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The geometric phase of the quantum systems with slow but finite rate of the external time-dependent field*

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With the help of the time-dependent gauge transformation technique, we have studied the geometric phase of a spin-half particle in a rotating magnetic field. We have found that the slow but finite frequency of the rotating magnetic field will make the difference between the adiabatic geometric phase and the exact geometric phase. When the frequency is much smaller than the energy space and the adiabatic condition is perfectly guaranteed, the adiabatic approximation geometric phase is exactly consistent with the adiabatic geometric phase. A simple relation for the accuracy of the adiabatic approximation is given in terms of the changing rate of the frequency of the rotating magnetic field and the energy level space.

Keywords: geometric phase, time-dependent gauge transformation
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1. Introduction

The investigation of the geometric phase has attracted much attention over the last few decades, since Berry[1] showed that a quantum system prepared in an eigenstate governed by the time-dependent Hamiltonian with a slowly varying parameter, will acquire a geometric phase in addition to the familiar dynamical phase. The importance of this basic concept has been accepted and generalized to various cases. Aharonov and Anandan[2] investigated the geometric phase in the case of the non-adiabatic cyclic evolution, which is called Aharonov–Anandan phase (AA phase). The relationship of the AA phase and its classical correspondence (Hannay angle) has been reported for a generalized harmonic oscillator with a squeezed state approach.[3] The noncyclic and nonunitary case has been studied by Samuel and Bhandari.[4] The geometric phase has been extended to more complex systems, for example the mixed states,[5] entangled quantum system,[6] composite system,[7] open system[8] and nonlinear system.[9,10] One more interesting result was pointed out that AA phases play the role of classical canonical actions and are conserved in the adiabatic evolution of a nonlinear system.[10] Furthermore, there are interesting new effects,[11−14] based on the geometric phase, in condensed matter physics[15] and some potential applications in quantum computation.[16,17] The adiabatic theorem[18] plays an important role in studying the geometric phase and should be employed carefully. In fact, Marzlin and Sanders[19] pointed out that a perfunctory application of this theorem is problematic if the change of the eigenstate is significant, regardless of how closely the evolution satisfies the requirements of the adiabatic theorem. Even if the adiabatic theory is carefully considered, an inconsistence between the adiabatic geometric phase and the exact one has been reported in more recent work.[20] This inconsistency may be considerable when the evolution time is large enough when investigating a solvable time-dependent model, a spin-half system driven by a rotating magnetic field.[20]

To understand this result better, we will investigate the same model by the invariant Hermitian operator method.[21−26] The invariant Hermitian operator method is one of the well-known effective meth-

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ods for solving the time-dependent systems.[21,22] Especially, for the systems with the Hamiltonian given by a linear combination of $SU(2)$ and $SU(1,1)$ generators, an invariant Hermitian operator can be constructed by the time-dependent gauge transformation technique.[23,24,26] An important feature of the invariant Hermitian operator method is that instead of the instantaneous eigenstates of the time-dependent Hamiltonian, we use the eigenstates of gauge invariant operators for solving the time-dependent systems. In the time-dependent gauge transformation technique the eigenstates of the Hamiltonian in the new gauge should be time-invariant.

In this paper, we will review the adiabatic theory, and consider the adiabatic geometric phase and the geometric phase for the time-dependent system. This paper is organized as follows: firstly, we will give a brief introduction of the time-dependent gauge transformation technique. And then an exactly solvable model, i.e. a spin-half particle driven by a rotating magnetic field,[20] is solved by using this method. Based on the exact solution of the time-dependent system, the geometric phase and the adiabatic condition are discussed and the conclusions will be given in the last section.

2. Time-dependent gauge transformation

In this section, we will give a brief introduction of the time-dependent gauge transformation technique.[23,24] For a time-dependent Hamiltonian $\hat{H}(t)$, the time evolution of the quantum state $|\psi(t)\rangle$ is governed by the time-dependent Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle, \tag{1}$$

where natural unit $\hbar = 1$ is used throughout. Introducing a time-dependent unitary transformation $\hat{R}(t)$, we then have $|\psi'(t)\rangle = \hat{R}(t)|\psi(t)\rangle$. In this new gauge the Schrödinger equation (1) can be rewritten as

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi'(t)\rangle = \hat{H}'(t)|\psi'(t)\rangle, \tag{2}$$

where $\hat{H}'(t)$ represents the Hamiltonian in the new gauge and is defined as

$$\hat{H}'(t) = \hat{R}(t)\hat{H}(t)\hat{R}^{\dagger}(t) - \mathrm{i}\hat{R}(t)\frac{\partial}{\partial t}\hat{R}^{\dagger}(t). \tag{3}$$

It is the key point of this method to find (or construct) the proper unitary transformation operator $\hat{R}(t)$, by which one can find the time-independent eigenstates of the Hamiltonian operator $\hat{H}'(t)$ in the new gauge

$$\hat{H}'(t)|E'_n\rangle = E'_n(t)|E'_n\rangle, \tag{4}$$

with $\partial E'_n(t)/\partial t = 0$. Then it is not difficult to find the solution of the time-dependent Schrödinger equation (2) with initial state $|E'_n\rangle$,

$$|\psi'_n(t)\rangle = \exp\Big[-\mathrm{i}\int_0^t E'_n(t')\mathrm{d}t'\Big]|E'_n\rangle. \tag{5}$$

After applying the inverse-time-dependent operator $\hat{R}^{\dagger}(t)$ to $|\psi'_n(t)\rangle$, we have

$$|\psi_n(t)\rangle = \hat{R}^{\dagger}(t)|\psi'_n(t)\rangle. \tag{6}$$

And therefore the general solution of the Schrödinger equation (1) with the initial state

$$|\psi(0)\rangle = \sum c_n\hat{R}^{\dagger}(0)|E'_n\rangle, \tag{7}$$

can be written as

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle, \tag{8}$$

where $\hat{U}(t,0)$, the time-dependent evolution operator, takes the form of

$$\hat{U}(t,0) = \hat{R}^{\dagger}(t)\mathrm{e}^{-\mathrm{i}\int_0^t \hat{H}'(t)\mathrm{d}t'}\hat{R}(0). \tag{9}$$

Thus the time evolution of arbitrary initial states, $|\psi(0)\rangle$, can be obtained by operating the time-evolution operator (9) to it. So far, once one found the proper unitary transformation operator $\hat{R}(t)$ to the time-dependent system $\hat{H}(t)$, we arrive at the exact solution of the Schrödinger equation (1) with the help of the time-evolution operators $\hat{U}(t,0)$. The advantage of this approach is that we can solve the time-dependent system, only using one complete set of time-independent eigenstates, which can be constructed by one unitary transformation operator $\hat{R}(t)$.

3. Model

In this section, we will illustrate how to find the unitary transformation operator $\hat{R}(t)$ for a simple but important solvable system of a spin-half particle driven by a rotating magnetic field.[20] The Hamiltonian can be written as

$$\hat{H}(t) = -\frac{\omega_0}{2}\boldsymbol{\sigma}\cdot\boldsymbol{n}, \tag{10}$$

where ω_0 is a constant parameter related to the magnetic moment of the spin and the intensity of the external magnetic field $\boldsymbol{B}$. The vector $\boldsymbol{\sigma}$ is $(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$, with $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ being Pauli matrices, and

$\boldsymbol{n} = (\sin\theta\cos\varphi(t), \sin\theta\sin\varphi(t), \cos\theta)$ denotes the unit vector of the rotating magnetic field $\boldsymbol{B}$. Here θ represents the angle between $\boldsymbol{\sigma}$ and $\boldsymbol{n}$, and $\varphi(t) = \omega t$, where ω is the rotating frequency of the external magnetic field $\boldsymbol{B}$. In this simple solvable time-dependent system, the instantaneous eigenstates are $|E_1(t)\rangle = (\mathrm{e}^{-\mathrm{i}\omega t/2}\sin\frac{\theta}{2}, -\mathrm{e}^{\mathrm{i}\omega t/2}\cos\frac{\theta}{2})^\dagger$ and $|E_2(t)\rangle = (\mathrm{e}^{-\mathrm{i}\omega t/2}\cos\frac{\theta}{2}, \mathrm{e}^{\mathrm{i}\omega t/2}\sin\frac{\theta}{2})^\dagger$. And the corresponding eigenvalues are $E_1 = \omega_0/2$, $E_2 = -\omega_0/2$.

3.1. Exact geometric phase

As mentioned in Section 2, the exact solutions of the Schrödinger equation can be obtained by means of the method of time-dependent gauge transformation. Like the procedures in Refs.[23,24], the Hamiltonian (10) can be rewritten as

$$\hat{H}(t) = -\frac{\omega_0}{2}\hat{\sigma}_z\cos\theta - \frac{\omega_0}{2}[\hat{\sigma}_+\exp(-\mathrm{i}\omega t) + \hat{\sigma}_-\exp(\mathrm{i}\omega t)]\sin\theta, \tag{11}$$

by introducing the operators $\hat{\sigma}_\pm = \frac{1}{2}(\hat{\sigma}_x \pm \mathrm{i}\hat{\sigma}_y)$. The time-dependent unitary transformation operator $\hat{R}(t)$ is set to be

$$\hat{R}(t) = \exp\Big[\frac{\lambda}{2}(\hat{\sigma}_-\exp(\mathrm{i}\omega t) - \hat{\sigma}_+\exp(-\mathrm{i}\omega t))\Big], \tag{12}$$

where λ is an auxiliary parameter and can be determined by letting the Hamiltonian operator $\hat{H}'$ be diagonal in the expression of $\hat{\sigma}_z$. Considering the following relations

$$\hat{R}(t)\widehat{\sigma}_z\hat{R}^\dagger(t) = \hat{\sigma}_z\cos\lambda + [\exp(\mathrm{i}\omega t)\hat{\sigma}_- + \exp(-\mathrm{i}\omega t)\hat{\sigma}_+]\sin\lambda, \tag{13}$$

$$\hat{R}(t)\widehat{\sigma}_+\hat{R}^\dagger(t) = -\frac{1}{2}\exp(\mathrm{i}\omega t)\hat{\sigma}_z\sin\lambda - \hat{\sigma}_-\exp(2\mathrm{i}\omega t)\sin^2\frac{\lambda}{2} + \hat{\sigma}_+\cos^2\frac{\lambda}{2}, \tag{14}$$

$$\hat{R}(t)\widehat{\sigma}_-\hat{R}^\dagger(t) = -\frac{1}{2}\exp(-\mathrm{i}\omega t)\hat{\sigma}_z\sin\lambda + \hat{\sigma}_-\cos^2\frac{\lambda}{2} - \hat{\sigma}_+\exp(-2\mathrm{i}\omega t)\sin^2\frac{\lambda}{2}, \tag{15}$$

and

$$\mathrm{i}\hat{R}(t)\frac{\mathrm{d}}{\mathrm{d}t}\hat{R}^\dagger(t) = -\omega\hat{\sigma}_z\sin^2\frac{\lambda}{2} + \frac{1}{2}\omega\hat{\sigma}_+\mathrm{e}^{-\mathrm{i}\varphi}\sin\lambda + \frac{1}{2}\omega\hat{\sigma}_-\mathrm{e}^{\mathrm{i}\varphi}\sin\lambda, \tag{16}$$

we can find a new Hamiltonian $\hat{H}'(t)$ from Eq.(3). Assuming the off-diagonal terms of $\hat{H}'(t)$ to be zero in the expression of $\widehat{\sigma}_z$, we have

$$\hat{H}' = -\frac{1}{2}(\bar{\omega} - \omega)\hat{\sigma}_z, \tag{17}$$

where $\cos\lambda = (\omega_0/\bar{\omega})\cos\theta + \omega/\bar{\omega}$, with

$$\bar{\omega} = \sqrt{\omega_0^2 + \omega^2 + 2\omega\omega_0\cos\theta}\,. \tag{18}$$

As stated above, the eigenstates of the time-independent Hamiltonian $\hat{H}'$ (17) in the new gauge are time-independent, $|E_1'\rangle = (1,0)^\dagger$, $|E_2'\rangle = (0,1)^\dagger$, and the corresponding eigenvalues are time-dependent, $E_{1,2}' = \mp\frac{1}{2}(\bar{\omega} - \omega)$. Combining Eqs.(12), (17) and (9), the evolution operator of the system in the new gauge is

$$\hat{U}(t,0) = \begin{pmatrix} \Big(\cos\frac{\bar{\omega}t}{2} + \mathrm{i}\frac{\omega + \omega_0\cos\theta}{\bar{\omega}}\sin\frac{\bar{\omega}t}{2}\Big)\mathrm{e}^{-\mathrm{i}\omega t/2}, & \mathrm{i}\frac{\omega_0\sin\theta}{\bar{\omega}}\sin\frac{\bar{\omega}t}{2}\mathrm{e}^{-\mathrm{i}\omega t/2} \\ \mathrm{i}\frac{\omega_0\sin\theta}{\bar{\omega}}\sin\frac{\bar{\omega}t}{2}\mathrm{e}^{\mathrm{i}\omega t/2}, & \Big(\cos\frac{\bar{\omega}t}{2} - \mathrm{i}\frac{\omega + \omega_0\cos\theta}{\bar{\omega}}\sin\frac{\bar{\omega}t}{2}\Big)\mathrm{e}^{\mathrm{i}\omega t/2} \end{pmatrix}. \tag{19}$$

Furthermore, we can have the exact solution of the Schrödinger equation $\psi(t)$ with initial state $\psi(0)$.

Assuming that our system starts with one of the eigenstates of the system, $\psi(0) = |E_1(0)\rangle = (\sin\frac{\theta}{2}, -\cos\frac{\theta}{2})^\dagger$, the exact solution of the system at any time t can be expressed as

$$\psi(t) = a(t)|E_1(t)\rangle + b(t)|E_2(t)\rangle, \tag{20}$$

where

$$a(t) = \cos\frac{\bar{\omega}t}{2} - \mathrm{i}\frac{\omega_0 + \omega\cos\theta}{\bar{\omega}}\sin\frac{\bar{\omega}t}{2},$$
$$b(t) = \mathrm{i}\frac{\omega\sin\theta}{\bar{\omega}}\sin\frac{\bar{\omega}t}{2}. \quad (21)$$

The solution (20) describes all of the information for the evolution of the system from the given initial state. Then geometric phase at some special time τ, arising from this evolution, can be found by the following definition[27]

$$\gamma(\tau) = \arg\langle\psi(0)|\psi(\tau)\rangle + \mathrm{i}\int_0^\tau \langle\psi(t)|\dot{\psi}(t)\rangle \mathrm{d}t. \quad (22)$$

Note that this geometric phase is calculated from the exact solution (20), so we called it an exact geometric phase. Substituting Eq.(20) into Eq.(22), we obtain the exact geometric phase of the system from $t = 0$ to $t = \tau$ with the initial state as one of the instantaneous eigenstate, $|E_1(0)\rangle$,

$$\begin{aligned}\gamma^{(\mathrm{exa})}(\tau) = \arg\Big[&\Big(\cos\frac{\bar{\omega}\tau}{2}\cos\frac{\omega\tau}{2} \\ &+ \frac{\omega + \omega_0\cos\theta}{\bar{\omega}}\sin\frac{\bar{\omega}\tau}{2}\sin\frac{\omega\tau}{2}\Big) \\ &+ \mathrm{i}\Big(\cos\theta\cos\frac{\bar{\omega}\tau}{2}\sin\frac{\omega\tau}{2} \\ &- \frac{\omega_0 + \omega\cos\theta}{\bar{\omega}}\sin\frac{\bar{\omega}\tau}{2}\cos\frac{\omega\tau}{2}\Big)\Big] \\ &+ \frac{\omega_0\omega^2\sin^2\theta}{2\bar{\omega}^3}\sin\bar{\omega}\tau + \frac{\omega_0\tau}{2} \\ &- \frac{\omega_0\omega^2\sin^2\theta}{2\bar{\omega}^2}\tau. \end{aligned} \quad (23)$$

3.2. Adiabatic geometric phase

Considering the adiabatic process, the initial state, prepared in $|E_1(0)\rangle$, will be transferred (by using the Schrödinger equation) into $|E_1(\tau)\rangle$.[28] In other words, we can have the adiabatic approximation condition, which means that the Hamiltonian of the system changes slowly enough, or equivalently, the rotating frequency (ω) of the magnetic field is small enough as compared with that of the energy level space (ω_0). This also implies that one can check if the system is good enough under the adiabatic process by how far $|b(t)|^2$ deviates from zero.

Under the adiabatic condition, we can also have the geometric phase, called adiabatic geometric phase,[20]

$$\begin{aligned}\gamma^{(\mathrm{adi})}(\tau) = &\arg[\langle E_1(0)|E_1(\tau)\rangle] \\ &+ \mathrm{i}\int_0^\tau \langle E_1(t)|\dot{E}_1(t)\rangle \mathrm{d}t, \\ = &\arg\Big(\cos\frac{\omega\tau}{2} + \mathrm{i}\cos\theta\sin\frac{\omega\tau}{2}\Big) \\ &- \frac{1}{2}\omega\tau\cos\theta. \end{aligned} \quad (24)$$

To derive Eq.(24), we have used the explicit formula of the instantaneous eigenstates.

4. Discussion

In this section, we will study systematically the adiabatic approximation and the difference between the exact geometric phase (23) and the adiabatic geometric phase (24). Generally, when the adiabatic approximation is perfectly guaranteed, i.e. $\omega/\omega_0 \to 0$, the exact geometric phase can be reduced to the adiabatic geometric one.

It is convenient to re-scale the frequency by that of the energy level space ω_0 and the time τ by $1/\omega_0$. Then we can rewrite the exact geometric phase (23) and the adiabatic phase (24) in terms of dimensionless parameters $\omega' \equiv \omega/\omega_0$, and $\tau' \equiv \omega_0\tau$:

$$\gamma^{(\mathrm{exa})}(\tau') = \gamma_1^{(\mathrm{exa})}(\tau') + \gamma_2^{(\mathrm{exa})}(\tau') + \gamma_3^{(\mathrm{exa})}(\tau') + \frac{\tau'}{2}, \quad (25)$$

where

$$\begin{aligned}\gamma_1^{(\mathrm{exa})}(\tau') \equiv \arg\Big[&\Big(\cos\frac{\tilde{\omega}\tau'}{2}\cos\frac{\omega'\tau'}{2} \\ &+ \frac{\omega' + \cos\theta}{\tilde{\omega}}\sin\frac{\tilde{\omega}\tau'}{2}\sin\frac{\omega'\tau'}{2}\Big) \\ &+ \mathrm{i}\cos\theta\cos\frac{\tilde{\omega}\tau'}{2}\sin\frac{\omega'\tau'}{2} \\ &- \frac{1 + \omega'\cos\theta}{\tilde{\omega}}\sin\frac{\tilde{\omega}\tau'}{2}\cos\frac{\omega'\tau'}{2}\Big], \end{aligned} \quad (26)$$

$$\gamma_2^{(\mathrm{exa})}(\tau') \equiv \frac{\omega'^2\sin^2\theta}{2\tilde{\omega}^3}\sin\tilde{\omega}\tau', \quad (27)$$

and

$$\gamma_3^{(\mathrm{exa})}(\tau') \equiv -\frac{\omega'^2\sin^2\theta}{2\tilde{\omega}^2}\tau', \quad (28)$$

with $\tilde{\omega} = \bar{\omega}/\omega_0 = \sqrt{1 + 2\omega'\cos\theta + \omega'^2}$, and

$$\begin{aligned}\gamma^{(\mathrm{adi})}(\tau') = &\arg\Big(\cos\frac{\omega'\tau'}{2} + \mathrm{i}\cos\theta\sin\frac{\omega'\tau'}{2}\Big) \\ &- \frac{1}{2}\omega'\tau'\cos\theta. \end{aligned} \quad (29)$$

Now we turn to the adiabatic approximation, $\omega' \ll 1$. In this case, we can expand the terms in exact geometric phase (25–28) with respect to the small quantity ω'. It is easy to obtain $\gamma_2^{(\mathrm{exa})}(\tau') = \frac{1}{2}\sin^2\theta\sin(\tilde{\omega}'\tau')\omega'^2$. Because it is always a quantity of second order of ω', we can safely neglect this term in the following consideration. While for $\gamma_3^{(\mathrm{exa})}(\tau') = -\frac{1}{2}\sin^2(\theta)\omega'^2\tau'$, it can be the first order of ω' once the

evolution time is long enough ($\tau' > 1/\omega'$).[20] Retaining the first order of ω' in $\gamma_1^{(\mathrm{exa})}(\tau')$ and $\gamma_3^{(\mathrm{exa})}(\tau')$, we can express the exact geometric phase as

$$\tilde{\gamma}(\tau') = \tilde{\gamma}_1(\tau') + \tilde{\gamma}_3(\tau') + \frac{\tau'}{2}, \tag{30}$$

where

$$\tilde{\gamma}_1(\tau') = \arg[\tilde{\gamma}_1[o(\omega'^0)] + \tilde{\gamma}_1[o(\omega')]], \tag{31}$$

with

$$\begin{aligned}\tilde{\gamma}_1[o(\omega'^0)] = &\Big(\cos\frac{\tau'\tilde{\omega}'}{2}\cos\frac{\omega'\tau'}{2} \\ &+ \cos\theta\sin\frac{\tau'\tilde{\omega}'}{2}\sin\frac{\omega'\tau'}{2}\Big) \\ &+ \mathrm{i}\Big(\cos\theta\cos\frac{\tau'\tilde{\omega}'}{2}\sin\frac{\omega'\tau'}{2} \\ &- \sin\frac{\tau'\tilde{\omega}'}{2}\cos\frac{\omega'\tau'}{2}\Big),\end{aligned} \tag{32}$$

$$\tilde{\gamma}_1[o(\omega')] = \Big(\sin^2\theta\sin\frac{\tau'\tilde{\omega}'}{2}\sin\frac{\omega'\tau'}{2}\Big)\omega', \tag{33}$$

and

$$\tilde{\gamma}_3(\tau') = -\frac{1}{2}\sin^2\theta\tau'\omega'^2, \tag{34}$$

where $\tilde{\omega}' = \sqrt{1 + 2\omega'\cos\theta}$.

To show this approximation is reasonable, we let $\theta = \pi/6$ and $\omega' = 10^{-6} \ll 1$. Firstly, we calculate all the terms and fill them in Table 1 at the given time τ'. Note that at $\tau' = 2\pi/\omega'$, the magnetic field completes one cycle and the Hamiltonian also comes back to the initial one.

We can see from Table 1 that the term $\gamma_2^{(\mathrm{exa})}(\tau')$ is always small and can then be neglected no matter how long the time τ' is. Meanwhile, $\gamma_3^{(\mathrm{exa})}(\tau')$ will increase as long as the evolution time τ increases. Especially, when τ' reaches the magnitude of $1/\omega'$ (τ' takes one period of time $2\pi/\omega'$), $\gamma_3^{(\mathrm{exa})}(\tau')$ becomes the first order of ω', i.e. 10^{-6}. When τ' reaches the magnitude of $1/\omega'^2$ (τ' takes $10^6 \cdot 2\pi/\omega'$), $\gamma_3^{(\mathrm{exa})}(\tau')$ is of the order of one. This means that $\gamma_3^{(\mathrm{exa})}(\tau')$ will contribute more and more to the geometric phase which will be inconsistent with the adiabatic geometric phase.[20] This result suggests that some external geometric phase could be created by the external magnetic field at a slow but finite rotating rate. It is not difficult to prove that in all of the time τ in Table 1, we have $|b(\tau)|^2 \sim 10^{-13}$, indicating that the adiabatic process is guaranteed.

Table 1. The terms of exact geometric phase for a given τ.

τ'	$\gamma_1^{(\mathrm{exa})}(\tau')$	$\gamma_2^{(\mathrm{exa})}(\tau')$	$\gamma_3^{(\mathrm{exa})}(\tau')$	$\tau'/2$
100	0.265482	-6.32862×10^{-14}	-1.245×10^{-11}	50
1000	2.65482	1.03421×10^{-13}	-1.245×10^{-10}	500
$2\pi/\omega'$	0.420893	-9.3229×10^{-14}	-0.785397×10^{-6}	3.14159×10^{6}
$10 \cdot (2\pi/\omega')$	-2.07425	-1.05648×10^{-13}	-0.785397×10^{-5}	3.14159×10^{7}
$100 \cdot (2\pi/\omega')$	-1.89298	-7.50855×10^{-14}	-0.785397×10^{-4}	3.14159×10^{8}
$10^4 \cdot (2\pi/\omega')$	-0.80201	1.24931×10^{-13}	-0.785397×10^{-2}	3.14159×10^{10}
$10^6 \cdot (2\pi/\omega')$	1.48023	-2.25169×10^{-14}	-0.785397	3.14159×10^{12}

Table 2. The values of $\gamma^{(\mathrm{exa})}(\tau')$, $\tilde{\gamma}(\tau')$ and $\gamma^{(\mathrm{adi})}(\tau')$ for given τ'.

τ'	$\gamma^{(\mathrm{exa})}(\tau')$	$\tilde{\gamma}(\tau')$	$\gamma^{(\mathrm{adi})}(\tau')$	the accuracy
100	-1.79767×10^{-11}	-1.16316×10^{-9}	9.0211×10^{-15}	10^{-9}
1000	-1.50941×10^{-10}	-8.86402×10^{-11}	9.0211×10^{-12}	10^{-10}
$2\pi/\omega'$	0.420892	0.420893	0.420894	10^{-6}
$10 \cdot (2\pi/\omega')$	4.20892	4.20893	4.20894	10^{-5}
$100 \cdot (2\pi/\omega')$	4.39013	4.39017	4.39025	10^{-4}
$10^4 \cdot (2\pi/\omega')$	5.47332	5.47725	5.4851	10^{-2}
$10^5 \cdot (2\pi/\omega')$	4.46777	4.50702	4.58554	10^{-1}
$10^6 \cdot (2\pi/\omega')$	0.695313	1.08789	1.87307	10^{-0}

In Table 2, we list the exact geometric phases $\gamma^{(\mathrm{exa})}(\tau')$, the exact phases $\tilde{\gamma}(\tau')$ in the adiabatic limit, and the adiabatic geometric phases $\gamma^{(\mathrm{adi})}(\tau')$ at the given time τ', where all of the values of $\gamma^{(\mathrm{exa})}(\tau')$, $\tilde{\gamma}(\tau')$ and $\gamma^{(\mathrm{adi})}(\tau')$ are re-scaled into the range of $(-2\pi, 2\pi)$. It is easily seen that the adiabatic ap-

proximation is excellently consistent for $\tau' < 2\pi/\omega'$ and good enough even at $\tau' = 10 \times 2\pi/\omega'$. The accuracy of the adiabatic approximation depends on the evolution time τ' of the system. The simple relation is $n \times \omega'$, where n denotes the period of the external magnetic field. This also implies that when $\omega \to 0$, the available range of the adiabatic approximation will be enlarged to infinity. Therefore, the difference between the adiabatic geometric phase and the exact geometric phase comes from the slow but finite rotations of the external magnetic field. When we design the quantum operator based on this geometric phase, we must consider the contribution from the finite rate change of the external time-dependent control field.

To illustrate this point clearly, we make two tables for two finite but 'slow' rotating rates $\omega' = 10^{-3}$ in Table 3 and $\omega' = 10^{-2}$ in Table 4. One can check again that the adiabatic condition, $|b(\tau')|^2 \sim 10^{-7}$, is still well satisfied for $\omega' = 10^{-3}$. While for $\omega' = 10^{-2}$, the adiabatic condition is kept not so good, $|b(\tau')|^2 \sim 10^{-4}$. Of course, compared with $\omega' = 10^{-6}$, for which $|b(\tau')|^2 \sim 10^{-13}$, the adiabatic condition cannot be perfectly guaranteed even for $\omega' = 10^{-3}$. It is interesting to see that the simple relation $n \times \omega'$ is still work even under these two conditions. Therefore, adiabatic geometric phase will give good result when the frequency of the external field is much smaller than that of the level space. In other words, the adiabatic geometric phase shows a good consistence with the exact geometric phase in the adiabatic limit, when the adiabatic condition is completely guaranteed. But for the slow but finite frequency, the adiabatic geometric phase will be somewhat inconsistent with the exact geometric phase in the adiabatic limit. Moreover, when the adiabatic condition is broken, the adiabatic geometric phase does not have any meanings.

Table 3. Same as Table 2 in the case of $\omega' = 10^{-3}$.

τ'	$\gamma^{(\mathrm{exa})}(\tau')$	$\tilde{\gamma}(\tau')$	$\gamma^{(\mathrm{adi})}(\tau')$
100	-9.06142×10^{-6}	-2.75738×10^{-6}	9.01997×10^{-6}
500	1.08441×10^{-3}	1.11572×10^{-3}	1.124×10^{-3}
$2\pi/\omega'$	0.419717	0.420108	0.420894
$10 \cdot (2\pi/\omega')$	4.19717	4.20108	4.20894
$100 \cdot (2\pi/\omega')$	4.27261	4.31171	4.39025
$10^3 \cdot (2\pi/\omega')$	5.02698	5.41798	6.20338

Table 4. Same as Table 2 in the case of $\omega' = 10^{-2}$.

τ'	$\gamma^{(\mathrm{exa})}(\tau')$	$\tilde{\gamma}(\tau')$	$\gamma^{(\mathrm{adi})}(\tau')$
50	1.96044×10^{-4}	4.92041×10^{-4}	1.124×10^{-3}
100	6.99639×10^{-3}	7.58817×10^{-3}	8.89393×10^{-3}
$2\pi/\omega'$	0.409267	0.41304	0.420894
$10 \cdot (2\pi/\omega')$	4.09279	4.1304	4.20894
$100 \cdot (2\pi/\omega')$	3.22896	3.60485	4.39025

5. Conclusion

In conclusion, we have investigated the geometric phase of a solvable quantum system, i.e. a spin-half particle driven by a rotating magnetic field, using a time-dependent gauge transformation technique. With the help of the exact solution of the system, one can not only study the exact geometric phase, but also check the adiabatic condition. Assuming that our system can be described in the adiabatic approximation, $\omega' \equiv \omega/\omega_0 \ll 1$, the exact geometric phase $\gamma^{\mathrm{exa}}(\tau)$ can be replaced by the approximative phase $\tilde{\gamma}(\tau)$ up to the second order of ω'. Even the external term $(\gamma_3^{\mathrm{exa}})$ has some contribution to the geometric phase when our evolution time is longer than $2\pi/\omega'$, as pointed out in Ref.[20]. Our results show that the adiabatic geometric phase is consistent with the approximative geometric phase when the adiabatic condition are perfectly guaranteed. But for the slow but finite frequency, the adiabatic geometric phase will be somewhat inconsistent with the exact geometric phase in the adiabatic limit. Furthermore, when the adiabatic condition is broken, the adiabatic geometric phase does not have any meanings. In addition, we have found that the accuracy of the adiabatic approximation depends on the evolution time τ' of the system and can be described by $n \times \omega'$, where n denotes the periods of the external magnetic field. This simple relation also supports the above conclusions.

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