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Evolution of soliton-like train in Klein–Gordon lattice system*

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This paper studies the evolution of wave in the system of a pure anharmonic lattice with a double well on-site potential by numerical calculation. It finds that an initial distribution of static or moving wave can evolve into two travelling soliton-like trains with contrary directions and a region of oscillation in this lattice system. It presents that some cases with cosine-square-shape and Gaussian-shape initial distribution of static or moving wave will produce ordered soliton-like train. Careful numerical observation shows that the centre oscillation region in this system may act as a resource of generating soliton-like train.

Keywords: Klein–Gordon lattice system, soliton-like train, numerical calculation

PACC: 6320P, 6320R

1. Introduction

In recent years, the study of soliton train has received a great deal of attention in optical fibres,[1,2] shallow water wave,[3] photonic lattices,[4,5] Bose–Einstein condensates,[6−13] magnetic films,[14] and so on.[15] Many models (including nonlinear Schrödinger equation,[1,2,4,15] the Gross–Pitaevskii equation[6−13] and shallow water wave equation[3]) and many approaches (such as numerical simulations,[1−3,8,9,11−13,15] experimental observation[1,2,5,6,14] and analytical analysis[3−5,7,9−11]) have been emphasized in connection with the formation and propagation of soliton trains.

Williams *et al* have generated up to 60 GHz quasi-continuous trains of dark soliton-like pulses by using the co-propagation of and interference between two bright pulses from a Ti: sapphire laser in 1.9 km of single mode optical fibre at 846 nm.[1] Atieh *et al*[2] have generated very stable train of bright and dark soliton pulses at 1553 nm at repetition rates close to 50 GHz by using comb-like dispersion profile fibre structures with a total length of approximately 6.6 km. Kamchatnov *et al*[3] have asymptotically and numerically investigated the soliton trains of Kaup–Boussinesq shallow water wave equations. Bronski and Kutz[15] have numerically simulated the semi-classical limit of the focusing nonlinear Schrödinger equation and observed the emergence of an ordered train of solitons. Kartashov *et al*[4] have addressed the formation and propagation of multi-spot soliton trains in saturable Kerr nonlinear media with an imprinted harmonic transverse modulation of the refractive index (photonic lattices), and discussed the potential applications of multicolour solitons (soliton train) supported by periodic lattices imprinted in quadratic nonlinear media. Chen *et al*[5] have demonstrated both experimentally and theoretically the formation of discrete soliton trains in two-dimensional optically-induced photonic lattices. Their results may pave the way towards the observation of similar phenomena in other relevant discrete nonlinear systems. Recently, Strecker *et al*[6] have reported the formation of bright solitons of ^{7}Li atoms created in a quasi-1D optical trap. The solitons are created from a stable Bose–Einstein condensate by magnetically tuning the interactions from repulsive to attractive. And the soliton train consisted of up to 10 solitons has been observed. Later, many theoretical works studied the dynamics of soliton train in Bose–Einstein condensate by numerical simulation and analytical analysis.[7−13] Wu *et al*[14] have observed for the first time the generation of dark and bright spin wave envelope soliton

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train from high dispersion dipole-exchange spin waves which propagated in a thin yttrium iron garnet film with pinned surface spins at frequencies situated near the dipole gaps in the dipole-exchange spin wave spectrum. However, to the best of our knowledge, the soliton train in pure anharmonic lattice systems has not been reported previously.

In this paper, we consider the lattice model of Ref.[16], where a system of atoms with unit mass is coupled anharmonically to their nearest-neighbours and interact with a double well on-site potential. This lattice model is different from the models of Refs.[17–22]. We numerically studied the dynamics of wave evolution in this system. An initial distribution of cosine-square-shape or Gaussian-shape type static or moving wave is chosen as an initial condition of the system to study the evolution and propagation of wave in this system. We found that an initial distribution of static or moving wave can evolve into two travelling orderly wave trains with contrary directions and a region of oscillation in this lattice system. The travelling orderly wave trains are something like the soliton train appeared in Refs.[1–15], and we call them soliton-like train in this paper. The paper is organized as follows. We present the lattice model and give the initial distribution in Section 2. Then, in Section 3, we numerically study the evolution and propagation of soliton-like train in this system. Discussions and conclusions are made in Section 4.

2. Model and initial distribution condition

We consider the nonlinear Klein–Gordon lattice system, its Hamiltonian is given by Ref.[16]

$$H = \sum_n \left[\frac{1}{2} \left(\frac{\mathrm{d}Q_n}{\mathrm{d}t} \right)^2 + \frac{1}{4} C_{nl} (Q_n - Q_{n-1})^4 + \frac{1}{2} V_0 (1 - Q_n^2)^2 \right], \tag{1}$$

where Q_n is the longitudinal displacement of the nth particle from its equilibrium position, C_{nl} and V_0 are parameters which control the strength of the nonlinear coupling between the nearest-neighbour lattice site and the potential barrier height of the double well on-site potential, respectively. And $(1 - Q_n^2)^2$ is the symmetric double well on-site potential of the system. The equation of motion for the system is

$$\frac{\mathrm{d}^2 Q_n}{\mathrm{d}t^2} = C_{nl} \left[(Q_{n+1} - Q_n)^3 + (Q_{n-1} - Q_n)^3 \right] + 2V_0 (Q_n - Q_n^3). \tag{2}$$

When some strict conditions are satisfied, Eq.(2) has a moving pulse or kink solution with compact support.[16]

The cosine-square-shape initial distribution wave used in this paper is

$$Q_n(t) = \cos^2[q(na - n_0 a \pm vt)] \quad \text{for} \quad |q(na - n_0 a \pm vt)| < \pi/2, \tag{3a}$$

$$Q_n = 0 \quad \text{otherwise}, \tag{3b}$$

and the Gaussian-shape initial distribution wave is

$$Q_n(t) = \exp\left[-q(na - n_0 a \pm vt)^2 \right], \tag{4}$$

where q is a positive parameter ($0 < q < \pi/2$) indicating the localization of the initial distribution, a is the lattice spacing and is assumed to be unity in what follows, n_0 is the centre site of the initial distribution, sign '+' and '−' correspond to left-travelling and right-travelling initial distribution wave respectively, and v is the propagation velocity of initial distribution wave.

3. Numerical studies

In order to investigate the evolution of soliton-like train in the lattice system, we have numerically integrated Eq.(2) by means of fourth-order Runge–Kutta scheme. The initial distribution waves [Eqs.(3) and (4)] stated above and their first derivatives with respect to time t at $t = 0$ are chosen as initial conditions of the system to study the dynamic behaviour of this lattice system. In this paper we take the parameters of the potential barrier height $V_0 = 2$, the nonlinear coupling strength $C_{nl} = 50$, the localized parameter $q = \pi/8$ with lattice chain $n = 1000$.

Firstly we consider the case $v = 0$. At the initial time $t = 0$, the cosine-square-shape initial distribution wave is centred on the centre of the chain, i.e. $n_0 = 500$. We systemically study the evolution of soliton-like train by changing the value of evolution time t step by step. Our numerical results are shown in Fig.1 for the time $t = 3, t = 6, t = 8, t = 10, t = 20$ and $t = 30$. The cosine-square-shape initial distribution turns to two symmetrical and orderly moving soliton-like trains with opposite velocity except that the centre of the chain is a region of oscillation. The amplitude of all the solitons is positive, and the number of the solitons increases as time goes on.

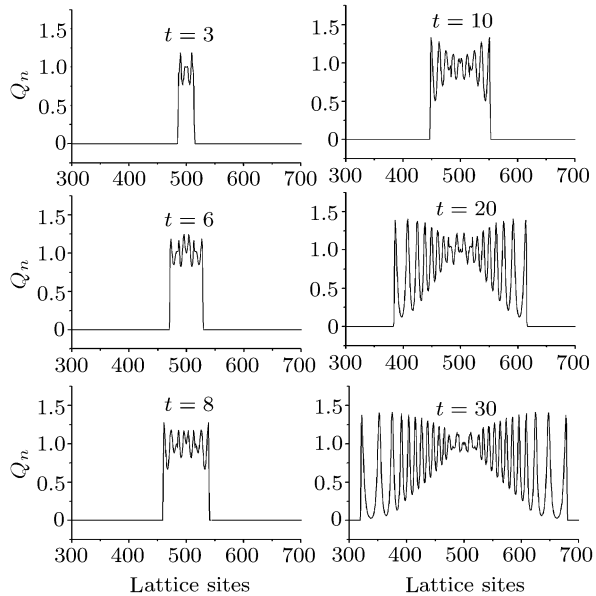


Fig.1. Plot showing the evolution of soliton-like train in an anharmonic lattice chain with parameter $q = \pi/8$, $v = 0$ at $t = 3$, $t = 6$, $t = 8$, $t = 10$, $t = 20$ and $t = 30$.

Next, we display the case $v \neq 0$. We study the evolution of soliton-like train by changing the integer value of velocity ν and the value of evolution time t stage by stage. The former two symmetrical soliton-like trains become somewhat asymmetrical. The amplitudes of one soliton-like train turns to negative when the value of initial velocity $v \geq 4$. And the centre of the chain is also a region of oscillation. The process of evolution of the soliton-like trains for right and left travelling cosine-square- shape initial distribution wave is shown in Fig.2 and Fig.3 respectively.

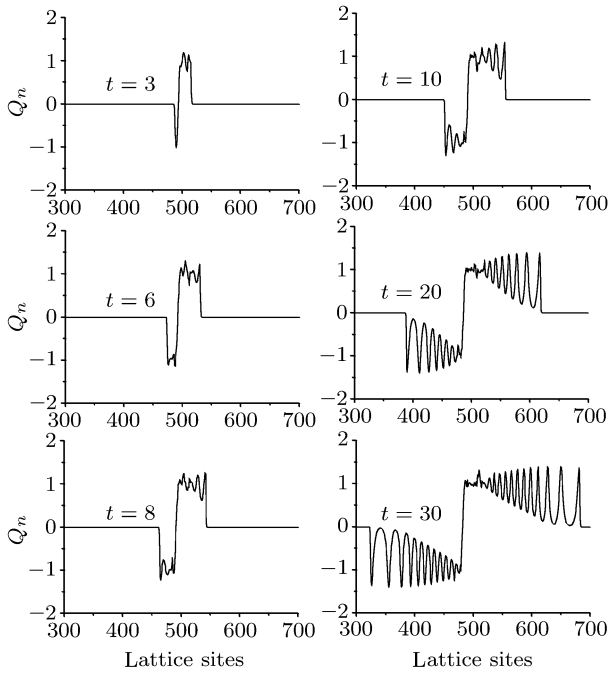


Fig.2. Plot showing the evolution of soliton-like train in an anharmonic lattice chain with parameter $q = \pi/8$, $v = 4$ at $t = 3$, $t = 6$, $t = 8$, $t = 10$, $t = 20$ and $t = 30$.

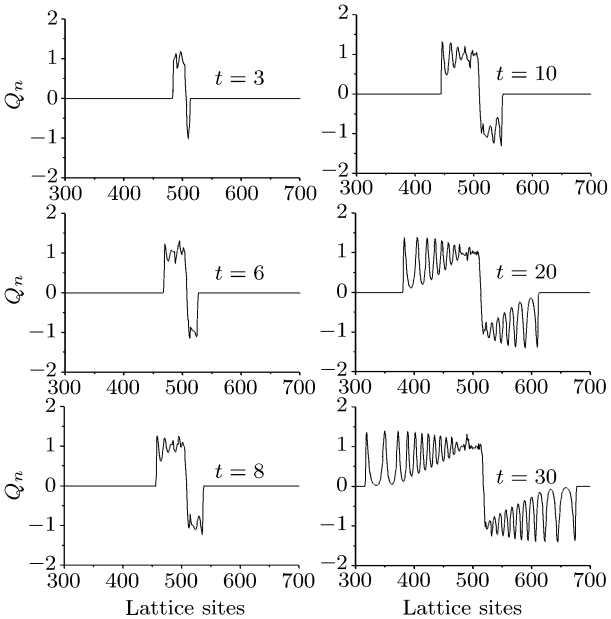


Fig.3. Plot showing the evolution of soliton-like train in an anharmonic lattice chain with parameter $q = \pi/8$, $v = -4$ at $t = 3$, $t = 6$, $t = 8$, $t = 10$, $t = 20$ and $t = 30$.

If we take the Gaussian-shape distribution wave as an initial condition to study the dynamic behaviour of this lattice system, we can obtain the similar results. And we omit the corresponding plots of the process of evolution of the soliton-like trains in this case for saving space.

4. Discussions and conclusions

In the above numerical experiments, we have observed the phenomenon that the centre of the initial disturbance wave is a region of oscillation and the number of the soliton-like in the soliton-like train increases as time goes on. By careful numerical experiments, we found that there may be some relationship between the two phenomena. We displayed the process of evolution of the soliton-like trains for the case $v = 0$ at $t = 21$ to $t = 26$ in Fig.4 (cosine-square-shape initial distribution wave) and Fig.5 (Gaussian-shape distribution wave). Careful observation shows that the centre oscillation region is a static breather. A similar situation can be observed in the case $\nu \neq 0$, but we omit it to save space. The period of the evolution of the breather is less than 3 s. After a period of evolution, the breather generates a pair of soliton-like which travel in contrary directions.

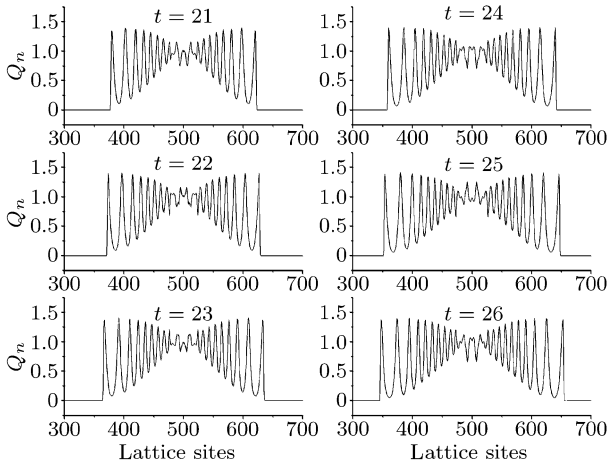


Fig.4. Plot showing the process of the centre oscillation breather generating a soliton train. The initial distribution wave is cosine-squared-shape with parameters $v = 0$ and $q = \pi/8$.

So the number of the soliton-likes in the soliton-like train increases as time goes on. The centre of the oscillation region in this system may act as a resource of generating the soliton-like train. We also found that different initial conditions (different value of velocity v) lead to different evolution instance of soliton-like train in this lattice system. This may be owing to the fact that the lattice system has different energy (Hamiltonian, see Eq.(1) at $t = 0$ when we take different value of the velocity v.

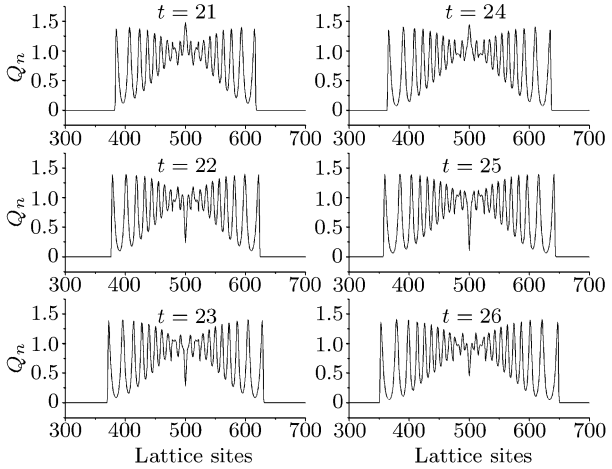


Fig.5. Plot showing the process of the centre oscillation breather generating soliton train. The initial distribution wave is Gaussian-shape with parameters $v = 0$ and $q = \pi/8$.

In summary, we have studied the evolution of soliton-like train in the system of a pure anharmonic lattice with a double well on-site potential by numerical experiments. It is found that the cosine-squared-shape initial distribution wave or the Gaussian-shape distribution wave evolves into two symmetrical and orderly moving soliton-like trains with contrary direction and the amplitude of all the soliton-likes is positive except that the centre of the chain is a region of oscillation when the initial velocity $v = 0$. When $v \neq 0$, the initial distribution wave turns to two asymmetrical and orderly moving soliton-like trains with contrary directions, the amplitude of one soliton-like train is positive and the other is negative, the centre of the chain is also a region of oscillation. Careful observation shows that the centre of the oscillation region in this system may act as a resource of generating soliton-like train.

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