Vol 16 No 1, January 2007
1009-1963/2007/16(01)/0042-05
Chinese Physics
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Time dependence of entropy flux and entropy production for a dynamical system driven by noises with coloured cross-correlation*

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(Received 13 November 2005; revised manuscript received 28 February 2006)

This paper shows the Fokker–Planck equation of a dynamical system driven by coloured cross-correlated white noises in the absence and presence of a small external force. Based on the Fokker–Planck equation and the definition of Shannon's information entropy, the time dependence of entropy flux and entropy production can be calculated. The present results can be used to explain the extremal behaviour of time dependence of entropy flux and entropy production in view of the dissipative parameter γ of the system, coloured cross-correlation time τ and coloured cross-correlation strength λ.

Keywords: information entropy, entropy flux, entropy production, coloured cross-correlation, Fokker–Planck equation

PACC: 0570L, 0520, 6550

1. Introduction

Shannon's information measure

$$S = -\int P(x,t)\ln P(x,t)\mathrm{d}x \tag{1}$$

is a very important tool for the study of nonequilibrium and stationary states of the stochastic processes. Here S is called information entropy, and $P(x,t)$ is the continuous probability distribution function in phase space. Recent studies[1−11] show that information entropy and related quantities are vital to the understanding of dynamical systems driven by noises in details.

Since the multiplicative and additive noises of a dynamical system may be taken as a common origin in certain situation, it is worthwhile to consider how the correlation of noises can influence information entropy and associated quantities in a dynamical system driven by noises with a common origin. The coloured cross-correlation of noises has been imported to study stochastic resonance in Refs.[12–14], etc. The effect of coloured cross-correlation of noises on the escape rate is shown in Ref.[15], however, a few results involve the influence of coloured cross-correlation of noises on information entropy. The white cross-correlation properties of noises have been considered to set an upper bound on the rate of entropy change in a nonequilibrium system,[1] and also used to examine some extremal qualities of entropy flux and production.[5] Hence, we have shown how the coloured cross-correlation properties of noises can set an upper bound on the rate of entropy change in a nonequilibrium system in Ref.[9].

This paper is to study time dependence of entropy flux and production of the dissipative dynamical system driven by coloured cross-correlated white noises (i.e. the coupling of the multiplicative and additive white noises is coloured with nonzero cross-correlation τ) in absence or presence of a small external force. The entropy flux and production are expected to depend on not only the dissipative parameter but also the coloured cross-correlation time and strength of noises.

2. The Fokker–Planck equation of the dynamical system driven by noises with coloured cross-correlation

We consider a simple dynamical system driven by both multiplicative and additive white noises

*Project supported by the National Natural Science Foundation of China (Grant No 10472091 and 10332030) and Natural Science Foundation of Shaanxi Province, China (Grant No 2003A03).
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with coloured cross-correlation time. The relevant Langevin equation is as follows:

$$\dot{X} = -\gamma X - X\zeta(t) + \eta(t), \tag{2}$$

where $\zeta(t)$ and $\eta(t)$ are multiplicative and additive white noises respectively. The two noise terms are characterized by their mean and variance

$$\begin{aligned}
&\langle\zeta(t)\rangle = \langle\eta(t)\rangle = 0,\\
&\langle\zeta(t)\zeta(t')\rangle = 2D'\delta(t-t'),\\
&\langle\eta(t)\eta(t')\rangle = 2D\delta(t-t'),\\
&\langle\zeta(t)\eta(t')\rangle = \langle\eta(t)\zeta(t')\rangle = \frac{\lambda\sqrt{DD'}}{\tau}\exp\left[-\frac{|t-t'|}{\tau}\right],
\end{aligned} \tag{3}$$

where D' and D correspond to intensities of multiplicative and additive white noises respectively. τ is the cross-correlation time of the coupling between the multiplicative and additive white noises. λ denotes the cross-correlation strength between these two noises with the limit $0 \le \lambda \le 1$.

Equation (3) shows that the coupling between multiplicative and additive noises is coloured cross-correlation.[9,12−14] The small noise approximation[2] and the result of Ref.[12] are important for treatment of our problem. We can write the Fokker–Planck equation corresponding to Langevin equation (2) and Eq.(3) as[9]

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[A(x)P(x,t)]}{\partial x} + B\frac{\partial^2 P(x,t)}{\partial x^2}. \tag{4}$$

The reports in Ref.[9] illuminate that the drift and diffusion terms of Eq.(4) are respectively,

$$\begin{aligned}
A(x) &= -\gamma x + D'x - \frac{\lambda\sqrt{DD'}}{1+\gamma\tau},\\
B &= \frac{[D'l^2 + 2l^2\Gamma + D\Gamma^2]}{\Gamma^2},
\end{aligned} \tag{5}$$

where

$$l = -\frac{\lambda\sqrt{DD'}}{(1+\gamma\tau)}, \quad \Gamma = \gamma - D'. \tag{6}$$

3. Time dependence of entropy flux and entropy production for dynamical system described by Eqs.(2) and (3)

Using Eq.(4) and the definition of the Shannon entropy Eq.(1), we quote the time evolution equation for entropy

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\int \mathrm{d}x\left[-\frac{\partial AP}{\partial x} + B\frac{\partial^2 P}{\partial x^2}\right]\ln P, \tag{7}$$

and the definitions of entropy flux and entropy production, respectively[1−6]

$$S_{\mathrm{F}} = \int P\frac{\partial A}{\partial x}\mathrm{d}x, \tag{8}$$

$$S_{\mathrm{P}} = B\int \frac{1}{P}\left(\frac{\partial P}{\partial x}\right)^2 \mathrm{d}x. \tag{9}$$

In view of the Fokker–Planck equation (4) is linear and can be exactly solvable, the time dependence of entropy flux and production is as follow:

$$S_{\mathrm{F}} = -\Gamma, \quad S_{\mathrm{P}} = \frac{2B}{\sigma(t)}. \tag{10}$$

The time evolution of $\sigma(t)$ can be written as

$$\sigma(t) = \frac{2B}{\Gamma}[1-\exp(-2\Gamma t)] + \sigma_0\exp(-2\Gamma t), \tag{11}$$

where σ_0 is constant.

Equation (10) depicts the interplay of the dissipative parameter γ of Eqs.(2) and (3), the cross-correlation time τ and cross-correlation strength λ of coloured cross-correlated multiplicative and additive white noises on entropy flux and production. In the limit $\tau \to 0$, i.e., in case of white cross-correlation between multiplicative and additive white noises, the details have discussed in Ref.[5]. The interest of this paper is the general case of nonzero cross-correlation time.

We consider the effect of τ on the time evolution of the entropy deviations. Figure 1 depicts entropy production S_{P} vs time t for different values of τ.

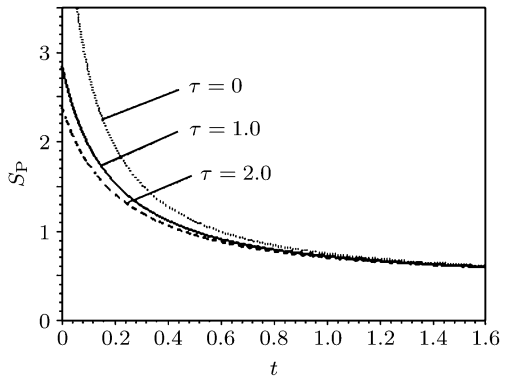


Fig.1. Plot of entropy production S_{P} vs time for $\sigma_0 = 0.5$, $D' = 0.5$, $D = 0.5$, $\gamma = 1.0$ and $\lambda = 0.75$.

As shown in Fig.1, S_{P} decreases monotonically almost exponentially with time and finally reaches to the limiting value for a given $\gamma - D'$ after long time because

of $S_{\rm P} = -S_{\rm F}$, however, the equilibration or relaxation time shortens with increase of τ.

4. The effects of coloured cross-correlated noises on the entropy flux and entropy production in the presence of external force $f_{\rm e}$

We now consider a constant drift $f_{\rm e}$ in Eq.(2) and examine the time dependence of entropy flux and production during the relaxation of steady state to a new steady state for the system by a weak external force,

$$\dot{X} = -\gamma X - X\zeta(t) + \eta(t) + cf_{\rm e}, \tag{12}$$

where c is a small parameter (here $c = 1.0$). The Fokker–Planck equation corresponding to Eq.(12) is

$$\frac{\partial P}{\partial t} = -\frac{\partial \phi P}{\partial x} - c\frac{\partial f_{\rm e} P}{\partial x} + B\frac{\partial}{\partial x}\left(P_{\rm s}\frac{\partial}{\partial x}\frac{P}{P_{\rm s}}\right), \tag{13}$$

where $\phi = A - B\dfrac{\partial \ln P_{\rm s}}{\partial x}$ and $P_{\rm s}$ is stationary solution of Eq.(4). Analogously, the time dependence of entropy flux and production can be derived as[3−5]

$$\Delta S_{\rm F} = \frac{\Gamma}{2B}\left[2B - \Gamma\sigma(t) + 2\left(\beta_{\rm c} - \frac{l}{\Gamma}\right)(-\beta_{\rm c}\Gamma + l + f_{\rm e})\right] + \frac{lf_{\rm e}}{B} - \frac{\Gamma}{B}f_{\rm e}\beta_{\rm c}, \tag{14}$$

$$\Delta S_{\rm P} = B\left(\frac{\Gamma}{B} - \frac{2}{\sigma(t)}\right)\left[\left(\frac{\Gamma}{B} - \frac{2}{\sigma(t)}\right)\left(\beta_{\rm c}^2 + \frac{\sigma(t)}{2}\right) + 2\left(\frac{2\beta_{\rm c}}{\sigma(t)} - \frac{l}{B}\right)\beta_{\rm c}\right] + B\left(2\frac{\beta_{\rm c}}{\sigma(t)} - \frac{l}{B}\right)^2, \tag{15}$$

where

$$\beta_{\rm c}(t) = \left(\beta_{\rm c}(0) - \frac{l + f_{\rm e}}{\Gamma}\right)\exp(-\Gamma t) + \frac{l + f_{\rm e}}{\Gamma}. \tag{16}$$

Now we choose $\sigma_0 = 0.5$, $\beta_{\rm c}(0) = 0.5$, $D' = 0.5$, $D = 0.5$ and $f_{\rm e} = 1.0$. It is interesting to examine the effects of noises on the entropy flux and production in the presence of external force $f_{\rm e}$. The entropy production $\Delta S_{\rm P}$ at first decreases with time and then passes through minima and finally reaches to a new steady state in Fig.2. when the cross-correlation time τ of noises becomes smaller, the external force becomes less effective. The deviation of the entropy flux $\Delta S_{\rm F}$ for different values of τ is shown clearly in Fig.3.

In Fig.4 we plot the variation of $\Delta S_{\rm P}$ with cross-correlation time τ at $t = 3$ and in presence of external force $f_{\rm e}$. $\Delta S_{\rm P}$ monotonically increases to a limiting value with the increase of τ. However, $\Delta S_{\rm F}$ monotonically decreases to a limiting value with the increase of τ in Fig.5.

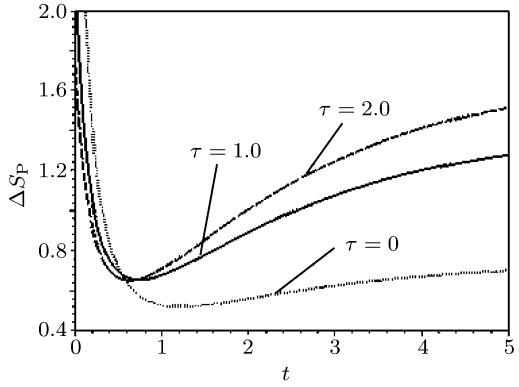


Fig.2. Plot of entropy production $\Delta S_{\rm P}$ vs time t for $\gamma = 1.0$, $\lambda = 0.75$ and $.f_{\rm e} = 1.0$.

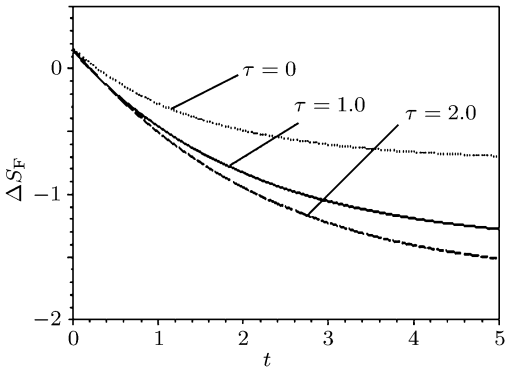


Fig.3. Plot of entropy flux $\Delta S_{\rm F}$ vs time t for the same parameter set as in Fig.2.

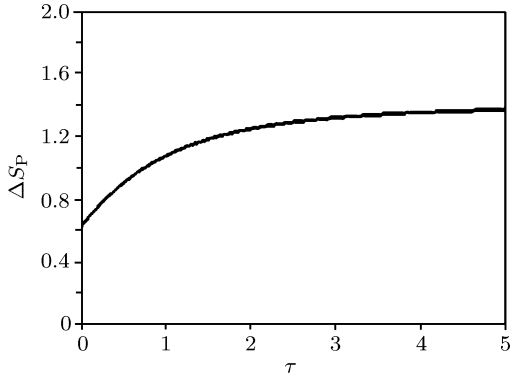


Fig.4. Plot of entropy production $\Delta S_{\rm P}$ vs τ for the same parameter set as in Fig.2 and $t = 3.0$.

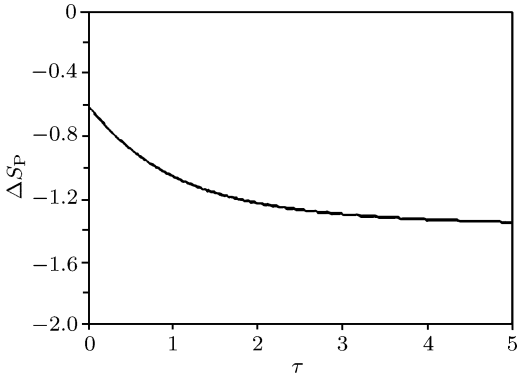


Fig.5. Plot of entropy flux $\Delta S_{\rm F}$ vs τ for the same parameter set as in Fig.2 and $t = 3.0$.

The variations of $\Delta S_{\rm P}$ and $\Delta S_{\rm F}$ with γ show extremal behaviour according to the presence of external force and coloured cross-correlation noises. At first $\Delta S_{\rm P}$ decreases and then increases to a limiting value after passing through a minimum in Fig.6. However, at first $\Delta S_{\rm F}$ increases and then decreases to a limiting value after passing through a maximum in Fig.7. Hence τ enhances the effect of γ on $\Delta S_{\rm P}$ and weakens the effect of γ on $\Delta S_{\rm F}$. This extremal behaviour is not observed for zero external force.

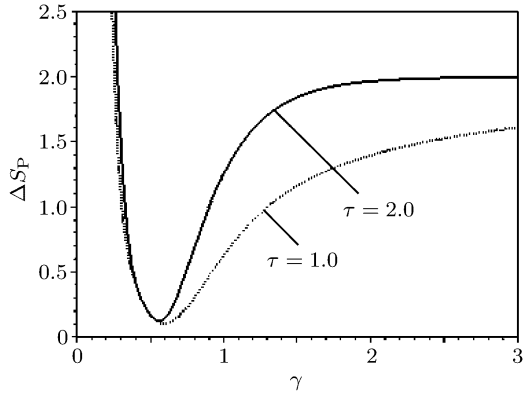


Fig.6. Plot of entropy production $\Delta S_{\rm P}$ vs γ for $\lambda = 0.75$ and $t = 3.0$.

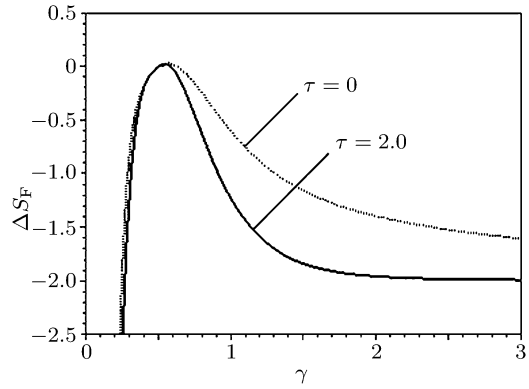


Fig.7. Plot of entropy flux $\Delta S_{\rm F}$ vs γ for the same parameter set as in Fig.6.

Now we examine how both $\Delta S_{\rm P}$ and $\Delta S_{\rm F}$ vary with the cross-correlation strength λ of multiplicative and additive noises. It is clearly seen that $\Delta S_{\rm P}$ decreases monotonically in Fig.8 and $\Delta S_{\rm F}$ increases monotonically in Fig.9. The increase of τ smoothes the variation of both $\Delta S_{\rm P}$ and $\Delta S_{\rm F}$ with λ. Units are arbitrary in all calculations.

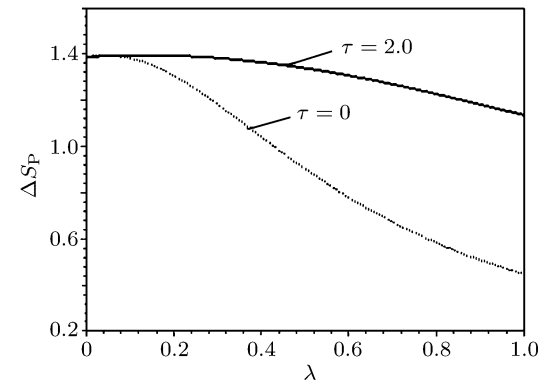


Fig.8. Plot of entropy production $\Delta S_{\rm P}$ vs λ for $\gamma = 1.0$ and $t = 3.0$.

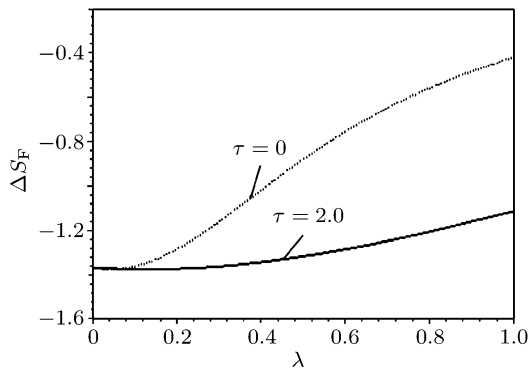


Fig.9. Plot of entropy flux $\Delta S_{\rm F}$ vs λ for the same parameter set as in Fig.8.

5. Conclusions

In this paper we have analysed the interaction between dissipative parameter and the coloured cross-correlation properties of white noises in absence and presence of a small external force. The coloured cross-correlation time τ is important for the time dependence of entropy flux and production in absence of a small force. The variation of entropy flux and production with dissipative parameter γ, coloured cross-correlation time τ and strength λ show extremal behaviour in the presence of a small external force. Specially τ strengthens the fluctuations of $\Delta S_{\rm P}$ and $\Delta S_{\rm F}$ with γ and weakens the fluctuations of $\Delta S_{\rm P}$ and $\Delta S_{\rm F}$ with λ.

Acknowledgement

The author gratefully acknowledges the support of Youth for NPU Teachers Scientific and Technological Innovation Foundation.

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