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Vectorial Hermite–Laguerre–Gaussian beams beyond the paraxial approximation*

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Starting from the vectorial Rayleigh–Sommerfeld integrals, the free-space propagation expressions for vectorial Hermite–Laguerre–Gaussian (HLG) beams beyond the paraxial approximation are derived. The far-field expressions and the scalar paraxial results are given as special cases of our general expressions. The intensity distributions of vectorial nonparaxial HLG beams are studied and illustrated with numerical examples.

Keywords: vectorial nonparaxial Hermite–Laguerre–Gaussian beam, vectorial Rayleigh–Sommerfeld integrals, propagation

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1. Introduction

The characterization of general laser beams is a topic that has been of considerable theoretical and practical interest for a long time. Recently, Abramochkin and Volostnikov[1] introduced a kind of generalized beams, which are the more general family of Hermite–Gaussian (HG) and Laguerre–Gaussian (LG) modes, named Hermite–Laguerre–Gaussian (HLG) beams. By changing a parameter α in HLG beams one can transform HLG beams from HG beams to LG beams in a continuous way. The experimental verification of HLG was also given.[1,2] However, the study is limited within the scalar paraxial framework. The purpose of this paper is to extend the concept of HLG beams to the vectorial nonparaxial regime. In Section 2, the vectorial propagation expressions of HLG beams beyond the paraxial approximation are derived. Numerical comparisons between the nonparaxial and paraxial results are given in Section 3. Finally, Section 4 summarizes the main results obtained in this paper.

2. Propagation of vectorial Hermite–Laguerre–Gaussian beams beyond the paraxial approximation

We start from the vectorial Rayleigh–Sommerfeld integrals,[3,4] which deliver the exact propagation of the optical field in the half space ($z > 0$) once the boundary condition at the initial plane ($z = 0$) is known, i.e.,

$$E_x(\boldsymbol{r}) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x(\boldsymbol{r}_0,0)\frac{\partial G(\boldsymbol{r},\boldsymbol{r}_0)}{\partial z}\mathrm{d}x_0\mathrm{d}y_0, \tag{1a}$$

$$E_y(\boldsymbol{r}) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y(\boldsymbol{r}_0,0)\frac{\partial G(\boldsymbol{r},\boldsymbol{r}_0)}{\partial z}\mathrm{d}x_0\mathrm{d}y_0, \tag{1b}$$

$$E_z(\boldsymbol{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[E_x(\boldsymbol{r}_0,0)\frac{\partial G(\boldsymbol{r},\boldsymbol{r}_0)}{\partial x} + E_y(\boldsymbol{r}_0)\frac{\partial G(\boldsymbol{r},\boldsymbol{r}_0)}{\partial y}\right]\mathrm{d}x_0\mathrm{d}y_0, \tag{1c}$$

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where

$$G(\boldsymbol{r},\boldsymbol{r}_0)=\frac{\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}_0|)}{|\boldsymbol{r}-\boldsymbol{r}_0|}, \tag{2}$$

$\boldsymbol{r}_0=x_0\boldsymbol{i}+y_0\boldsymbol{j}$ and $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ being the unit vectors in the x, y and z directions respectively. In dealing with the beam propagation beyond the paraxial approximation, it is feasible to make use of the following approximation:[5−7]

$$|\boldsymbol{r}-\boldsymbol{r}_0|\approx r+\frac{x_0^2+y_0^2-2xx_0-2yy_0}{2r}. \tag{3}$$

Thus, Eqs.(1) can be approximated as:

$$E_x(\boldsymbol{r})=-\frac{\mathrm{i}z}{\lambda r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_x(\boldsymbol{r}_0,0)\exp\left(\mathrm{i}k\frac{x_0^2+y_0^2-2xx_0-2yy_0}{2r}\right)\mathrm{d}x_0\mathrm{d}y_0, \tag{4a}$$

$$E_y(\boldsymbol{r})=-\frac{\mathrm{i}z}{\lambda r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_y(\boldsymbol{r}_0,0)\exp\left(\mathrm{i}k\frac{x_0^2+y_0^2-2xx_0-2yy_0}{2r}\right)\mathrm{d}x_0\mathrm{d}y_0, \tag{4b}$$

$$E_z(\boldsymbol{r})=\frac{\mathrm{i}}{\lambda r}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[E_x(\boldsymbol{r}_0,0)(x-x_0)+E_y(\boldsymbol{r}_0,0)(y-y_0)]\exp\left(\mathrm{i}k\frac{x_0^2+y_0^2-2xx_0-2yy_0}{2r}\right)\mathrm{d}x_0\mathrm{d}y_0. \tag{4c}$$

Equations (4) make it easy to deal with the propagation of vectorial beams beyond the paraxial approximation once the boundary condition at the plane $z=0$ is given.

Assume that the field of HLG beams at plane $z=0$ is polarized in the x direction and takes the form[1]

$$\Psi_{n,mx}(x,y,0|\alpha)=\sum_{h=0}^{n+m}\mathrm{i}^h(\cos\alpha)^{n-h}(\sin\alpha)^{m-h}P_h^{(n-h,m-h)}(-\cos 2\alpha)H^{\mathrm{G}}_{n+m-h,h}(x,y,0), \tag{5a}$$

$$(n,m=0,1,\cdots),$$

$$\Psi_{n,m}(x,y,0|\alpha)=0, \tag{5b}$$

where $P_k^{(u,v)}(t)$ is the Jacobi polynomial, α is a parameter which allows us to make a continuous transition from HG to LG beams,

$$H^{\mathrm{G}}_{n+m-h,h}(x,y,0)=\exp\left(-\frac{x^2+y^2}{2w_0^2}\right)H_{n+m-h}\left(\frac{x}{w_0}\right)H_h\left(\frac{y}{w_0}\right). \tag{6}$$

$H_m(\bullet)$ denotes the Hermite polynomial, and w_0 is the waist width of the corresponding fundamental Gaussian beam. HG and LG beams are special cases of Eq.(5a). For example, when $\alpha=0,\pi/4$, we have:

$$\Psi_{n,m}(x,y,0|0)=(-\mathrm{i})^m\exp\left(-\frac{x^2+y^2}{2w_0^2}\right)H_n\left(\frac{x}{w_0}\right)H_m\left(\frac{y}{w_0}\right), \tag{7}$$

$$\Psi_{n,m}\left(x,y,0|\frac{\pi}{4}\right)=\begin{cases}(-1)^m2^nm!\exp\left(-\dfrac{x^2+y^2}{2w_0^2}\right)\left(\dfrac{x+\mathrm{i}y}{\sqrt{2}w_0}\right)^{n-m}L_m^{n-m}\left(\dfrac{x^2+y^2}{w_0^2}\right), & n\geq m,\\ (-1)^n2^mn!\exp\left(-\dfrac{x^2+y^2}{2w_0^2}\right)\left(\dfrac{x+\mathrm{i}y}{\sqrt{2}w_0}\right)^{m-n}L_n^{m-n}\left(\dfrac{x^2+y^2}{w_0^2}\right), & n\leq m,\end{cases} \tag{8}$$

where $L_m^n(\bullet)$ is the associated Laguerre polynomial. On substituting Eqs.(5) into Eqs.(4) and using the formula:

$$\int_{-\infty}^{\infty}\exp\left[-\frac{(\xi-a)^2}{b}\right]H_n(\xi)\mathrm{d}\xi=\sqrt{\pi b}(1-b)^{n/2}H_n\left(\frac{a}{\sqrt{1-b}}\right), \tag{9}$$

and after tedious mathematical manipulations, we obtain the free-space propagation expressions for vectorial

nonparaxial HLG beams:

$$E_x^{nm}(x,y,z)=\frac{z}{r(1+\mathrm{i}kf^2r)}\exp(\mathrm{i}kr)\sum_{h=0}^{n+m}\mathrm{i}^h(\cos\alpha)^{n-h}(\sin\alpha)^{m-h}P_h^{(n-h,m-h)}(-\cos 2\alpha)Q_x^{n+m-h}Q_y^h, \tag{10a}$$

$$E_y^{nm}(x,y,z)=0, \tag{10b}$$

$$E_z^{nm}(x,y,z)=\frac{\mathrm{i}B}{2k^2f^3r^2}\exp(\mathrm{i}kr)\sum_{h=0}^{n+m}\mathrm{i}^h(\cos\alpha)^{n-h}(\sin\alpha)^{m-h}P_h^{(n-h,m-h)}(-\cos 2\alpha)Q_y^h \times\left[kfxQ_x^{n+m-h}-(n+m-h)Q_x^{n+m-h-1}-\frac{1}{2}Q_x^{n+m-h+1}\right], \tag{10c}$$

where

$$Q_\xi^u=(1-B)^{u/2}H_u\left(\frac{A\xi}{\sqrt{1-B}}\right)\exp\left(\frac{A^2\xi^2}{B}\right), \tag{11}$$

and $f=1/kw_0, A=kf/(\mathrm{i}df^2r+1), B=2\mathrm{i}rfA$.

Equations (10) are the main analytical results obtained in this paper, which enable us to evaluate the propagation of HLG beams beyond the paraxial approximation. From Eqs.(10) it can be seen that, because of the existence of the longitudinal component of the field, the intensity distribution of vectorial nonparaxial HLG beams is not symmetric about the x and y directions even when $m=n$. Additionally, vectorial nonparaxial HLG beams are structurally stable gainst propagation because the sum of indices of each HG component in Eqs.(10) is the same. In the following we present some special cases of Eqs.(10).

In the far field Eq.(3) is further approximated as[8]

$$|\boldsymbol{r}-\boldsymbol{r}_0|\approx r-\frac{xx_0+yy_0}{r}. \tag{12}$$

A similar procedure to the above leads to the far-field expressions for vectorial nonparaxial HLG beams:

$$E_{xf}^{nm}(x,y,z)=-(-1)^{(m+n)/2}\frac{\mathrm{i}z}{kf^2r^2}\exp(\mathrm{i}kr)\exp\left(-\frac{1}{2f^2}\frac{x^2+y^2}{r^2}\right)\sum_{h=0}^{n+m}\mathrm{i}^h(\cos\alpha)^{n-h}(\sin\alpha)^{m-h}P_h^{(n-h,m-h)} \times(-\cos 2\alpha)H_{n+m-h}\left(-\frac{x}{rf}\right)H_h\left(-\frac{y}{rf}\right), \tag{13a}$$

$$E_{yf}^{nm}(x,u,z)=0, \tag{13b}$$

$$E_{zf}^{nm}(x,y,z)=(-1)^{(m+n)/2}\frac{\mathrm{i}}{k^2f^3r^2}\exp(\mathrm{i}kr)\exp\left(-\frac{1}{2f^2}\frac{x^2+y^2}{r^2}\right)\sum_{h=0}^{n+m}\mathrm{i}^h(\cos\alpha)^{n-h}(\sin\alpha)^{m-h}P_h^{(n-h,m-h)} \times(-\cos 2\alpha)H_h\left(-\frac{y}{rf}\right)\left[kfxH_{m+n-h}\left(-\frac{x}{rf}\right)+\mathrm{i}(n+m-h)H_{n+m-h-1}\left(-\frac{x}{rf}\right) -\mathrm{i}\frac{1}{2}H_{m+n-h+1}\left(-\frac{x}{rf}\right)\right]. \tag{13c}$$

By letting $m=n=0$ in Eqs.(10), we have propagation expressions for vectorial nonparaxial Gaussian beams:

$$E_x(x,y,z)=\frac{1}{(1+\mathrm{i}kf^2r)}\frac{z}{r}\exp(\mathrm{i}kr)\exp\left[-\frac{\mathrm{i}k}{2(1+\mathrm{i}kf^2r)r}(x^2+y^2)\right], \tag{14a}$$

$$E_y(x,y,z)=0, \tag{14b}$$

$$E_z(x,y,z)=-\frac{\mathrm{i}kf^2x}{(1+\mathrm{i}kf^2r)^2}\exp(\mathrm{i}kr)\exp\left[-\frac{\mathrm{i}k}{2(1+\mathrm{i}kf^2r)r}(x^2+y^2)\right]. \tag{14c}$$

In the far field r (i.e., $r\gg kw_0^2/2$), Eqs.(14a) and (14c) reduce to

$$E_{xf}(x,y,z)=\frac{1}{\mathrm{i}kf^2}\frac{z}{r^2}\exp(\mathrm{i}kr)\exp\left(-\frac{1}{2f^2}\frac{x^2+y^2}{r^2}\right), \tag{15a}$$

$$E_{zf}(x,y,z)=-\frac{1}{\mathrm{i}kf^2}\frac{x}{r^2}\exp(\mathrm{i}kr)\exp\left(-\frac{1}{2f^2}\frac{x^2+y^2}{r^2}\right), \tag{15b}$$

respectively. Equation (15) are consistent with the results in Ref.[9] except for unimportant factors.

In the paraxial regime, we have

$$r \approx z + \frac{x^2+y^2}{2z}. \tag{16}$$

Equations (3) and (12) evolve into the corresponding Fresnel and Fraunhofer approximations[10] in the paraxial theory. Then, substituting Eq.(16) for r in 'exp(ikr)' of Eq.(10a) and z for other r in Eq.(10a) yields the propagation expression for scalar HLG beams within the paraxial approximation:

$$E_p^{nm}(x,y,z) = \frac{kw_0^2}{(kw_0^2+\mathrm{i}z)}\exp(\mathrm{i}kz)\exp\left(\mathrm{i}k\frac{x^2+y^2}{2z}\right)\sum_{h=0}^{n+m}\mathrm{i}^h(\cos\alpha)^{n-h}(\sin\alpha)^{m-h}P_h^{(n-h,m-h)} \times(-\cos 2\alpha)Q_{xp}^{n+m-h}Q_{yp}^h, \tag{17}$$

where

$$Q_{\xi p}^u = D^u H_u\left(\frac{kw_0}{D(kw_0^2+\mathrm{i}z)}\xi\right)\exp\left(-\frac{\mathrm{i}k^2w_0^2}{kw_0^2+\mathrm{i}z}\frac{\xi^2}{2z}\right), \tag{18}$$

$$D = \left(\frac{kw_0^2-\mathrm{i}z}{kw_0^2+\mathrm{i}z}\right)^{1/2}. \tag{19}$$

Similarly, from Eq.(13a) we obtain the far-field expression for scalar paraxial HLG beams:

$$E_{fp}^{nm}(x,y,z) = -(-1)^{(m+n)/2}\frac{\mathrm{i}kw_0^2}{z}\exp(\mathrm{i}kz)\exp\left(\mathrm{i}k\frac{x^2+y^2}{2z}\right)\exp\left(-\frac{k^2w_0^2}{2}\frac{x^2+y^2}{z^2}\right) \times\sum_{h=0}^{n+m}\mathrm{i}^h(\cos\alpha)^{n-h}(\sin\alpha)^{m-h}P_h^{(n-h,m-h)}(-\cos 2\alpha)H_{n+m-h}\left(-\frac{kw_0}{z}x\right)H_h\left(-\frac{kw_0}{z}y\right). \tag{20}$$

Equations (17) and (20) are the free-space propagation expressions for HLG beams corresponding to the Fresnel and Fraunhofer diffraction,[10] respectively.

3. Numerical calculations and discussions

The intensity distribution of vectorial HLG beams beyond the paraxial approximation can be evaluated by using Eqs.(10). Figure 1 shows the intensity distributions of $|E_x^{nm}|^2+|E_z^{nm}|^2$ (solid line) at the plane $z = 40z_0$ ($z_0 = kw_0^2/2$ is the Rayleigh length), and the comparison of the paraxial results of $|E_p^{nm}|^2$ obtained by using Eq.(17) (dashed line) and $|E_z^{nm}|^2$ by using Eq.(10c) only (dotted line) are plotted together. From Fig.1(a) it can be seen that, for paraxial HLG beams (i.e., f is very small,[11] e.g., $f = 0.01$), $|E_x^{nm}|^2+|E_z^{nm}|^2$ is in good agreement with $|E_p^{nm}|^2$; $|E_z^{nm}|^2$ is very small and is negligible. For nonparaxial HLG beams, from Fig.1(b) ($f = 0.4$) we see that there is a remarkable difference between $|E_x^{nm}|^2+|E_z^{nm}|^2$ and $|E_p^{nm}|^2$, and $|E_z^{nm}|^2$ becomes noticeable. Therefore, unlike the paraxial case, the propagation of HLG beams beyond the paraxial approximation should be treated in the vector framework.

Figure 2 displays the intensity patterns calculated by using Eqs.(10) for HLG beams of orders $n = 5$, $m = 3$, (a) $f = 0.01$ and (b) $f = 0.4$. The parameter α is chosen from 0 to $\pi/4$ with a step of $\pi/16$. From Fig.2(a) it can be seen that when α changes from 0 to $\pi/4$, the intensity pattern of the HLG beam in the paraxial regime changes from a Hermite pattern to a Laguerre pattern. While Fig.2(b) shows that the intensity patterns of nonparaxial HLG beams are notably different from those of paraxial HLG beams of the same orders, and are, as discussed above, no longer symmetric.

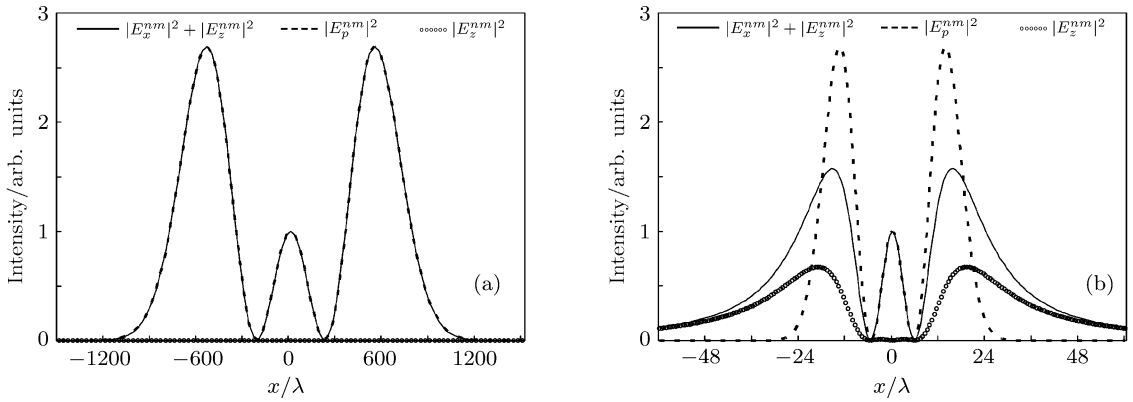


Fig.1. Intensity distributions $|E_x^{nm}|^2 + |E_z^{nm}|^2$ (solid line) of HLG beams. For comparison $|E_z^{nm}|^2$ (dotted line) and the paraxial results $|E_p^{nm}|^2$ (dashed line) are plotted together. (a) $f = 0.01$, (b) $f = 0.4$. The other calculation parameters are $m = 2$, $n = 2$, $\alpha = \pi/16, y = 0$, and $z = 40z_0$.

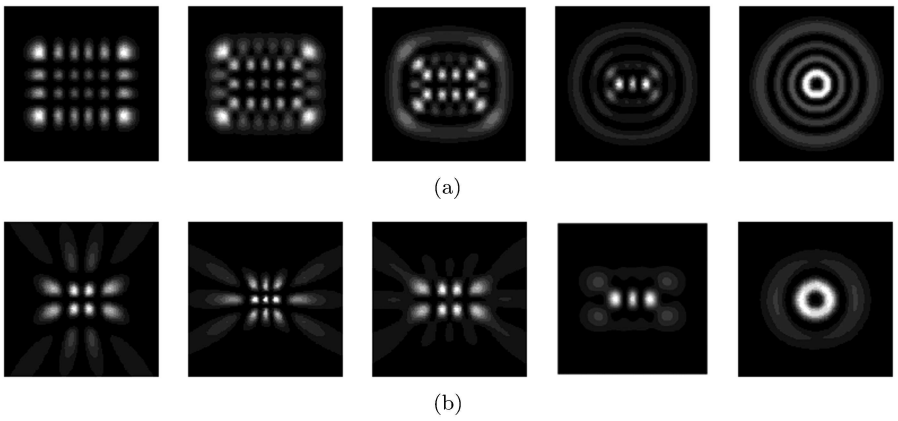


Fig.2. Intensity patterns calculated by using Eqs.(10) for HLG beams of orders $n = 5$, $m = 3$. (a) $f = 0.01$ and (b) $f = 0.4$.

4. Conclusion

In this study, the concept of HLG beams has been extended to the general vectorial nonparaxial regime. The free-space propagation expressions for vectorial nonparaxial HLG beams have been derived for the first time, so far as we know, and written in a closed form. The far-field expressions and the scalar paraxial results have been given and treated as special cases of our general expressions. Numerical calculations have shown that in the paraxial regime, the vectorial nonparaxial results are in good agreement with the scalar paraxial ones and the longitudinal components are negligible. However, if the paraxial approximation fails, the vectorial nonparaxial results are notably different from the scalar paraxial ones, and the longitudinal components can not be neglected in general. Therefore, the propagation of HLG beams beyond the paraxial approximation should be treated in the vector framework and with nonparaxial theory. Furthermore, we would like to point out that the propagation of vectorial nonparaxial beams in the apertured case is of interest.[4] The relevant problem for apertured HLG beams will be dealt with in a separate paper.

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