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Strength dynamics of weighted evolving networks*

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In this paper, a simple model for the strength dynamics of weighted evolving networks is proposed to characterize the weighted networks. By considering the congestion effects, this approach can yield power law strength distribution appeared on the many real weighted networks, such as traffic networks, internet networks. Besides, the relationship between strength and degree is given. Numerical simulations indicate that the strength distribution is strongly related to the strength dynamics decline. The model also provides us with a better description of the real weighted networks.

Keywords: strength dynamics, weighted, complex networks
PACC: 0590, 0200

Since the seminal paper by Watts and Strogatz,[1] more and more researches have focused on the properties of small-world and scale-free networks,[2−7] including social networks, biological networks and man-made networks. Recently, traffic and transportation systems (such as the subway system,[8] the urban transit system [9] or the railway system [10]) have been modelled as complex networks. In addition, real networks often exhibit a power-law degree distribution $p(k) \sim k^{-\lambda}$, where k denotes vertex degree and λ denotes the degree exponent. The original explanation of an emergent power-law is proposed by Barabási and Albert (BA), who introduced two mechanisms as network growth and preferential attachment (PA) represented by equation $\Pi^{\mathrm{BA}} = k_i/\sum_j k_j$.[11]

However, many networks are intrinsically weighted: their edges (or nodes) have different weights (or strength). In a traffic network, there is different volume of traffic between two sites. In a social network there may be stronger or weaker social ties among individuals. In a metabolic network there may be more or less flux along particular reaction pathways. Barrat *et al* [12] made the first detailed comparative analysis on the structure of weighted networks of the real world. These networks exhibit the following characteristics or properties: power law distribution for the degree, the vertex strength and the link weight, and together with the high clustering coefficient. The need to explain these complex topological features has led to a new modelling framework which focuses on the dynamical evolution and growth of networks. [13] In this paper, we consider three items—strength dynamics, weight dynamics enhancing and congestion-dependent dynamic decrease caused by the growth of weights. These three parts contribute to the evolving of complex networks together. The PA rule is always incorporated with the growth of complex networks in the previous models, which may produce scale-free properties. The main differences with respect to previous models are as follows:

(1) Both random and PA chosen rules are considered in our model. In the process of weight dynamics enhancing, one node is chosen randomly and another one is chosen by PA rule.

(2) Overload leads to the decrease of the node strength. In the decrease item caused by congestion, two nodes are chosen by PA rule, the reason being that the congestion always occurs in the larger strength nodes.

Considering the effect of congestion, from our model we can yield power law strength distributions as appearing on the many real weighted traffic networks. Many phenomena (such as the diversity of scale-free characteristics, nontrivial clustering and assortativ-

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ity coefficient, which have been empirically observed) can be well explained by our microscopic mechanisms. Moreover, in our model, we allow that congestion has occurred.

The network growth starts from an initial seed of N_0 nodes, and a new node adds into the network per unit time, until the sum of nodes N is reached. As we know, the evolving time is presented by the number of nodes added to the network, i.e. $t = N - N_0$. Denote w as the weight of the link, s as the strength of the node of the networks. In this paper, the parameters n, w, s and t are treated as continuous variables using the continuous approximation. The detail evolving processes mentioned above can be described as follows.

(1) *Strength dynamics.* In each time interval, a new node with m links adding into the network leads to the increment of strength s_i of the node i. This process is performed according to the strength PA probability, which is denoted as

$$\Pi_{n\to i} = \frac{s_i}{\sum_j s_j}.$$

(2) *Weight dynamics enhancing.* In the network, we choose a node randomly as the one end node of edge, and the other end node is obtained by PA rule until l links are assigned to the current network. This process can be understood as the reinforcement in the current network. For example, in a traffic-driven network, traffic flows between two nodes i and j drive the network evolution. The operation is repeated l times. That is

$$\frac{\mathrm{d}w_{ij}^{w}}{\mathrm{d}t} = l\frac{1}{t} \times l\frac{s_j}{\sum_n s_n}. \tag{1}$$

(3) *Congestion-dependent dynamic decline.* The increment of the links may lead to the overload (congestion) of the node i. Take the traffic network for example: with the increase of link cost, the traffic flow in this link will decrease, which will cause the decline of the weight on the edge ij. In this paper, we assume that there are h links deleted in this process, and two end nodes are selected by PA rule. The operation is repeated h times until h links are decreased. That is

$$\frac{\mathrm{d}w_{ij}^{\mathrm{c}}}{\mathrm{d}t} = -h\frac{s_i}{\sum_n s_n} \times h\frac{s_j}{\sum_n s_n}. \tag{2}$$

Hence, the total evolving process can be described as

$$\begin{aligned}\frac{\mathrm{d}s_i}{\mathrm{d}t} = {} & m\frac{s_i}{\sum_n s_n} + \sum_j l\frac{1}{t} \times l\frac{s_j}{\sum_n s_n} \\ & - \sum_j h\frac{s_i}{\sum_n s_n} \times h\frac{s_j}{\sum_n s_n},\end{aligned} \tag{3}$$

where

$$\sum_n s_n \approx (2m + l^2 - h^2)t.$$

Let the initial condition of Eq.(2) be $s_i(t_i) = m$, we then have

$$s_i(t) = \frac{1}{b}(l^2 + mb)\Big(\frac{t}{t_i}\Big)^b - \frac{l^2}{b},$$

where

$$b = \frac{2m - h^2}{2(l^2 + 2m - h^2)}.$$

The probability that a node has strength $s_i(t)$ smaller than s is

$$P[s_i(t) < s] = P\Big[t_i > t\frac{bs + l^2}{(l^2 + mb)}\Big]. \tag{4}$$

Assume that all nodes are added to the network at equal time interval, then t_i takes values with a constant probability density:

$$p(t_i) = \frac{1}{N_0 + t}. \tag{5}$$

Then, we can obtain the strength distribution $P(s) \sim s^{-1/b-1}$ if $l^2/(l^2 + mb) \to \varepsilon$, where ε is a given positive constant. Let $\theta = 1/b+1 = (b+1)/b$, the distribution $P(s) \sim s^{-\theta}$ is held.

In the paper, all the simulations start with $m_0 = 10$. Clearly, when the probability $h = 0$ and $l = 0$, the model is topologically equivalent to the famous BA network and the exponent $\theta = 3$ is satisfied as expected (see Fig.1).

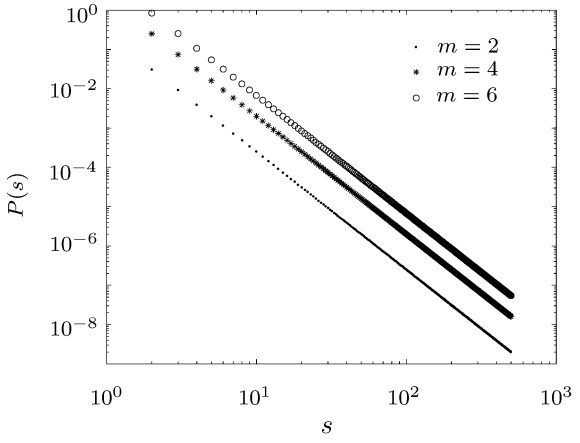


Fig.1. Double logarithm plot of the probability distribution $P(s)$ versus the strength s for different m. The data are plotted with $h = 0$, $l = 0$ and size $N = 500$.

Figure 2 displays the double logarithm plot of the probability distribution $P(s)$ versus the strength s for various m values. It shows that the exponent is independent on the parameter m, and the strength distribution will follow power-law scaling in deed. We also

give the double logarithm plot of the probability distribution $P(s)$ versus the strength s for different h as shown in Fig.3. Although there is the effect of congestion on the network, the power-law decay of the tail of strength distribution is produced.

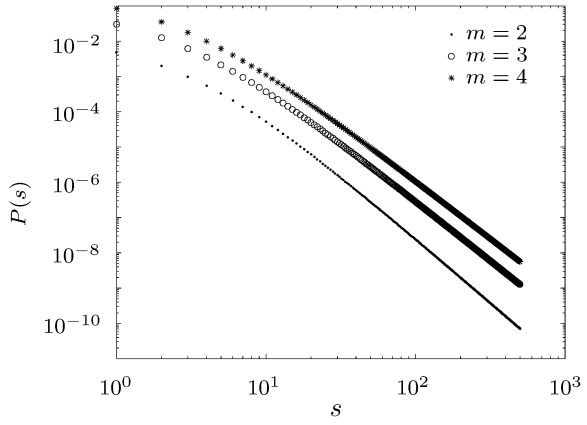


Fig.2. Double logarithm plot of the probability distribution $P(s)$ versus the strength s for different m. The data are plotted with $h = 1$, $l = 1$ and size $N = 500$.

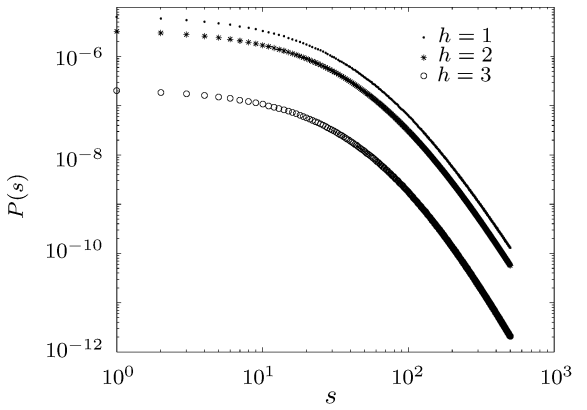


Fig.3. Double logarithm plot of the probability distribution $P(s)$ versus the strength s for different h. The data are plotted with $m = 8$, $l = 3$ and size $N = 500$.

Figure 4 is the plot of the strength s_i versus the time t for different links h when $t_i = 5$. It shows that the strength strongly relies on the parameter h. And the larger of h value is, the smaller of strength will be.

Empirical evidence indicates that in most cases the strength distribution has a fat tail, similar to the power law of degree distribution. Highly correlated with the degree, the strength usually displays scale-free property $s \sim k^{\beta}$.[13,14] In this paper, we plot the strength and the degree in Fig.5. Moreover, the degree distribution can be obtained by the results $s \sim k^{\beta}$ and $P(s) \sim s^{-\theta}$. And this is a more important characteristic of reflecting the scale free property. Consider the following equation:

$$P(k) = P(s)\frac{\mathrm{d}s}{\mathrm{d}k} = \beta k^{-[\beta(\theta-1)+1]}. \tag{6}$$

The degree distribution exponent $\lambda = \beta(\theta-1)+1$ can be obtained by Eq.(6), where $\theta = (b+1)/b$ and $b = (2m-h^2)/2(l^2+2m-h^2)$. So $\lambda = \beta/b+1$ is given. Now we take the world-wide airport network (WAN) for example, the WAN exhibits a nonlinear scaling with $\beta \approx 1.5$.[15]

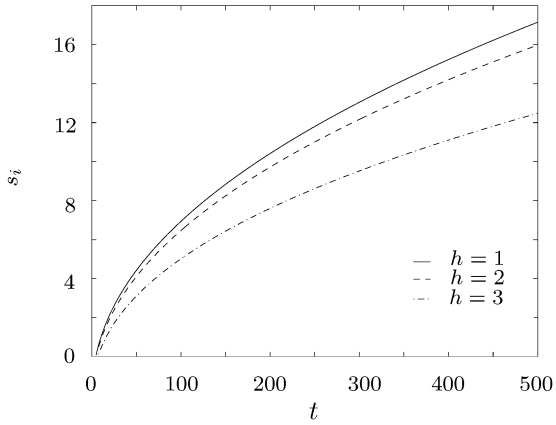


Fig.4. Plot of the strength s_i versus the time t for different probability h when $t_i = 5$. The larger of the links h is, the smaller of the strength s will be.

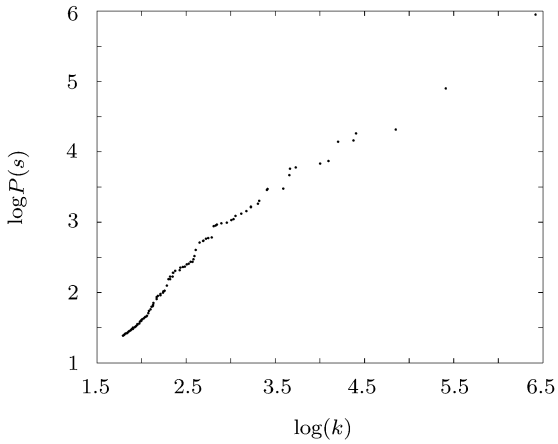


Fig.5. Double logarithm plot of the strength s and the degree k. Highly correlated with the degree, the strength usually displays scale-free property $s \sim k^{\beta}$.

Figures 6 and 7 show that the double logarithm plot of the degree distribution $P(k)$ versus the strength k for different m and h respectively. These two graphs show that the scale-free characteristics of degree will display on the network, and the network

will be grown into power-law distribution of degree by our model.

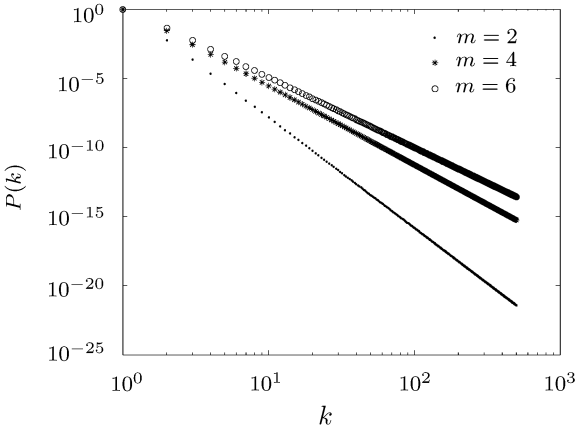


Fig.6. Double logarithm plot of the degree distribution $P(k)$ versus the degree k for different m. The data are plotted with $l = 2$, $\beta = 1.5$, $h = 1$ and size $N = 500$.

In this paper, we define a simple model for weighted network which considers the basic evolution rule of the traffic system as that driven by the weight properties, and the congestion (overload) is accounted in the evolving mechanisms. The generated networks exhibit power law (power-law decay of the tail) behaviour for the strength distribution as statistical characteristics of many real networks. Moreover, the degree distribution $P(k)$ is given in this paper. However, the real networks are more complex than what we studied, and the model cannot reflect the properties completely. Therefore, more reasonable model should be introduced for the evolving weighted networks.

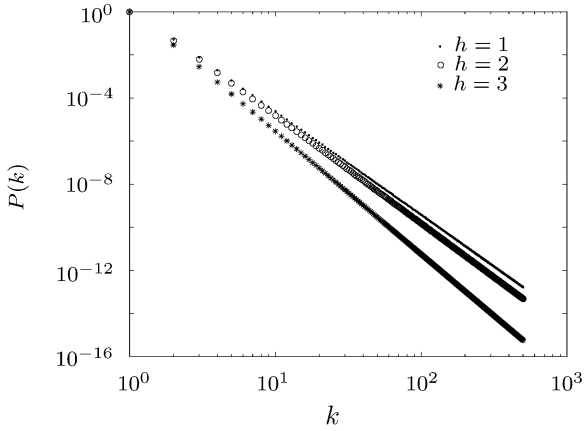


Fig.7. Double logarithm plot of the degree distribution $P(k)$ versus the degree k for different h. The data are plotted with $m = 8$, $l = 2$, $\beta = 1.5$ and size $N = 500$.

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