, the normal mode decomposition of the scalar field Ψ must be

$$\Psi = \psi(z)\mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}k_1 x + \mathrm{i}k_2 y} \tag{9}$$

with ψ satisfying the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[z\frac{d\psi}{dz}\right] - \frac{\omega^2}{z}\psi + z[\mu^2 + k_1^2 + k_2^2]\psi = 0. \tag{10}$$

Equation (10) is Bessel's equation with order being $\pm \mathrm{i}\omega$.[8] Its solutions are the modified Bessel function $I_{\pm \mathrm{i}\omega}(\sqrt{\mu^2+k_1^2+k_2^2}z)$.‡ Suppose the frequency $\omega = \alpha + \mathrm{i}\beta$ with α and β being real and $\beta > 0$, then $I_{-\mathrm{i}\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z)$ will go to infinity exponentially as $z \to \infty$ and vanish at $z = 0$. On the contrary, $I_{+\mathrm{i}\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z)$ goes to infinity at $z = 0$ and falls to zero exponentially at $z \to \infty$.[8]

The boundaries of Eq.(10) consist of $z = 0$, the horizon $H^{\pm}$ of the Rindler spacetime, and $z \to \infty$ of the infinity. We must demand that the scalar field function Ψ initially well-behave at the boundaries. For the infinity, we can require the field Ψ to fall to zero initially. So we just select $I_{\mathrm{i}\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z)$ to satisfy the infinite boundary condition initially with the frequency whose imaginary is positive, that is

$$\Psi = \mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}k_1 x+\mathrm{i}k_2 y} I_{\mathrm{i}\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z). \quad (11)$$

Generally, researchers use the Rindler coordinate time $t = 0$ as the initial time to study the problem. By using expression (11), the scalar field is regular at infinity ($z \to \infty$) when $t = 0$. But $I_{\mathrm{i}\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z)$ blows up at the horizon $z = 0$ when $t = 0$, that is, Ψ initially blows up at the horizon for the positive imaginary number of the frequency ($\beta > 0$). This shows that $\beta > 0$ is unacceptable for the scalar field because it behaves badly at the initial time $t = 0$. Therefore the scalar field is stable and the Rindler spacetime is also stable to some extent (confining to the perturbation of the scalar field).

But the metric is singular at the horizon $z = 0$, and the time coordinate t even loses its meaning there due to the fact $g_{00}|_{z=0} = 0$. In reality, the past and future horizons correspond to $t \to -\infty$ and $t \to +\infty$, respectively. Their intersection is at $T = Z = 0$ in Minkowski coordinates, but it corresponds to $z = 0$ and t from $-\infty$ to $+\infty$ in the Rindler coordinates. This example clearly shows that the Rindler time coordinate t is meaningless and $t = 0$ should not be used as initial time containing the horizon. Therefore, we cannot be sure whether the above conclusion about stability is really right; we should also employ the Minkowski time coordinate to treat the problem again.

Now, we make use of the Minkowski time T to define the initial time. By using the transformations (3) and (4), we can rewrite the scalar field Ψ as a function of the independent variables T, Z, x and y, i.e.

$$\Psi = A\mathrm{e}^{-(\mathrm{i}\omega t/2)\ln[(Z+T)/(Z-T)]}\mathrm{e}^{\mathrm{i}k_1 x+\mathrm{i}k_2 y} \times I_{\mathrm{i}\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\sqrt{Z^2-T^2}). \quad (12)$$

By excluding the horizon, the scalar field Ψ is bounded at the initial time $T = C = \text{const}$ for the positive frequency $\mathrm{Im}\omega = \beta > 0$. At the horizon $Z^2 - T^2 = 0$, the asymptotic form of the scalar field Ψ in expression (12) is

$$\Psi \propto A\Big(\frac{Z+T}{Z-T}\Big)^{-\mathrm{i}\omega/2}(Z^2-T^2)^{\mathrm{i}\omega/2} = (Z-T)^{\mathrm{i}\omega}. \quad (13)$$

Whether expression (13) is initially divergent or not at the horizon really depends on the sign of the initial time $T = C$. When we select the initial time $T = C \geq 0$, the initial time slice $T = C \geq 0$ intersects the future horizon that corresponds to $Z = T = C$. So, the scalar field Ψ goes to infinity initially at $Z = T = C$ (the horizon). This excludes the possibility of the positive imaginary frequency, and subsequently ensures its stability and the Rindler spacetime to some extent.

Alternatively, we can also select the initial time $T = C < 0$. The initial time slice $T = C$ intersects the past horizon that is denoted by $Z = -T = -C$, so the scalar field Ψ is also bounded initially at the horizon for the positive imaginary frequency $\beta > 0$. In this very case, the scalar field Ψ is not stable, and consequently the Rindler spacetime is not stable with respect to the scalar field perturbation either.

Now, we come to some discussion.

It has been shown that the stability properties of the Rindler spacetime vitally depend on what time coordinate is used to define the initial time. If we use the Rindler time, the scalar field is stable. On the ground of that, the Rindler time coordinate loses its time meaning in the horizon, and thus one should choose 'good' time to define the initial time. Of course, the Minkowski time is one of the 'good' time. When we choose the Minkowski time to define the initial times, the stability properties of the scalar field become very intricate and depend on where the initial time slice intersects the horizon. The Rindler spacetime is stable when the initial time slice intersects the future horizon, while it is unstable as the initial time slice intersects the past horizon. The controversial conclusions are almost the same as those for the Schwarzschild black hole.[3−6] This problem is

‡When $\mathrm{i}\omega$ is an positive integer, $I_{-\mathrm{i}\omega}$ is replaced by $K_{-\mathrm{i}\omega}$.

unsolved in the Schwarzschild black hole. It is suggested that one should study the stability problem of the Schwarzschild black hole in the whole Kruskal coordinate system to obtain a definite answer to the controversy.[2] But it becomes almost impossible because the background metric in the Kruskal coordinate system is not even stationary. So, one has not obtained an absolutely and convincingly unambiguous answer to the stability problem of the Schwarzschild black hole.

The results are similar in the two cases. This is the truly good news. We can really obtain something interesting for the Schwarzschild black-hole from the present results. The Minkowski spacetime is the extension of the Rindler spacetime, and is luckily stationary in nature. This stationariness property of the Minkowski spacetime presents a striking contrast to that of the whole Kruskal spacetime, i.e. the extension of the Schwarzschild black hole. We can easily investigate the perturbation equation of the Rindler spacetime in the whole Minkowski coordinates. It is not difficult to obtain the result. It might give some clues to the stability problem of the Schwarzschild black hole.

3. The scalar field equation in the Minkowski spacetime

The scalar field equation in the Minkowski spacetime is

$$-\frac{\partial^2 \Psi}{\partial T^2}+\frac{\partial^2 \Psi}{\partial Z^2}+\frac{\partial^2 \Psi}{\partial x^2}+\frac{\partial^2 \Psi}{\partial y^2}-m^2\Psi=0. \quad (14)$$

The Rindler spacetime corresponds to the region where the coordinates T, x and y range from $-\infty$ to $+\infty$ and $Z>0,\ Z^2-T^2>0$. Therefore, the normal mode decomposition of the scalar field Ψ must be

$$\Psi=A\mathrm{e}^{-\mathrm{i}\omega T+\mathrm{i}k_3 Z+\mathrm{i}k_1 x+\mathrm{i}k_2 y}, \quad (15)$$

where k_3 must satisfy the following relationship:

$$k_3=\pm\sqrt{\omega^2-k_1^2-k_2^2-m^2}. \quad (16)$$

In order to make the scalar field Ψ initially finite over $-\infty<x$ and $y<+\infty$, k_1 and k_2, of course, are real numbers. If we similarly choose the frequency $\omega=\mathrm{i}\beta$ with $\beta>0$, the above equation becomes

$$k_3=\pm\mathrm{i}\sqrt{\beta^2+k_1^2+k_2^2+m^2}. \quad (17)$$

We can select $k_3=\mathrm{i}\gamma$ with

$$\gamma=\sqrt{\beta^2+k_1^2+k_2^2+m^2}>0,$$

then the scalar field Ψ is

$$\Psi=A\mathrm{e}^{\beta T}\mathrm{e}^{-\gamma Z+\mathrm{i}k_1 x+\mathrm{i}k_2 y}. \quad (18)$$

The boundaries of the Rindler spacetime consist of the infinity $Z\to+\infty$ and the horizon $Z^2-T^2=0$. Because the Minkowski metric is regular everywhere including the horizon, we can directly use the Minkowski time $T=0$ to define the initial time. By using expression (18), we can see that the scalar field Ψ well behaves everywhere corresponding to the Rindler spacetime at the initial time $T=0$. But it blows up to infinity as the time T goes to infinity for $\beta>0$.

Directly, it is easy to see that the Rindler spacetime as a whole is unstable with respect to the scalar field perturbation.

4. Arguments and conclusion

We have just obtained the conclusive instability of the whole Rindler spacetime by using the Minkowski coordinates. We now check the study in the Rindler coordinates in Section 2. The result definitely supports the view that the Rindler time coordinate t is not suitable for one to study the stability problem containing the horizon where t loses its meaning. This might suggest that the Schwarzschild time $t=0$ should not be qualified as initial time for the study of the Regge-Wheeler equation in the Schwarzschild black hole.

The result also supports that only the instability in Section 2 obtained by using the Minkowski time $T=C<0$ is correct. This can be explained easily. The Minkowski spacetime consists of four parts: the right region R, i.e. the Rindler spacetime $Z>0$ and $Z^2-T^2>0$; the left region L, i.e. $Z<0$ and $Z^2-T^2>0$; the past region P, i.e. $T<0$ and $Z^2-T^2<0$; the future region F, i.e. $T>0$ and $Z^2-T^2<0$. At the initial time slice $T=C<0$, the Rindler spacetime can be influenced by the part P where every particle must leave and come into either the Rindler spacetime or the other parts L and F. In this sense, the Rindler spacetime is unstable. Similarly, when the initial time slice intersects the past horizon of the Schwarzschild black hole, it easy to see that its instability comes from the influence of the white hole. Though the Kruskal metric of the Schwarzschild black hole is not stationary, and one

has not obtained the same conclusive result as that for the Rindler spacetime in the Minkowski coordinates; nevertheless, it is not unreasonable to infer that the Schwarzschild black hole as a whole might really be unstable in comparison with the case in Rindler spacetime. Of course, one must go further to come to the conclusion.

Why is the conclusion about stability wrong when using the Minkowski time $T = C > 0$? Furthermore, it is well-known that the Minkowski spacetime is stable, how does one harmonize our instability conclusion of the Rindler spacetime with this fact? Now we once again investigate the process for our conclusion. By using expression (18), only when the scalar field Ψ is completely confined to the Rindler spacetime, can it have positive imaginary frequency. This in turn shows that the Rindler spacetime as a whole is unstable. On the contrary, when the whole Minkowski spacetime is served as its background (Z ranges from $-\infty$ to $+\infty$), expression (18) clearly excludes the possibility of the positive imaginary frequency of the scalar field Ψ. Therefore, the Minkowski spacetime is definitely stable. Actually, this is just the well-known fact of the Minkowski spacetime; no new conclusion comes out. Really, there appears some inconsistency or contradiction in these conclusions. How are they explained? We give some naive arguments below.

When the Minkowski time $T = C > 0$ is the initial time, only the part F borders upon the Rindler spacetime initially. Classically, the part F cannot influence the Rindler spacetime, but this may not hold true from the quantum theory. It is well-known from the de-Broglie's theory of matter wave that the phase speed of the relativistic particles may be greater than the speed of light. The scalar wave equation can also be regarded as a quantum wave equation for a freely moving particle with a relativistic velocity. From the wave equations (10) and expressions (11)–(13) and by using the WKB method, it is not difficult to see that the phase speed of the wave is not greater than the speed of light at the horizon. So, the part F cannot have any effect on the Rindler spacetime in quantum way (or by wave theory). This ensures the conclusion about stability. But now we know that the conclusion is wrong, there must exist something wrong in wave equations (10) or (8). To see this clearly, we inspect wave equations (14) and (15) in the Minkowski coordinates. Expression (15) presents a striking contrast to Eq.(10): its phase speed can easily greater than the speed of light at the horizon.[9−13] Therefore the part F definitely has an influence on the Rindler spacetime, so does the part L. The Rindler spacetime as a whole is impossible to be stable, only the whole Minkowski spacetime is stable. From the above argument, we see that wave equation (8) loses some useful information at the horizon and might result in error in the stability study for the Rindler spacetime.

Therefore, we reinforce our conclusion again: the whole Minkowski spacetime is a real entity, while the Rindler spacetime cannot exist independently. Actually, the Rindler spacetime cannot exist if the left region L does not exist.

By comparing with the stability study of the Schwarzschild black hole, one can reasonably infer as follows: (1) the perturbation equation should be in the Kruskal spacetime, though it may be difficult; (2)the Regge-Wheeler equation is possibly not sufficient to determine the stable properties, and it surely loses some useful information at the horizon, which we will further study; (3) if the Schwarzschild black hole exists stably, the other regions without communicating with it must exist, that is, the Kruskal spacetime is a real physical entity.

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