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The periodic solution to a delayed sea–air oscillator coupling model for the El Niño–Southern Oscillation (ENSO)*

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A time-delay sea–air oscillator coupling model is studied. Using Mawhin's continuation theorem, the result on the existence of periodic solutions for the sea–air oscillator model is obtained.

Keywords: El Niño-Southern Oscillation, sea–air oscillator, periodic solution
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1. Introduction

During the last decade, many studies have focused on interannual climate variability associated with the El Niño–Southern Oscillation (ENSO). Progress has been made in understanding and simulating the coupled tropical ocean-atmosphere system by using models with complexity varying.[1] A mechanism for the oscillatory nature of ENSO was first proposed by McCreary,[2] based on the reflection of subtropical oceanic upwelling Rossby waves at the western boundary. Suarez and Schopf[3] introduced the conceptual delayed oscillator as a candidate of the mechanism for ENSO, by considering the effects of equatorially trapped oceanic waves propagating in a closed basin through a delay term. The conceptual delayed oscillator model is represented by a single ordinary differential delay equation with both positive and negative feedbacks. The positive feedback is represented by the local ocean–atmosphere coupling in the equatorial eastern Pacific. The delayed negative feedback is represented by free Rossby waves generated in the eastern Pacific coupling region, which propagate to and reflect from the western boundary, returning as Kelvin waves to reverse the anomalies in the eastern Pacific coupling region.

The ENSO is a natural phenomenon involved in the tropical Pacific ocean-atmosphere interactions. The phenomenon of ENSO is a very attractive subject of study in the international academic circles.[4−13] Mo et al[14−27] considered also a kind of nonlinear problems involved in atmospheric physics, ocean weather, dynamic system and so on.

2. A delayed model for the ENSO

Recently, many scholars have investigated the approximation theory on nonlinear problems.[28−33] However, the existence of periodic solutions of the corresponding problems has been studied little. We now consider the following nonlinear time-delay model of equatorial Pacific:[13]

$$\frac{\mathrm{d}T}{\mathrm{d}t} = a\tau_1 - b_1\tau_1(t-\eta) - \epsilon T^3, \tag{1}$$

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}t} = dT - R_{\tau_1}\tau_1, \tag{2}$$

where T is the Niño-3 sea surface temperature (SST) anomaly; τ_1 is the Niño-4 zonal wind stress anomaly; η is the time taken by waves to travel to the western boundary and return to the eastern Pacific; a is the coefficient representing the positive feedback between the Niño-3 SST anomalies; b_1 is the coefficient representing the negative feedback due to wave reflection at the western boundary; d is the positive coefficient that relates the Niño-3 SST anomalies to the Niño-4 zonal wind stress anomalies; R_{τ_1} is the damping coefficient; ϵ is the cubic damping coefficient. Using the perturbation theory and corresponding method, the asymptotic expansion of the solution for the sea–air oscillator model is obtained.[18]

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In this paper, we continue to study the nonlinear time-delay model of equatorial Pacific i.e. Eqs.(1) and (2). By using the Mawhin's continuation theorem, the result on the existence of periodic solutions for the sea–air oscillator model is obtained. Throughout this paper we assume that a, b_1, ϵ, d, and R_{τ_1} are positive constants.

3. The periodic solutions of SST equation

Theorem 3.1 If $a - b_1 < \epsilon R_{\tau_1}^3/d^3$ and $6R_{\tau_1}l + 12R_{\tau_1}^2 l^2 < 1$, then the system of Eqs. (1) and (2) has at least one $2l-$ periodic solution.

Proof In order to use Mawhin's continuation theorem to investigate the existence of $2l$-periodic solutions for the system of Eqs.(1) and (2), we should consider the following equation:

$$\begin{aligned}\tau_1''(t) = & -R_{\tau_1}\tau_1'(t) + ad\tau_1(t) - b_1 d\tau_1(t-\eta) \\ & - \frac{\epsilon}{d^2}(\tau_1'(t) + R_{\tau_1}\tau_1(t))^3. \end{aligned} \tag{3}$$

Clearly, if $\tau_1(t)$ is a $2l-$periodic solution of Eq.(3), then $((\tau_1'(t) + R_{\tau_1}\tau_1(t))/d, \tau_1(t))^{\mathrm{T}}$ must be a $2l-$periodic solution of the system of Eqs.(1) and (2).

Obviously, 0 is a trivial periodic solution of Eq.(3). Without loss of generality, we may assume that $\tau_1(t)$ is a nontrivial solution of Eq.(3).

Now, let

$$L : D(L) \subset X \longrightarrow Y, L\tau_1 = \tau_1''$$

and $N : X \longrightarrow Y$,

$$\begin{aligned}[N\tau_1](t) = & -R_{\tau_1}\tau_1'(t) + ad\tau_1(t) - b_1 d\tau_1(t-\eta) \\ & - \frac{\epsilon}{d^2}(\tau_1'(t) + R_{\tau_1}\tau_1(t))^3, \end{aligned} \tag{4}$$

where $Y = C_{2l} = \{x | x \in C(\boldsymbol{R}, \boldsymbol{R}) : x(t+2l) \equiv x(t)\}$ with the norm $|\varphi|_0 = \max\limits_{t\in[0,2l]} |\varphi(t)|, \forall \varphi \in C_{2l}$, $X = C_{2l}^1 = \{x | x \in C^1(\boldsymbol{R}, \boldsymbol{R}) : x(t+2l) \equiv x(t)\}$ with the norm $\|\varphi\| = \max\{|\varphi|_0, |\varphi'|_0\}, \forall \varphi \in C_{2l}^1$, and $D(L) = \{\tau_1 \in C_{2l} : \tau_1'' \in C(\boldsymbol{R}, \boldsymbol{R})\}$. It is easy to see that Eq.(3) can be converted to an abstract equation $L\tau_1 = N\tau_1$. Moreover, from the definition of L, we see that $\ker L = \boldsymbol{R}$, $\mathrm{Im}L = \{\tau_1 : \tau_1 \in Y, \int_0^{2l} \tau_1(t)\mathrm{d}t = 0\}$. So L is a Fredholm operator with index 0. Also, let projectors $P : X \longrightarrow \ker L$ and $Q : Y \longrightarrow \mathrm{Im}Q$ be defined as

$$P\tau_1 = \tau_1(0)\ ;\ Q\tau_1 = \frac{1}{2l}\int_0^{2l} \tau_1(t)\mathrm{d}t,$$

and let K represent the inverse of $L|_{\ker P \bigcap D(L)}$, then obviously, $\ker L = \mathrm{Im}\, P = \boldsymbol{R}$, $\ker Q = \mathrm{Im}\, L$; and $\forall y \in \mathrm{Im}L$,

$$\begin{aligned}[Ky](t) = & -\frac{t}{2l}\int_0^{2l} (2l-s)y(s)\mathrm{d}s \\ & + \int_0^t (t-s)y(s)\mathrm{d}s. \end{aligned} \tag{5}$$

From Eqs.(4) and (5), one can easily see that N is $L-$compact on $\overline{\Omega}$, where Ω is an arbitrary open, bounded subset of X. Meanwhile, take $\Omega_1 = \{\tau_1 : \tau_1 \in D(L) \subset X, L\tau_1 = \lambda N\tau_1, \lambda \in (0,1)\}$, $\forall \tau_1 \in \Omega_1$, then τ_1 must satisfy

$$\begin{aligned}\tau_1''(t) = & -\lambda R_{\tau_1}\tau_1'(t) + \lambda ad\tau_1(t) - \lambda b_1 d\tau_1(t-\eta) \\ & - \lambda\frac{\epsilon}{d^2}(\tau_1'(t) + R_{\tau_1}\tau_1(t))^3. \end{aligned} \tag{6}$$

Set

$$\tau_1(t_{\max}) = \max_{t\in\boldsymbol{R}} \tau_1(t),$$

and

$$\tau_1(t_{\min}) = \min_{t\in\boldsymbol{R}} \tau_1(t),$$

where $t_{\max}, t_{\min} \in \boldsymbol{R}$, then we will have

$$\begin{aligned} &\tau_1'(t_{\max}) = 0, \quad \tau_1''(t_{\max}) \le 0, \\ &\tau_1'(t_{\min}) = 0, \quad \tau_1''(t_{\min}) \ge 0, \end{aligned} \tag{7}$$

which, together with Eq.(6) and the intermediate value theorem, reveals that there exists a constant $t_1 \in \boldsymbol{R}$ such that

$$ad\tau_1(t_1) - \frac{\epsilon R_{\tau_1}^3}{d^2}\tau_1^3(t_1) - b_1 d\tau_1(t_1-\eta) = 0. \tag{8}$$

We now prove that if $\tau_1(t)$ is a $2l-$periodic solution of Eq.(6), then there exists a constant $t_2 \in \boldsymbol{R}$ such that

$$|\tau_1(t_2)| \le \max\left\{\sqrt{\frac{ad^3}{3\epsilon R_{\tau_1}^3}}, 1\right\} := \bar{d}. \tag{9}$$

Now we adopt a contradictory method. Assume that inequality (9) does not hold, then we will have

$$|\tau_1(t)| > \bar{d}, \quad \text{for all} \quad t \in \boldsymbol{R},$$

which, together with Eq.(8), implies that one of the following relations holds:

$$\tau_1(t_1) > \tau_1(t_1-\eta) > \bar{d}; \tag{10}$$

$$\tau_1(t_1-\eta) > \tau_1(t_1) > \bar{d}; \tag{11}$$

$$\tau_1(t_1) < \tau_1(t_1-\eta) < -\bar{d}; \tag{12}$$

$$\tau_1(t_1-\eta) < \tau_1(t_1) < -\bar{d}. \tag{13}$$

Suppose that inequality (10) holds, then we will obtain

$$\begin{aligned} & ad\tau_1(t_1) - \frac{\epsilon R_{\tau_1}^3}{d^2}\tau_1^3(t_1) - b_1 d\tau_1(t_1-\eta) \\ < & ad\tau_1(t_1-\eta) - \frac{\epsilon R_{\tau_1}^3}{d^2}\tau_1^3(t_1-\eta) \\ & - b_1 d\tau_1(t_1-\eta) < 0, \end{aligned}$$

which is in contradiction with Eq.(8). This contradiction means that inequality (9) holds true.

Suppose that inequality (11) (or (12), or (13)) holds, by using a similar method, we can also show that inequality (9) is correct.

Let $t_2 = 2ml + t_0$, where $t_0 \in [0, 2l]$ and m is an integer, then $|\tau_1(t_0)| \le \bar{d}$. Hence we have

$$|\tau_1|_0 \le \bar{d} + \int_0^{2l} |\tau_1'(t)|\mathrm{d}t. \tag{14}$$

Multiplying the two sides of Eq.(6) by $\tau_1'(t)$ and integrating them over $[0, 2l]$, we have

$$\begin{aligned} 0 = & R_{\tau_1}\int_0^{2l} |\tau_1'(t)|^2\mathrm{d}t + b_1 d\int_0^{2l} \tau_1(t-\eta)\tau_1'(t)\mathrm{d}t \\ & + \frac{\epsilon}{d^2}\int_0^{2l} |\tau_1'(t)|^4\mathrm{d}t + \frac{3\epsilon R_{\tau_1}}{d^2}\int_0^{2l} \tau_1(t)(\tau_1'(t))^3\mathrm{d}t \\ & + \frac{3\epsilon R_{\tau_1}^2}{d^2}\int_0^{2l} |\tau_1'(t)|^2|\tau_1(t)|^2\mathrm{d}t, \end{aligned}$$

which, together with inequality (14) and Hölder inequality, yields

$$\begin{aligned} & \Big(\frac{\epsilon}{d^2} - \frac{6\epsilon R_{\tau_1} l}{d^2} - \frac{12\epsilon R_{\tau_1}^2 l^2}{d^2}\Big)\int_0^{2l} |\tau_1'(t)|^4\mathrm{d}t \\ & \le [R_{\tau_1}(2l)^{1/2} + b_1 d(2l)^{3/2}]\Big(\int_0^{2l} |\tau_1'(t)|^4\mathrm{d}t\Big)^{1/2} \\ & \quad + \frac{3\epsilon R_{\tau_1}^2 \bar{d}^2(2l)^{1/2}}{d^2}\Big(\int_0^{2l} |\tau_1'(t)|^4\mathrm{d}t\Big)^{1/2} \\ & \quad + b_1 d\bar{d}(2l)^{3/4}\Big(\int_0^{2l} |\tau_1'(t)|^4\mathrm{d}t\Big)^{1/4} \\ & \quad + \Big[\frac{3\epsilon R_{\tau_1}\bar{d}(2l)^{1/4}}{d^2} + \frac{6\epsilon R_{\tau_1}^2\bar{d}(2l)^{3/4}}{d^2}\Big] \\ & \quad \times \Big(\int_0^{2l} |\tau_1'(t)|^4\mathrm{d}t\Big)^{3/4}. \end{aligned}$$

As $6R_{\tau_1}l + 12R_{\tau_1}^2 l^2 < 1$, there exists a constant $M > 0$ such that

$$\int_0^{2l} |\tau_1'(t)|^4\mathrm{d}t < M. \tag{15}$$

From inequalities (14) and (15), we can obtain two constants M_0 and M_1 independent of λ and $\tau_1(t)$ such that

$$|\tau_1|_0 < M_0 \text{ , and } |\tau_1'|_0 < M_1.$$

Let $\widetilde{M} = \max\{M_0, M_1\}$, $\Omega = \{\tau_1 : \|\tau_1\| \le \widetilde{M}\}$ and $\Omega_2 = \{\tau_1 \in \partial\Omega : \tau_1 \in \ker L\}$, then $\forall \tau_1 \in \Omega_2$,

$$QN\tau_1 = \frac{1}{2l}\int_0^{2l}\Big[ad\tau_1 - b_1 d\tau_1 - \frac{\epsilon R_{\tau_1}^3}{d^2}\tau_1^3\Big]\mathrm{d}t \ne 0.$$

On the other hand, let $J : \mathrm{Im}Q \longrightarrow \ker L$ as follows:

$$H(\tau_1, \mu) = -\mu\tau_1 + (1-\mu)JQN\tau_1,\ (\tau_1, \mu) \in \Omega \times [0,1],$$

then we will have $\forall(\tau_1, \mu) \in (\partial\Omega \bigcap \ker L) \times [0,1]$,

$$\begin{aligned} H(\tau_1, \mu) = & \frac{1-\mu}{2l}\int_0^{2l}\Big[ad\tau_1 - b_1 d\tau_1 - \frac{\epsilon R_{\tau_1}^3}{d^2}\tau_1^3\Big]\mathrm{d}t \\ & - \mu\tau_1 \ne 0. \end{aligned}$$

Hence,

$$\begin{aligned} & \deg\{JQN, \Omega \cap \ker L, 0\} \\ = & \deg\{H(\tau_1, 0), \Omega \cap \ker L, 0\} \\ = & \deg\{H(\tau_1, 1), \Omega \cap \ker L, 0\} \\ = & \deg\{-I, \Omega \cap \ker L, 0\} \ne 0. \end{aligned}$$

Now we can conclude that[34] equation $L\tau_1 = N\tau_1$ has at least one $2l-$periodic solution $\tau_1(t)$, i.e. the system of Eqs.(1) and (2) has at least one $2l-$periodic solution

$$(T(t), \tau_1(t))^{\mathrm{T}} = \Big(\frac{\tau_1'(t) + R_{\tau_1}\tau_1(t)}{d}, \tau_1(t)\Big)^{\mathrm{T}}.$$

4. Conclusions

The ENSO is a complicated natural phenomenon, and it describes an interannual climate variability. Hence we need to reduce basic models for the sea–air oscillator and investigate the periodic behaviour of the temperature anomaly and the zonal wind stress anomaly in the equatorial eastern Pacific. Qualitative method is a simple and valid method. Furthermore, qualitative analysis does not require the expression of solution, which is different from asymptotic methods and general numerical methods.

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