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Analytical mechanics methods for solving Whittaker equations*

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The purpose of this paper is to study the solution of the celebrated Whittaker equations by using analytical mechanics methods, including the Lagrange–Noether method, Hamilton–Poisson method and potential integral method.

Keywords: Whittaker equations, Lagrange–Noether method, Hamilton–Poisson method, potential integral method

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1. Introduction

In the 1930s Tolman presented a problem: a set of equations, which are existent or not, cannot be gained by using a Lagrange function. Whittaker, who is a famous specialist in classical mechanics, gave the answer to Tolman's problem. Subsequently the answer is called Whittaker equations[1,2] and importance was attached to these by many physicists. So the Whittaker equations occupy a significant position in the phylogeny of classical mechanics, especially in the study of inverse problem of Lagrange mechanics. To the category of classical integral methods of analytical mechanics belong mainly the Routh reduction method, Whittaker reduction method, Hamilton–Jacobi method, Poisson method, canonical transformation, integral invariant, etc.[3] Modern integral methods include primarily the field method,[4−6] Noether symmetry method,[7−11] Lie symmetry method[12−21] and so on. Ref.[16] has studied the field method of the Whittaker equations. This paper uses some integral methods of analytical mechanics to study the integral of Whittaker equations.

2. Whittaker equations

Whittaker gave the following equations[1,2] for the answer of Tolman's problem:

$$\ddot{q}_1 = q_1, \quad \ddot{q}_2 = \dot{q}_1; \tag{1}$$

and made sure that Eqs.(1) cannot be deduced by using any Lagrange function. But the solution to Eqs.(1) in mathematics does not present any difficulties.

In the following, we use some methods of analytical mechanics to solve Eqs.(1).

3. Lagrange-Noether method

The basic idea of the Lagrange–Noether method is firstly to transform ordinary differential equations, all or partly, into equations expressed by the Lagrange function, and to then make use of Noether theory to find the conserved quantities. Equations (1) cannot be expressed by a Lagrangian, however it partly can be shown as a Lagrangian, therefore we have

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}q_1^2,$$
$$Q_1 = 0, \quad Q_2 = \dot{q}_1; \tag{2}$$

which is a holonomic non-conservative system. According to Noether theory, the Noether identity will be[7−9]

$$\left\{\frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}q_1^2\right\}\dot{\xi}_0 + \dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)$$
$$+q_1\xi_1 + \dot{q}_1(\xi_2 - \dot{q}_2\xi_0) + \dot{G}_N = 0. \tag{3}$$

In this way, we obtain the following solutions:

$$\xi_0 = \xi_2 = 0, \quad \xi_1 = \exp(-t), \quad G_N = q_1\exp(-t), \tag{4}$$

$$\xi_0 = \xi_2 = 0, \quad \xi_1 = -\exp t, \quad G_N = q_1\exp t, \tag{5}$$

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$$\xi_0 = \xi_1 = 0, \quad \xi_2 = 1, \quad G_N = -q_1, \tag{6}$$

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = t, \quad G_N = -q_2 - tq_1. \tag{7}$$

And the corresponding Noether conserved quantities are

$$\begin{aligned} I_{\mathrm{N}_1} &= \dot{q}_1 \exp(-t) + q_1 \exp(-t) = C_1, \\ I_{\mathrm{N}_2} &= -\dot{q}_1 \exp t + q_1 \exp t = C_2, \\ I_{\mathrm{N}_3} &= \dot{q}_2 - q_1 = C_3, \\ I_{\mathrm{N}_4} &= \dot{q}_1 + \dot{q}_2 t - q_2 - q_1 t = C_4, \end{aligned}$$

where C_1, C_2, C_3, C_4 are arbitrary constants. Consequently we obtain

$$\begin{aligned} q_1 &= \frac{1}{2} C_1 \exp t + \frac{1}{2} C_2 \exp(-t), \\ q_2 &= \frac{1}{2} C_1 \exp t - \frac{1}{2} C_2 \exp(-t) + C_3 t - C_4. \end{aligned} \tag{8}$$

4. Hamilton–Poisson method

In the Hamilton–Poisson method, one first transforms ordinary differential equations, all or partly, into equations expressed by Hamilton function, then makes use of Poisson condition or generalized Poisson condition of the first integral to find the integrals.

Let

$$\begin{aligned} &a^1 = q_1, \quad a^2 = q_2, \quad a^3 = p_1 = \dot{q}_1, \quad a^4 = p_2 = \dot{q}_2, \\ &H = \frac{1}{2}\Big\{(a^3)^2 + (a^4)^2\Big\} - \frac{1}{2}(a^1)^2. \end{aligned} \tag{9}$$

Then Eqs.(1) can be written as

$$\dot{a}^\mu = S^{\mu\gamma} \frac{\partial H}{\partial a^\gamma} \quad (\mu, \gamma = 1, 2, 3, 4), \tag{10}$$

where

$$(S^{\mu\gamma}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & a^3/a^4 \end{pmatrix}. \tag{11}$$

The Generalized Poisson condition of first integral gives[9]

$$\frac{\partial I}{\partial t} + I \circ H = 0. \tag{12}$$

Here

$$I \circ H = \frac{\partial I}{\partial a^\mu} S^{\mu\gamma} \frac{\partial H}{\partial a^\gamma}. \tag{13}$$

Substituting (13) into Eq.(12) yields

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^1} a^3 + \frac{\partial I}{\partial a^2} a^4 + \frac{\partial I}{\partial a^3} a^1 + \frac{\partial I}{\partial a^4} a^3 = 0. \tag{14}$$

And ordinary differential equations which correspond to partial differential equation (14) are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}a^1}{a^3} = \frac{\mathrm{d}a^2}{a^4} = \frac{\mathrm{d}a^3}{a^1} = \frac{\mathrm{d}a^4}{a^3}. \tag{15}$$

Integrating the above equations, we have

$$\begin{aligned} &a^4 - a^1 = D_1, \quad (a^1)^2 - (a^3)^2 = D_2, \\ &(a^1 + a^3)\exp(-t) = D_3, \\ &a^3 + t a^4 - a^2 - t a^1 = D_4, \end{aligned} \tag{16}$$

where D_1, D_2, D_3, D_4 are integral constants. From this we obtain

$$\begin{aligned} a^1 &= \frac{1}{2}\Big\{D_3 \exp t + \frac{D_2}{D_3} \exp(-t)\Big\}, \\ a^2 &= \frac{1}{2}\Big\{D_3 \exp t - \frac{D_2}{D_3} \exp(-t)\Big\} + D_1 t - D_4, \end{aligned} \tag{17}$$

which are general solutions of Whittaker equations.

5. Potential integral method

The basic idea of the potential integral method is that the problem of solving ordinary differential equation is translated into the seeking of complete integral of partial differential equation.[22,23]

Let

$$x_1 = q_1, \quad x_2 = q_2, \quad x_3 = \dot{q}_1, \quad x_4 = \dot{q}_2, \tag{18}$$

then Eqs.(1) can be expressed as

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = x_1, \quad \dot{x}_4 = x_3. \tag{19}$$

According to potential integral method, we have the partial differential equation

$$\frac{\partial \psi}{\partial t} = p_1 \frac{\partial \psi}{\partial p_3} + p_2 \frac{\partial \psi}{\partial p_4} + p_3 \frac{\partial \psi}{\partial p_1} + p_4 \frac{\partial \psi}{\partial p_3}. \tag{20}$$

And the corresponding ordinary differential equation is

$$\frac{\mathrm{d}p_1}{p_3} = \frac{\mathrm{d}p_2}{0} = \frac{\mathrm{d}p_3}{p_1 + p_4} = \frac{\mathrm{d}p_4}{p_2} = \frac{\mathrm{d}t}{-1}. \tag{21}$$

In this case, we have

$$p_2 = C_2, \tag{22}$$

$$p_4 + p_2 t = C_4, \tag{23}$$

$$p_1 + p_4 = C_1 \exp t + C_3 \exp(-t), \tag{24}$$

$$p_2 + p_3 = -C_1 \exp t + C_3 \exp(-t). \tag{25}$$

From formula (24) and (25), we have

$$(-p_1 + p_2 + p_3 - p_4)\exp(-t) = 2C_1, \tag{26}$$

$$(p_1 + p_2 + p_3 + p_4)\exp t = 2C_3. \tag{27}$$

So the complete integral of Eq.(20) is

$$\psi = a_1(-p_1+p_2+p_3-p_4)\exp(-t) + a_2p_2 + a_3(p_1+p_2+p_3+p_4)\exp t + a_4(p_4+p_2t). \quad (28)$$

Hence the solutions of the problem given by the potential integral method are as follows:

$$x_1 = \frac{\partial\psi}{\partial p_1} = -a_1\exp(-t) + a_3\exp t, \quad (29)$$

$$x_2 = \frac{\partial\psi}{\partial p_2} = a_1\exp(-t) + a_3\exp t + a_2 + a_4t, \quad (30)$$

$$x_3 = \frac{\partial\psi}{\partial p_3} = a_1\exp(-t) + a_3\exp t, \quad (32)$$

$$x_4 = \frac{\partial\psi}{\partial p_4} = -a_1\exp(-t) + a_3\exp t + a_4, \quad (32)$$

where a_1, a_2, a_3, a_4 are arbitrary constants, and formula (29) and (30) are just the general solutions of Whittaker equations.

6. Conclusions

Integral methods presented in this paper, such as the Lagrange-Noether method, Hamilton-Poisson method, potential integral method, etc., are all analytical mechanics methods. In virtue of these methods, one can solve general ordinary differential equations.

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