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Unified symmetry of non-holonomic singular systems

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In this paper, the unified symmetry of non-holonomic singular systems is studied. The differential equations of motion of the systems are given. The definition and the criterion of the unified symmetry for the systems are presented. The Noether conserved quantity, the Hojman conserved quantity and the Mei conserved quantity are obtained. An example is given to illustrate the application of the results.

Keywords: singular system, non-holonomic constraint, unified symmetry, conserved quantity
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1. Introduction

The study of symmetries and conserved quantities is of theoretical and practical significance in mechanics. The symmetry method is a modern method for finding conserved quantities of constrained mechanical systems. There are mainly three kinds of symmetries, namely, the Noether symmetry,[1−3] the Lie symmetry[2−10] and the Mei symmetry, i.e. the form invariance[11−16]. There are mainly three kinds of conserved quantities either, i.e. the Noether conserved quantity,[1−3] the Hojman conserved quantity[17,18] and the Mei conserved quantity,[19] which can, respectively, be deduced from the Noether symmetry, the special Lie symmetry and the Mei symmetry. The system described by a singular Lagrangian is called the singular system, and the singular system is a physical one. In field theory, many important physical systems are singular systems, so it's necessary to study the singular systems. Dirac[20] and Li[1] studied the Noether symmetry of the singular systems, Mei studied the Lie symmetry of the singular Lagrange systems,[3] Zhang studied the Lie symmetry of Hamilton canonical equations in a singular system,[21] Li studied the Lie symmetry of a type of non-holonomic singular systems,[22] Luo studied the Mei symmetry, the Noether symmetry and the Lie symmetry of Hamilton canonical equations in a singular system,[23] Mei studied the form invariance, i.e. the Mei symmetry of non-holonomic singular systems of non-Chetaev type.[24] Mei studied the unified symmetry of Lagrangian systems.[25] Xia studied the unified symmetry of the non-holonomic systems with variable mass.[26] In this paper, the unified symmetry of non-holonomic singular systems is studied, from which the Noether conserved quantity, the Hojman conserved quantity and the Mei conserved quantity are found.

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2. Equations of motion of the system

Suppose that the position of a system is determined by n generalized coordinates $q_s\,(s=1,\ldots,n)$. The system is subjected to g ideal non-holonomic constraints of Chetaev type

$$f_\beta = f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0, \quad (\beta = 1,\ldots,g). \tag{1}$$

Limitations of the non-holonomic constraints (1) added on the virtual displacements are shown as

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0\,. \tag{2}$$

Then, the differential equations of the system can be expressed in the form

$$E_s(L) = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \quad (s=1,\ldots,n), \tag{3}$$

where $L = T - V$ is the Lagrangian of the system. Q_s are the generalized non-potential forces, and λ_β are the constraint multipliers. E_s is Euler operator,

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}\,.$$

Before we integrate the differential equations of motion, we can obtain λ_β from Eqs.(1) and (3), and

express it as a function of $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$. Thus Eqs.(3) can be expressed as

$$E_s(L) = Q_s + \Lambda_s, \quad (s = 1, \ldots, n), \tag{4}$$

where $\Lambda_s = \lambda_\beta(\partial f_\beta/\partial \dot{q}_s)$ and their expansions are

$$h_{sk}\ddot{q}_k = \frac{\partial L}{\partial q_s} - \frac{\partial^2 L}{\partial \dot{q}_s \partial q_k}\dot{q}_k - \frac{\partial^2 L}{\partial \dot{q}_s \partial t} + Q_s + \Lambda_s. \tag{5}$$

If

$$\det(h_{sk}) = \det\Big(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\Big) = 0. \tag{6}$$

We cannot derive all the generalized accelerations from Eqs.(5), but we can derive a part of the generalized accelerations

$$\ddot{q}_i = \alpha_i(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (i = 1, \ldots, r; 0 \le r \le n), \tag{7}$$

and $(n - r)$ relations

$$\phi_j = \phi_j(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \quad (j = r + 1, \ldots, n). \tag{8}$$

3. Definition and criterion of the unified symmetry

Definition If a symmetry is a Noether one as well as a Lie one and is also a Mei one at one time, the symmetry is called a unified symmetry.[25]

We consider the infinitesimal transformations of time and generalized coordinates as

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{9}$$

where ε is the infinitesimal parameter, ξ_0, ξ_s are the generators of infinitesimal transformations.

The invariance of Eqs.(8) under the infinitesimal transformations (9) leads to the following restriction equations:

$$X^{(1)}(\phi_j) = 0, \quad (j = r + 1, \ldots, n), \tag{10}$$

the constrained Eqs.(1) retain the form invariance

$$X^{(1)}(f_\beta) = 0, \quad (\beta = 1, \ldots, g), \tag{11}$$

and the restriction equations are

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0, \quad (\beta = 1, \ldots, g). \tag{12}$$

For non-holonomic singular systems, the Noether identity takes the form

$$L\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + X^{(1)}(L) + (Q_s + \Lambda_s) \times (\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}} = 0, \tag{13}$$

the determining equations of the Lie symmetry are

$$X^{(2)}\{\bar{E}_s(L)\} = X^{(1)}(Q_s + \Lambda_s), \tag{14}$$

and the criterion equations of the Mei symmetry are

$$\bar{E}_s\{X^{(1)}(L)\} = X^{(1)}(Q_s + \Lambda_s), \tag{15}$$

where

$$\begin{aligned}
&X^{(1)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\Big)\frac{\partial}{\partial \dot{q}_s},\\
&X^{(2)} = X^{(1)} + \Big(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - 2\ddot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\Big)\frac{\partial}{\partial \ddot{q}_s},\\
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s},\\
&\bar{E}_s = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}.
\end{aligned} \tag{16}$$

Criterion For the non-holonomic singular system, if there exists a function $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that the generators of infinitesimal transformations ξ_0 and ξ_s satisfy Eqs.(10)–(12) and

$$\begin{aligned}
&\Big[L\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + X^{(1)}(L) + (Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}}\Big]^2\\
&\quad + \{X^{(2)}[\bar{E}_s(L)] - X^{(1)}(Q_s + \Lambda_s)\}^2\\
&\quad + \{E_s[X^{(1)}(L)] - X^{(1)}(Q_s + \Lambda_s)\}^2 = 0,
\end{aligned} \tag{17}$$

then the corresponding symmetry is a unified symmetry.

4. The conserved quantity deduced from the unified symmetry

Under some conditions, by using the unified symmetry, the Noether conserved quantity , the Hojman conserved quantity and the Mei conserved quantity can all be obtained.

Proposition 1 For non-holonomic singular systems, the unified symmetry will lead to the Noether conserved quantity

$$I_{\mathrm{N}} = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{N}} = \mathrm{const}. \tag{18}$$

Proof Because the unified symmetry is certainly a Noether symmetry, there exists such a gauge function $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ that the identity (13) holds. By using the Noether theorem, the system has the conserved quantity (18).

Proposition 2 For non-holonomic singular systems, under the special infinitesimal transformations in which time is not varied, if there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\sum_{i=1}^{r} \frac{\partial \alpha_i}{\partial \dot{q}_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \tag{19}$$

then the unified symmetry will lead to the Hojman conserved quantity

$$I_{\mathrm{H}} = \frac{1}{\mu} \frac{\partial}{\partial q_i} (\mu \xi_i) + \frac{1}{\mu} \frac{\partial}{\partial \dot{q}_i} \Big(\mu \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_i \Big) = \text{const.} \tag{20}$$

Proof Because the unified symmetry is certainly a Lie symmetry, so we have the determining equations (14). By using Eqs.(14) and (19), proposition 2 is verified.

Proposition 3 For non-holonomic singular systems, if there is a function $G_{\mathrm{M}} = G_{\mathrm{M}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$X^{(1)}\{X^{(1)}(L)\} + X^{(1)}(L) \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_0 + X^{(1)}(Q_s + \Lambda_s) \times (\xi_s - \dot{q}_s \xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} G_{\mathrm{M}} = 0, \tag{21}$$

then the unified symmetry will lead to the Mei conserved quantity

$$I_{\mathrm{M}} = X^{(1)}(L) \xi_0 + \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} (\xi_s - \dot{q}_s \xi_0) + G_{\mathrm{M}} = \text{const.} \tag{22}$$

Proof Because the unified symmetry is certainly a Mei symmetry, so the criterion equations (15) hold. By using Eqs.(15) and (21), we can prove that proposition 3 is true.

5. An illustrative example

Let the Lagrangian be

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \dot{q}_1 q_3 + \frac{1}{2}(q_1 - q_2)^2, \tag{23}$$

the non-potential generalized forces be

$$Q_1 = Q_2 = 0, Q_3 = \dot{q}_3, \tag{24}$$

and the non-holonomic constraint acting on the system be

$$f = \dot{q}_1 - \dot{q}_2 + q_3 = 0, \tag{25}$$

we study the unified symmetry and conserved quantities of the system.

Eqs.(3) give the equations of motion as follows:

$$\begin{aligned} &\ddot{q}_1 + \dot{q}_3 - (q_1 - q_2) = \lambda, \\ &\ddot{q}_2 + (q_1 - q_2) = -\lambda, \\ &\dot{q}_1 + \dot{q}_3 = 0. \end{aligned} \tag{26}$$

Form Eqs.(25) and (26), we obtain

$$\lambda = q_2 - q_1, \quad \ddot{q}_1 = -\dot{q}_3 = \dot{q}_1, \quad \ddot{q}_2 = 0. \tag{27}$$

Take generators as follows:

$$\xi_0 = \xi_3 = 0, \quad \xi_1 = \xi_2 = 1, \tag{28}$$

It is easy to verify that the generators (28) satisfy Eqs.(10)–(12) and (17), and the transformations are unified symmetrical transformations.

Substituting (28) into the Noether identity (13), we obtain the gauge function

$$G_{\mathrm{N}} = 0, \tag{29}$$

and Proposition 1 gives

$$I_{\mathrm{N}} = \dot{q}_1 + \dot{q}_2 + q_3 = \text{const.} \tag{30}$$

Formula (19) gives

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = -1, \tag{31}$$

which has the solutions

$$\mu = \exp(-t), \tag{32}$$

$$\mu = (q_1 + q_3)\exp(-t). \tag{33}$$

From relations (28) and (33), using proposition 2, we obtain

$$I_{\mathrm{H}} = (q_1 + q_3)^{-1} = \text{const.} \tag{34}$$

Expression (21) gives

$$G_{\mathrm{M}} = q_1 + q_3, \tag{35}$$

and Proposition 3 gives

$$I_{\mathrm{M}} = q_1 + q_3 = \text{const.} \tag{36}$$

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