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Entropy as a measure of the noise extent in a two-level quantum feedback controlled system*

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By introducing the von Neumann entropy as a measure of the extent of noise, this paper discusses the entropy evolution in a two-level quantum feedback controlled system. The results show that the feedback control can induce the reduction of the degree of noise, and different control schemes exhibit different noise controlling ability, the extent of the reduction also related with the position of the target state on the Bloch sphere. It is shown that the evolution of entropy can provide a real time noise observation and a systematic guideline to make reasonable choice of control strategy.

Keywords: quantum entropy, quantum feedback
PACC: 4250

1. Introduction

Since the quantum theory was proposed at the beginning of last century, numerous works have been done to explore the newly founded field. Recently, with the development of quantum computation and quantum information, the control of the quantum mechanical system has attracted much attention; so far as we know, Hang and Tarn first discussed the problem of controllability of the quantum mechanical system.[1] With the advance of experimental technology, it is possible to observe the individual quantum system in real time,[2,3] and the possibility of controlling the quantum system in real time by using feedback has also been considered,[4] and much literature has analysed the real-time Markovian feedback and experimental implementation in certain optical systems.[5]

Feedback control is the process of controlling the evolution of a system by using information obtained from the system through measurement; its object is usually to limit the deviations of a system from a desired state.[6] A central problem of feedback control is designing a control strategy to achieve our goal in the presence of noise, which we must take into consideration in practice.[7−10] So it is necessary for us to introduce a physical entity to characterize the extent of noise emerging in the evolution of quantum system, and then the entity can provide us with a systematic guideline to the design of our feedback control. Liu *et al*[11] have studied a control scheme which controls the localization of state in a two-level system via homodyne detection Markovian feedback,[12] and have given the parameters of the feedback and driving field for control of the system state to localization at any desired state on the Bloch sphere.

In this paper, we will follow the feedback scheme in Ref.[11] to investigate the quantum feedback controlled two-level system by introducing the entity von Neumann entropy as a measure of the extent of noise emerged in this system,[13] and study the entropies with two different feedback control schemes. We have demonstrated that the feedback control can induce the reduction in the von Neumann entropy, or in other words, result in the noise reduction, and we also show that different choices of the feedback control strategy can exhibit some differences about the way of the noise reduction.

2. Quantum feedback master equation

Considering the evolution of state which can be represented by a matrix operator, we can describe the dynamical process by using the master equation. When the system is subjected to a weak continuous

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measurement, the master equation is[14,15]

$$\begin{aligned} \mathrm{d}\rho = & -\mathrm{i}[\hat{H},\rho]\mathrm{d}t + \mathcal{D}[\mathrm{e}^{-\mathrm{i}\phi}\hat{c}]\rho\mathrm{d}t \\ & + \mathcal{H}[\mathrm{e}^{-\mathrm{i}\phi}\hat{c}]\rho\mathrm{d}W(t), \end{aligned} \tag{1}$$

where $\hat{c}$ is a jump operator describing the interaction between the system and its environment, $\mathrm{d}W(t)$ is an infinitesimal Wiener increment[16] satisfying

$$\begin{aligned} E[\mathrm{d}W(t)] &= 0, \\ \mathrm{d}W(t)^2 &= \mathrm{d}t. \end{aligned} \tag{2}$$

The super-operators $\mathcal{D}$ and $\mathcal{H}$ are defined by

$$\begin{aligned} \mathcal{D}[\hat{r}]\rho &= \hat{r}\rho\hat{r}^\dagger - \frac{1}{2}\hat{r}^\dagger\hat{r}\rho - \frac{1}{2}\rho\hat{r}^\dagger\hat{r}, \\ \mathcal{H}[\hat{r}]\rho &= \hat{r}\rho + \rho\hat{r}^\dagger - \mathrm{Tr}(\hat{r}\rho + \rho\hat{r}^\dagger)\rho. \end{aligned} \tag{3}$$

As mentioned above, in feedback control the dynamics of a system is manipulated by using information obtained about the system through measurement; here we will focus on continuous feedback control, and take the Markovian feedback of the white-noise measurement record via a Hamiltonian. As explained in Ref.[17], the continuous measurement record can be described by the homodyne detection photocurrent:

$$I(t) = \langle \mathrm{e}^{-\mathrm{i}\phi}\hat{c} + \hat{c}^\dagger\mathrm{e}^{\mathrm{i}\phi}\rangle + \frac{\mathrm{d}W(t)}{\mathrm{d}t}\,. \tag{4}$$

Then the strength of the feedback can be written as $\hat{H}_{\mathrm{fb}}(t) = I(t)\hat{A}$, where $\hat{A}$ is a Hermitian operator. Adding this feedback to the master equation given by Eq.(1), we can get the feedback master equation[18]

$$\begin{aligned} \mathrm{d}\rho = & -\mathrm{i}\Big[\hat{H} + \frac{\mathrm{e}^{\mathrm{i}\phi}\hat{c}^\dagger\hat{A} + \mathrm{e}^{-\mathrm{i}\phi}\hat{A}\hat{c}}{2}, \rho\Big]\mathrm{d}t \\ & + \mathcal{D}[\mathrm{e}^{-\mathrm{i}\phi}\hat{c} - \mathrm{i}\hat{A}]\rho\mathrm{d}t \\ & + \mathcal{H}[\mathrm{e}^{-\mathrm{i}\phi}\hat{c} - \mathrm{i}\hat{A}]\rho\mathrm{d}W(t)\,. \end{aligned} \tag{5}$$

When we take the ensemble average of the master equation, the last term can be dropped. For the two-level quantum system, the $\hat{c}$ operator is

$$\hat{c} \equiv \frac{\sqrt{\gamma}(\sigma_x - \mathrm{i}\sigma_y)}{2},$$

and the system Hamiltonian can be chosen as

$$\hat{H}_1 = a_2\sigma_y + a_3\sigma_z,$$

or

$$\hat{H}_2 = a_3\sigma_x + a_2\sigma_z$$

as discussed in Ref.[11], where σ_x, σ_y and σ_z are the usual Pauli matrices and γ is the decoherence rate. Accordingly, we can consider the feedback via the Hamiltonian $\hat{H}_{\mathrm{fb1}}(t) = I(t)(m\sigma_y + n\sigma_z)$ or $\hat{H}_{\mathrm{fb2}}(t) = I(t)(n\sigma_x + m\sigma_z)$. For simplicity, it makes sense to set $\phi = 0$ in Eq.(5) by adjusting the parameters in the device.

3. Entropy evolution of the quantum feedback control two-level system

As for the two-level system, the density matrix of the state can be expressed as

$$\rho = \frac{1}{2}(I + x\sigma_x + y\sigma_y + z\sigma_z)\,. \tag{6}$$

The above expression is a three-dimensional vector $(x, y, z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ in the Bloch sphere, then we can get the eigenvalues of the above density operator

$$\lambda^{\pm}(t) = \frac{1}{2}(1 \pm \sqrt{x^2 + y^2 + z^2}). \tag{7}$$

In the following, we will use the von Neumann entropy to measure the degree of noise emerged in the process of controlling the system state to a target state. The von Neumann entropy of a quantum state ρ can be defined by the formula:[19−22]

$$S(\rho) = -\mathrm{Tr}(\rho\log\rho)\,. \tag{8}$$

In this formula logarithms are taken to base on 2. If λ_x are the eigenvalues of ρ, then von Neumann's definition can be re-expressed as

$$S(\rho) = -\Sigma_x\lambda_x\log\lambda_x\,. \tag{9}$$

Concerning our cases, we can write explicitly the entropy of the state in the system as

$$S(\rho) = -\lambda^+(t)\ln\lambda^+(t) - \lambda^-(t)\ln\lambda^-(t)\,. \tag{10}$$

3.1. Entropy evolution via the quantum feedback $m\sigma_y + n\sigma_z$

We combine the expression Eqs.(5) and (6), and substitute the Hamiltonian $\hat{H}_1 = a_2\sigma_y + a_3\sigma_z$ and $\hat{H}_{\mathrm{fb1}}(t) = I(t)(m\sigma_y + n\sigma_z)$ into the master equation, the equation can be written in the following form:

$$\mathrm{d}X = (A_1X + C_1)\mathrm{d}t + S_1\mathrm{d}W\,. \tag{11}$$

Where

$$X=\begin{pmatrix}x\\y\\z\end{pmatrix},\qquad C_1=\begin{pmatrix}0\\2n\sqrt{\gamma}\\-\gamma-2m\sqrt{\gamma}\end{pmatrix},$$

$$A_1=\begin{pmatrix}-2n^2-2m^2-2m\sqrt{\gamma}-\dfrac{\gamma}{2} & -2a_3 & 2a_2\\ 2a_3 & -2n^2-\dfrac{\gamma}{2} & 2nm+2n\sqrt{\gamma}\\ -2a_2 & 2nm & -2m\sqrt{\gamma}-\gamma-2m^2\end{pmatrix},$$

$$S_1=\begin{pmatrix}(2m+\sqrt{\gamma})z-2ny+\sqrt{\gamma}-x^2\sqrt{\gamma}\\ 2nx-xy\sqrt{\gamma}\\ -(2m+\sqrt{\gamma})x-\sqrt{\gamma}xz\end{pmatrix}. \tag{12}$$

It is shown that almost any initial state can reach state $|\theta,\varphi\rangle$ by choosing appropriate driving and feedback control parameters a_2, a_3, m, n, which can be got from the condition for a stationary state, exactly when $S=0$, and $AX+C=0$, those parameters are

$$\begin{aligned}
m&=-\frac{1}{2}\sqrt{\gamma}(1+\cos\theta),\\
n&=\frac{1}{2}\sqrt{\gamma}\sin\theta\sin\varphi,\\
a_2&=\frac{1}{4}\gamma\sin\theta(\cos^2\varphi\cos\theta-\sin^2\varphi)/\cos\varphi,\\
a_3&=-\frac{1}{4}\gamma\sin\varphi(\cos\theta+\cos^2\varphi\sin^2\theta)/\cos\varphi\,.
\end{aligned}\tag{13}$$

Making use of those parameters, we can get the evolution of an individual system to $|\theta,\varphi\rangle$. The specific steps are as follows: first we chose a target state $|\theta,\varphi\rangle=\cos\frac{\theta}{2}|0\rangle+\mathrm{e}^{\mathrm{i}\varphi}\sin\frac{\theta}{2}|1\rangle$ on the Bloch sphere, substitute the two parameters θ,φ into Eq.(13), then we can get the specific form of A_1 and C_1. It is not difficult to get the numerical solution of $x(t),y(t),z(t)$ from Eq.(11), in other words, these solutions are the dynamical evolution of an individual system to the target state $|\theta,\varphi\rangle$. In this process, we have dropped the last stochastic term and chosen the initial state $(x(0),y(0),z(0))=(0.3,0.4,0.866)$ in Eq.(11). Since we have obtained $x(t),y(t),z(t)$, hence according to Eqs.(7) and (10), we can get the evolution of the von Neumann entropy of the feedback controlled two-level system.

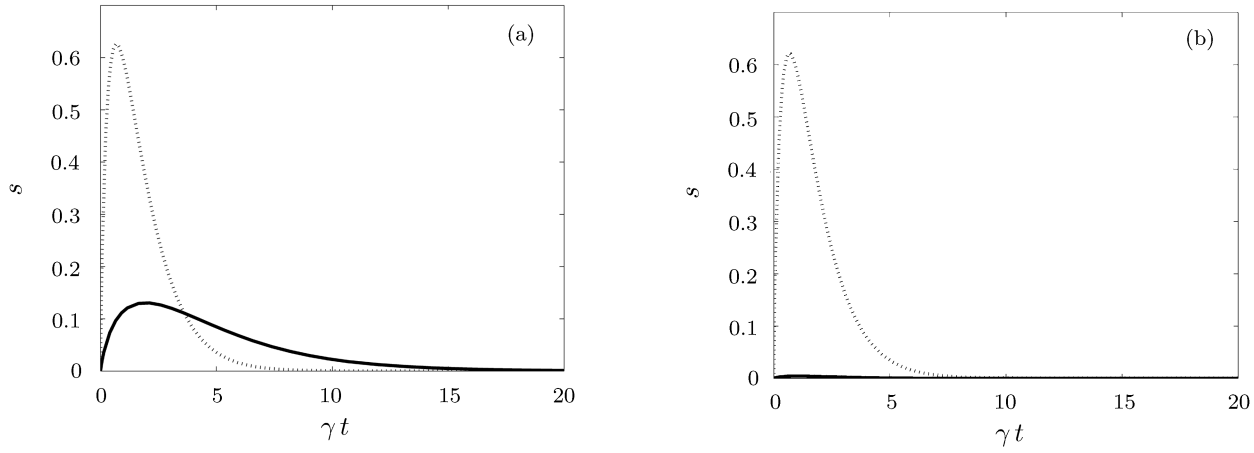


Fig.1. With fixed $\varphi=\pi/6$, the entropy evolution via the quantum feedback $m\sigma_y+n\sigma_z$ (solid) compared with feedback off (dotted). (a) $\theta=3\pi/8$; (b) $\theta=\pi/8$.

In Fig.1 and Fig.2, we plotted the quantum entropy of the two-level system versus the scale time γt when choosing different target state $|\theta, \varphi\rangle$, and compared the entropy with the feedback off. From these figures, we notice that when the feedback is off, the atomic quantum entropy at early times increased rapidly from zero to a maximum, which indicates that the system was affected greatly by the environment, then the entropy gradually decreased to small value. When we turn on the feedback, the maximum of entropy decreased and the entropy gradually decreased to zero, that is to say, we can get our target state in a steady pure state on the Bloch sphere. In Fig.1, we fixed $\varphi = \pi/6$, and in Fig.2 we fixed $\theta = \pi/4$, comparing the two cases we can see that when choosing the different position of our target state on the Bloch sphere, the effect of feedback on the entropy evolution is different, that is, the degree of noise reduction depends on the position of our target state.

We interpret this phenomenon as being due to the different ability of the state on the Bloch sphere protecting against noise.[5] So from these figures, we can see that the feedback can inhibit the noise very well, and we can also clearly know the extent of noise via the quantum entropy.

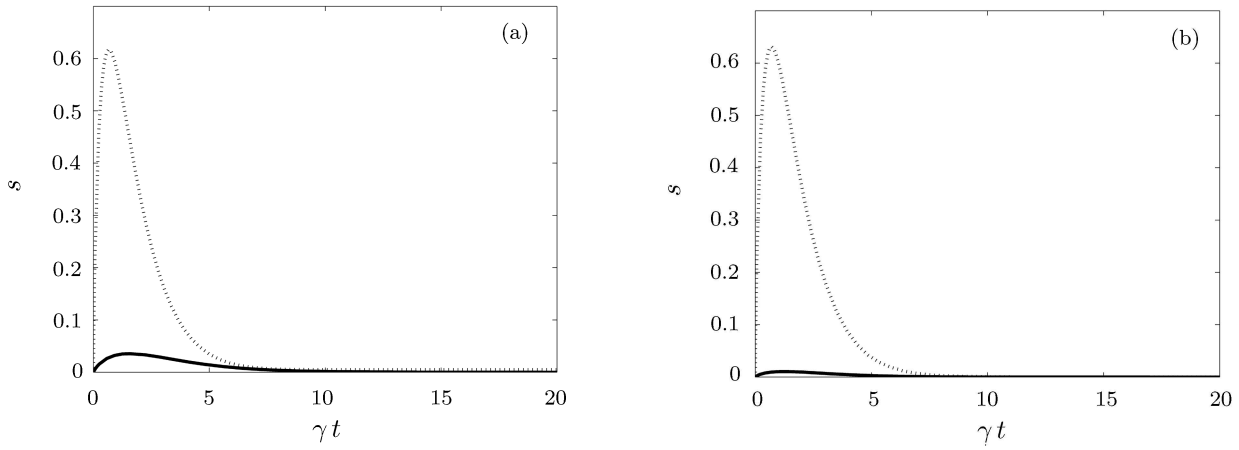


Fig.2. With fixed $\theta = \pi/4$, the entropy evolution via the quantum feedback $m\sigma_y + n\sigma_z$ (solid) compared with feedback off (dotted). (a) $\varphi = \pi/8$; (b) $\varphi = \pi/4$.

3.2. Entropy evolution via the quantum feedback $n\sigma_x + m\sigma_z$

When the driving Hamiltonian is taken as $\hat{H}_2 = a_3\sigma_x + a_2\sigma_z$ and the feedback control is chosen as $\hat{H}_{\rm fb2}(t) = I(t)(n\sigma_x + m\sigma_z)$, similarly we can get

$$\mathrm{d}X = (A_2X + C_2)\mathrm{d}t + S_2\mathrm{d}W\,, \tag{14}$$

where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 0 \\ 2m\sqrt{\gamma} \\ -\gamma \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -2m^2 - \dfrac{\gamma}{2} & -2a_2 & 2mn \\ 2a_2 + 2n\sqrt{\gamma} & -2n^2 - 2m^2 - \dfrac{\gamma}{2} & -2a_3 + 2m\sqrt{\gamma} \\ 2mn & 2a_3 & -\gamma - 2n^2 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} z\sqrt{\gamma} - 2my + \sqrt{\gamma} - x^2\sqrt{\gamma} \\ -2nz - xy\sqrt{\gamma} \\ -x\sqrt{\gamma} + 2ny - xz\sqrt{\gamma} \end{pmatrix}. \tag{15}$$

The driving and feedback control interaction parameters m, n, a_2, a_3,

$$m = \frac{\sqrt{\gamma}}{2\sin\theta\sin\varphi}(1+\cos\theta-\cos^2\varphi\sin^2\theta),$$

$$n = \frac{\sqrt{\gamma}}{2\sin\varphi}\cos\varphi(1+\cos\theta),$$

$$a_2 = -\frac{\gamma\cos\varphi}{4\sin\varphi}(2+\cos\theta-\cos^2\varphi\sin^2\theta),$$

$$a_3 = \frac{\gamma}{4\sin\theta\sin\varphi}(2-\cos^2\varphi\sin^2\theta)(1+\cos\theta). \quad (16)$$

We plotted the entropy in Fig.3 (fixed $\varphi = \pi/6$) and Fig.4 (fixed $\theta = \pi/4$). From these figures, we found that the feedback decreased the entropy, and the degree of reduction depended on the position of our target state on the Bloch sphere, just as that in Section 3.1. But due to the different choice of the driving and feedback Hamiltonian, there also exist differences. We notice that when the feedback is off, the entropy evolves to different steady values for different target states. When the effect of the feedback is taken into account, the entropy of an individual system to the target state $|\theta,\varphi\rangle$ decreases more rapidly than that in Section 3.1, which means that the feedback control strategy is much more effective.

In this section, through discussing the von Neumann entropy, we get a much clearer insight into the evolution of a quantum system which is under the feedback control. Different choices of driving and feedback Hamiltonian exhibit different ability of controlling noise, thus when we plan to control a certain state on the Bloch sphere, we can make more reasonable choice of the control parameters by considering the quantum entropy.

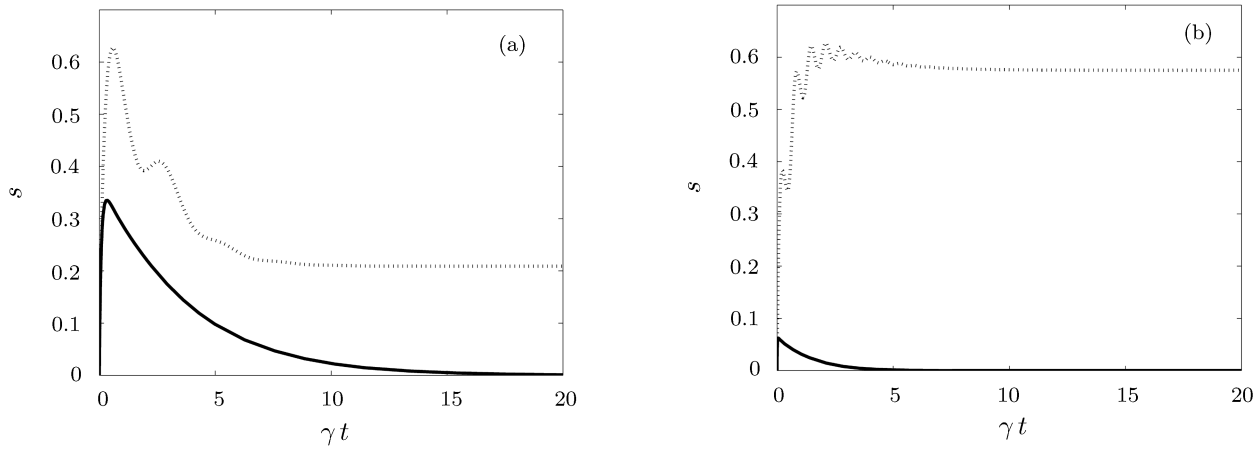


Fig.3. With fixed $\varphi = \pi/6$, the entropy evolution via the quantum feedback $n\sigma_x + m\sigma_z$ (solid) compared with feedback off (dotted). (a) $\theta = 3\pi/8$; (b) $\theta = \pi/8$.

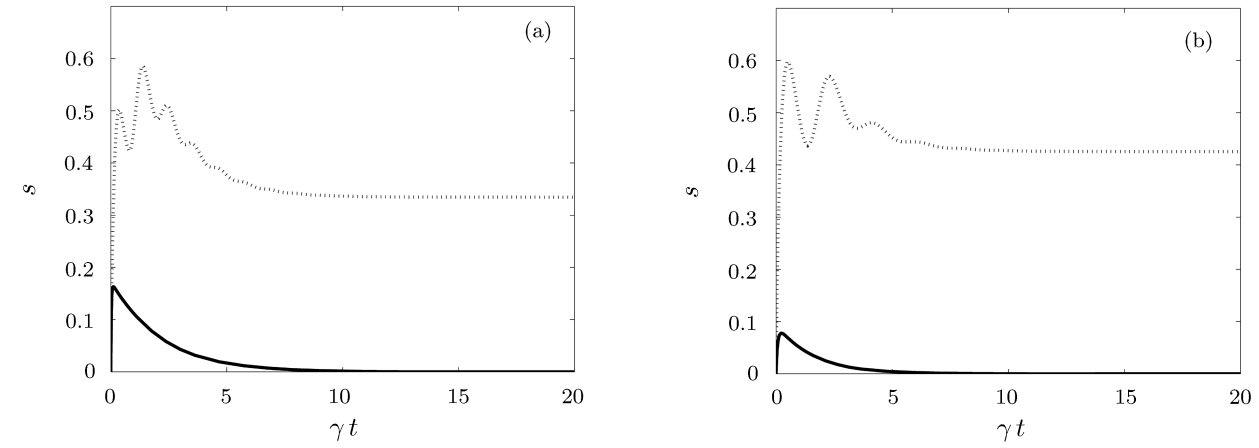


Fig.4. With fixed $\theta = \pi/4$, the entropy evolution via the quantum feedback $n\sigma_x + m\sigma_z$ (solid) compared with feedback off (dotted). (a) $\varphi = \pi/8$; (b) $\varphi = \pi/4$.

4. The influence of special initial states on the entropy evolution

In this section, we discuss the effect of some special initial states on the entropy evolution; here the quantum feedback is chosen as $n\sigma_x + m\sigma_z$. We choose the target state $|\theta = \pi/4, \varphi = \pi/4\rangle$ and consider three different initial states: the ground state $|0\rangle$, the excited state $|1\rangle$ and the initial state which is the same as the target state. In Fig.5 we get the explicit comparison of the effect of different initial states on the entropy evolution. Comparing Fig.5(a) with Fig.5(b), we find that it takes a much longer time to control the state when the initial state is the excited state $|1\rangle$. Figure 5(c) exhibits that when the target state is the same as the initial state, the entropy with feedback on nearly always equals zero, that is to say the state under feedback control cannot be much affected by the noise.

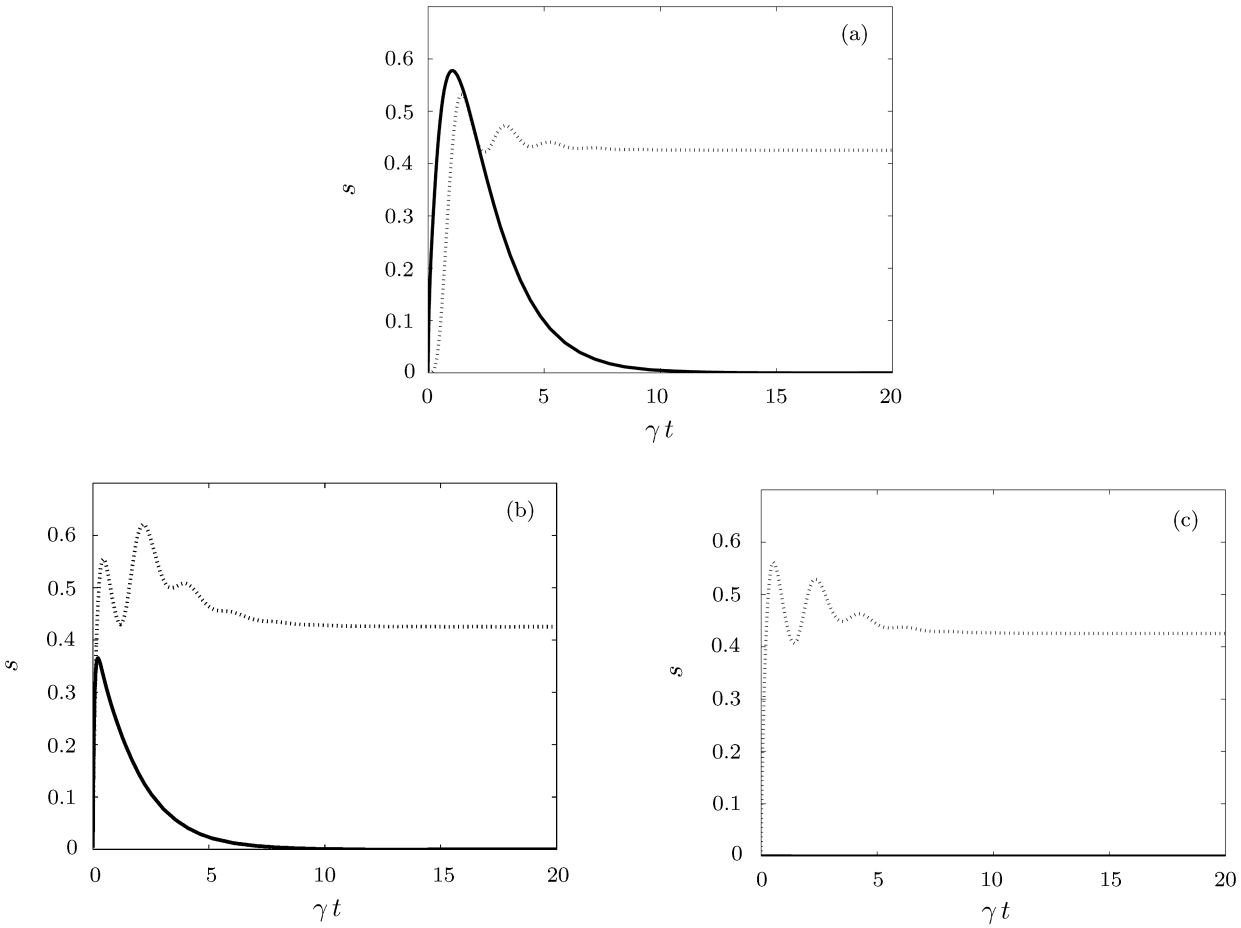


Fig.5. When the target state is $|\theta = \pi/4, \varphi = \pi/4\rangle$, the entropy evolution via the quantum feedback $n\sigma_x + m\sigma_z$ (solid) compared with feedback off (dotted). Different initial states: (a) excited state $|1\rangle$; (b) ground state $|0\rangle$; (c) the initial state is the same as the target state.

5. Summary

In this paper, we have discussed the evolution of von Neumann entropy in a two-level quantum feedback controlled system. We have shown that different control strategies exhibit different noise controlling ability. The evolution of entropy can provide us with a clear real time noise observation of the system, and a simple systematic approach which can help to make reasonable choice of control strategy to produce ideal results. Since the question of how to best design feedback strategies to control noisy quantum system is a complex one, our discussion of the entropy evolution will be useful to understand the feedback control process.

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