Vol 16 No 1, January 2007
1009-1963/2007/16(01)/0023-09

Chinese Physics

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Establishing path integral in the entangled state representation for Hamiltonians in quantum optics*

Wang Ji-Suo(王继锁)†, Meng Xiang-Guo(孟祥国),
Feng Jian(冯 健), and Gao Yun-Feng(高云峰)

Department of Physics, Liaocheng University, Liaocheng 252059, *China*

(Received 6 January 2006; revised manuscript received 18 May 2006)

Based on two mutually conjugate entangled state representations, we establish the path integral formalism for some Hamiltonians of quantum optics in entangled state representations. The Wigner operator in the entangled state representation is presented. Its advantages are explained.

Keywords: path integral, Hamiltonian, entangled state representation

PACC: 0365

1. Introduction

Feynman's path integral representation of the propagator gives us new insight of the connection between the classical and quantum dynamics of systems. As Feynman and Hibbs wrote in Ref.[1], in developing a conceptual understanding of quantum mechanics, one may prefer to take path integral formalism. Moreover, semiclassical approximation led by path integral expressions is related to Euler Lagrangian equation in a natural way. In Ref.[2] Klauder and Skargerstam first introduced coherent state representation for studying the path integral, and later Hillery and Zubairy applied the coherent state path integral approach to some problems in quantum optics.[3] On the other hand, in recent years quantum entanglement and entangled states have been paid much attention by physicists due to their wide use in quantum information and quantum communication. The quantum entanglement involved in a bipartite state of continuum variables can be traced back to the pioneering work of Einstein–Podolsky–Rosens (EPR)[4] in which they noticed that two particles' relative coordinate operator X_1–X_2 commutes with the total momentum operator $P_1 + P_2$. On the basis of EPR's analysis the bipartite entangled state of continuum variables, denoted as $|\eta\rangle$, is constructed which is the common eigenvector of $(X_1$–$X_2)$ and $(P_1 + P_2)$. In this work we intend to apply the entangled state representation $|\eta >$ to studying path integrals. Two questions thus naturally arise: 1) Can we define 'path' in the phase space where the entangled state is defined? 2) What is the advantage of working with the entangled state? In Section 2 we briefly review the two mutually conjugate entangled states of continuum variables. Then in Section 3 we establish a general path integral formalism for the matrix element of the time evolution transform between two entangled states. In Sections 4 and 5 we take the Hamiltonian of nondegenerate parametric amplifier (NDPA) as an example to show the advantages of the formalism in Section 3, i.e. this formalism leads to the correct Euler equation which corresponds to the Heisenberg equation, and the energy level formula of the NDPA.

2. $|\chi\rangle - |\eta\rangle$ representation

In Ref.[5] the bipartite EPR entangled state is constructed in the two-mode Fock space

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2 + \eta a^+ - \eta^* b^+ + a^+ b^+\right]|00\rangle, \quad \eta = |\eta|\mathrm{e}^{\mathrm{i}\varphi}, \tag{1}$$

*Project supported by the National Natural Science Foundation of China (Grant No 10574060) and the Natural Science Foundation of Shandong Province of China (Grant No Y2004A09).

†Corresponding author. E-mail: jswang@lctu.edu.cn

which obeys the eigenvalue equations

$$
\begin{aligned}
&(a-b^+)|\eta\rangle=\eta|\eta\rangle, \quad (a^+-b)|\eta\rangle=\eta^*|\eta\rangle,\\
&(X_1-X_2)|\eta\rangle=\sqrt{2}\eta_1|\eta\rangle, \quad (P_1+P_2)|\eta\rangle=\sqrt{2}\eta_1|\eta\rangle, \quad \eta_1+\mathrm{i}\eta_2=\eta,\\
&X_1=\frac{1}{\sqrt{2}}(a^++a), \quad X_2=\frac{1}{\sqrt{2}}(b^++b),\\
&P_1=\frac{\mathrm{i}}{\sqrt{2}}(a^+-a), \quad P_2=\frac{\mathrm{i}}{\sqrt{2}}(b^+-b).
\end{aligned}
\tag{2}
$$

Using the technique of integration within an ordered product (IWOP) of operators[6,7] we can prove that the set of $|\eta\rangle$ makes up a complete and orthonormal set

$$
\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta|=1, \quad \langle\eta|\eta'\rangle=\pi\delta(\eta-\eta')\delta(\eta^*-\eta'^*). \tag{3}
$$

Defining a new state vector $|\chi\rangle$ as the Fourier transform of $|\eta\rangle$

$$
|\chi\rangle=\int\frac{\mathrm{d}^2\eta}{2\pi}|\eta\rangle\exp\left[\frac{\mathrm{i}}{2}(\chi\eta^*+\chi^*\eta)\right]=\exp\left(-\frac{1}{2}|\chi|^2+\mathrm{i}\chi a^+-\mathrm{i}\chi^*b^+-a^+b^+\right)|00\rangle, \tag{4}
$$

where χ is a complex number too. Then $|\chi\rangle$ satisfies

$$
(a+b^+)|\chi\rangle=\mathrm{i}\chi|\chi\rangle, \quad (a^++b)|\chi\rangle=-\mathrm{i}\chi^*|\chi\rangle. \tag{5}
$$

We can also prove

$$
\int\frac{\mathrm{d}^2\chi}{\pi}|\chi\rangle\langle\chi|=1, \quad \langle\chi|\chi'\rangle=\pi\delta(\chi-\chi')\delta(\chi^*-\chi'^*). \tag{6}
$$

The overlap $\langle\chi|\eta\rangle$ is

$$
\langle\chi|\eta\rangle=\frac{1}{2}\exp\left[-\frac{\mathrm{i}}{2}(\chi\eta^*+\chi^*\eta)\right]. \tag{7}
$$

Since $|\chi\rangle$ and $|\eta\rangle$ are qualified to be quantum mechanical representations, and they are mutually canonical conjugate, so we have confidence to establish a path integral theory based on them. The transition amplitude is defined as the matrix element of the time evolution operator between the two entangled states $|\chi''\rangle\to|\chi'\rangle$ with all the paths going from χ'' to χ'.

3. Path integral formalism in $|\chi\rangle-|\eta\rangle$ representation

For a Hamiltonian operator we can write

$$
\begin{aligned}
H(a^+,a;b^+,b)=&\int\frac{\mathrm{d}^2\eta'\mathrm{d}^2\chi'\mathrm{d}^2\eta''\mathrm{d}^2\chi''}{\pi^4}|\chi'\rangle\langle\chi'|\eta'\rangle\langle\eta'|H|\eta''\rangle\langle\eta''|\chi''\rangle\langle\chi''|\\
=&\frac{1}{4}\int\frac{\mathrm{d}^2\eta'\mathrm{d}^2\chi'\mathrm{d}^2\eta''\mathrm{d}^2\chi''}{\pi^4}|\chi'\rangle\langle\chi''|\langle\eta'|H|\eta''\rangle\\
&\times\exp\left[-\frac{\mathrm{i}}{2}(\chi'\eta'^*+\chi'^*\eta'-\chi''^*\eta''-\eta''^*\chi'')\right],
\end{aligned}
\tag{8}
$$

where the completeness relations (3) and (6) are used. Making the integral variable transform

$$
\eta'=\eta+\frac{\sigma}{2}, \quad \chi'=\chi+\frac{\lambda}{2}, \quad \chi''=\chi-\frac{\lambda}{2}, \quad \eta''=\eta-\frac{\sigma}{2}, \tag{9}
$$

such that

$$
\mathrm{d}^2\chi'\mathrm{d}^2\chi''=\mathrm{d}^2\chi\mathrm{d}^2\lambda, \quad \mathrm{d}^2\eta'\mathrm{d}^2\eta''=\mathrm{d}^2\eta\mathrm{d}^2\sigma, \tag{10}
$$

we have

$$H = \int \frac{\mathrm{d}^2\chi \mathrm{d}^2\eta \mathrm{d}^2\lambda \mathrm{d}^2\sigma}{4\pi^4} \left|\chi + \frac{\lambda}{2}\right\rangle \left\langle \chi - \frac{\lambda}{2}\right| H \left|\eta + \frac{\sigma}{2}\right\rangle \left\langle \eta - \frac{\sigma}{2}\right| \times \exp[-\mathrm{i}(\lambda\eta^* + \lambda^*\eta + \chi^*\sigma + \chi\sigma^*)/2]. \tag{11}$$

By introducing

$$H = \int \frac{\mathrm{d}^2\lambda}{\pi} \left|\chi + \frac{\lambda}{2}\right\rangle \left\langle \chi - \frac{\lambda}{2}\right| \exp[-[\mathrm{i}(\lambda\eta^* + \lambda^*\eta)/2] \equiv \Delta(\chi, \eta), \tag{12}$$

where $\Delta(\chi, \eta)$ can be looked on as the 2-mode Wigner operator, and

$$\int \frac{\mathrm{d}^2\sigma}{\pi} \left\langle \eta + \frac{\sigma}{2}\right| H \left|\eta - \frac{\sigma}{2}\right\rangle \exp[-\mathrm{i}(\chi^*\sigma + \chi\sigma^*)/2] \equiv h(\chi, \eta), \tag{13}$$

we can express Eq.(11) in the compact form

$$H = \int \frac{\mathrm{d}^2\chi \mathrm{d}^2\eta}{4\pi^2} h(\chi, \eta)\Delta(\chi, \eta), \tag{14}$$

which can be looked on as Weyl correspondence between $h(\chi, \eta)$ and H in the entangled state representation. Then H's matrix element in the $\langle\chi|$ representation, as a result of Eqs.(11) and (6), is

$$\begin{aligned} \langle\chi'|H|\chi''\rangle = & \int \frac{\mathrm{d}^2\chi \mathrm{d}^2\eta}{4} h(\chi, \eta) \int \frac{\mathrm{d}^2\lambda}{\pi} \exp[-\mathrm{i}(\lambda\eta^* + \lambda^*\eta)/2]\delta^{(2)}\left(\chi' - \chi - \frac{\lambda}{2}\right)\delta^{(2)}\left(\chi - \frac{\lambda}{2} - \chi''\right) \\ = & \int \frac{\mathrm{d}^2\chi \mathrm{d}^2\eta}{\pi} h(\chi, \eta)\delta^{(2)}(\chi'' + \chi' - 2\chi) \exp\{-\mathrm{i}[(\chi' - \chi)\eta^* + (\chi' - \chi)^*\eta]\} \\ = & \int \frac{\mathrm{d}^2\eta}{4} h\left(\frac{\chi'' + \chi'}{2}, 2\eta\right) \exp\{-\mathrm{i}[\eta^*(\chi' - \chi'') + (\chi'^* - \chi''^*)\eta]\}. \end{aligned} \tag{15}$$

According to Feynman's path integral theory, we consider the transition amplitude,

$$\langle\chi'', t''|\chi', t'\rangle = \langle\chi''| \exp[-\mathrm{i}H(t'' - t')]|\chi'\rangle = \int \prod_{i=1}^{n} \left[\frac{\mathrm{d}^2\chi_i}{\pi}\right] \prod_{j=1}^{n+1} \langle\chi_j| \exp[-\mathrm{i}\varepsilon H]|\chi_{j-1}\rangle, \tag{16}$$

where $\chi'' = \chi_{n+1}, \chi' = \chi_0$, and $\varepsilon = (t'' - t')/(n + 1)$. When $\varepsilon \to 0$, the monomial in Eq.(16) can be evaluated by using Eqs.(6) and (15)

$$\begin{aligned} & \langle\chi_j| \exp[-\mathrm{i}\varepsilon H]|\chi_{j-1}\rangle \\ = & \pi\delta^{(2)}(\chi_j - \chi_{j-1}) - \mathrm{i}\varepsilon\langle\chi_j|H|\chi_{j-1}\rangle \\ = & \int \frac{\mathrm{d}^2\eta_j}{\pi} \exp\left\{-\mathrm{i}[\eta_j^*(\chi_j - \chi_{j-1}) + \eta_j(\chi_j^* - \chi_{j-1}^*)]\left[1 - \mathrm{i}\varepsilon h\left(\frac{\chi_j + \chi_{j-1}}{2}, 2\eta_j\right)\right]\right\} \\ = & \int \frac{\mathrm{d}^2\eta_j}{\pi} \exp\left\{-\mathrm{i}[\eta_j^*(\chi_j - \chi_{j-1}) + \eta_j(\chi_j^* - \chi_{j-1}^*)] - \mathrm{i}\varepsilon h\left(\frac{\chi_j + \chi_{j-1}}{2}, 2\eta_j\right)\right\}. \end{aligned} \tag{17}$$

Substituting Eq.(17) into Eq.(16) we have the formalism of path integral in the $\langle\chi|$ representation

$$\begin{aligned} \langle\chi'', t''|\chi', t'\rangle = & \int \prod_t \frac{\mathrm{d}^2\chi(t)\mathrm{d}^2\eta(t)}{\pi^2} \exp\left\{\mathrm{i}\int_{t'}^{t''} \mathrm{d}tL\right\}, \\ L = & [-\dot{\chi}^*(t)\eta(t) - \dot{\chi}(t)\eta^*(t)] - h(\chi(t), 2\eta(t)). \end{aligned} \tag{18}$$

Here $\exp\{\mathrm{i}\int_{t'}^{t''} \mathrm{d}tL\}$ is the action, and L is the Lagrangian. Thus we have defined "paths" in the phase space where the entangled state is defined.

4. Calculating $h(\chi,\eta)$ for the Hamiltonian of the generalized parametric amplifier

The following is a general Hamiltonian in the two-photon process,

$$\begin{aligned} H = \ & \omega_1 a^+ a + \omega_2 b^+ b + g(ab + a^+ b^+) \\ & + f(a^2 + a^{+2}) + k(b^2 + b^{+2}), \end{aligned} \tag{19}$$

where the ω_1, ω_2, g, f and k are real parameters to guarantee the Hamiltonian to be hermitian, when f and k are zero, Eq.(19) reduces to the Hamiltonian of optical parametric amplifier. For convenience, letting

$$\begin{aligned} H_1 &= \omega_1 a^+ a + \omega_2 b^+ b, \\ H_2 &= g(ab + a^+ b^+), \\ H_3 &= f(a^2 + a^{+2}), \\ H_4 &= k(b^2 + b^{+2}), \end{aligned}$$

We have $H = \sum_{i=1}^{4} H_i$. First of all, we calculate

$$h_1 = \int \frac{\mathrm{d}^2\sigma}{\pi} \exp[-\mathrm{i}(\chi^*\sigma + \chi\sigma^*)/2] \Big\langle \eta + \frac{\sigma}{2} \Big| H_1 \Big| \eta - \frac{\sigma}{2} \Big\rangle. \tag{20}$$

Let $\alpha = \eta - \dfrac{\sigma}{2}$, Eq.(20) becomes

$$h_1 = 4\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \int \frac{\mathrm{d}^2\alpha}{\pi} \exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)] \langle 2\eta - \alpha | H_1 | \alpha \rangle. \tag{21}$$

Note that $|\alpha\rangle$ belongs to the entangled state $|\eta\rangle$, from Eq.(1) we see

$$\begin{aligned} a^+|\alpha\rangle &= \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right)|\alpha\rangle, \quad a|\alpha\rangle = -\left(\frac{\partial}{\partial\alpha^*} - \frac{\alpha}{2}\right)|\alpha\rangle, \\ b^+|\alpha\rangle &= -\left(\frac{\partial}{\partial\alpha^*} + \frac{\alpha}{2}\right)|\alpha\rangle, \quad b|\alpha\rangle = \left(\frac{\partial}{\partial\alpha} - \frac{\alpha^*}{2}\right)|\alpha\rangle. \end{aligned} \tag{22}$$

Then follows

$$H_1|\alpha\rangle = \left[(\omega_1 + \omega_2)\left(\frac{|\alpha|^2}{4} - \frac{1}{2} - \frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) + \frac{\omega_1 - \omega_2}{2}\left(\alpha\frac{\partial}{\partial\alpha} - \alpha^*\frac{\partial}{\partial\alpha^*}\right)\right]|\alpha\rangle \equiv \mathcal{H}_1|\alpha\rangle. \tag{23}$$

We let $\mathcal{H}_1 = \mathcal{H}_{11} + \mathcal{H}_{12}$, where

$$\mathcal{H}_{11} = (\omega_1 + \omega_2)\left(\frac{|\alpha|^2}{4} - \frac{1}{2} - \frac{\partial^2}{\partial\alpha\partial\alpha^*}\right), \quad \mathcal{H}_{12} = \frac{\omega_1 - \omega_2}{2}\left(\alpha\frac{\partial}{\partial\alpha} - \alpha^*\frac{\partial}{\partial\alpha^*}\right). \tag{24}$$

Inserting the completeness relation of $|\chi\rangle|_{\chi=\xi}$ into Eq.(21) and using the overlap relation $\langle\xi|\alpha\rangle = \dfrac{1}{2}\exp[-\mathrm{i}(\xi\alpha^* + \alpha\xi^*)/2]$, we have $h_1 = h_{11} + h_{12}$, where

$$\begin{aligned} h_{11} \equiv \ & 4\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \iint \frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2} \exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)] \langle 2\eta - \alpha|\xi\rangle\langle\xi|\mathcal{H}_{11}|\alpha\rangle \\ = \ & \exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \iint \frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2} \exp\left[\frac{\mathrm{i}}{2}(\xi(2\eta - \alpha)^* + \xi^*(2\eta - \alpha))\right] \\ & \times \exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)] \times (\omega_1 + \omega_2)\left(\frac{|\alpha|^2}{4} - \frac{1}{2} - \frac{\partial^2}{\partial\alpha\partial\alpha^*}\right) \exp\left[-\frac{\mathrm{i}}{2}(\alpha\xi^* + \xi\alpha^*)\right] \\ = \ & \exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \iint \frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}\left(\frac{|\alpha|^2 + |\xi|^2}{4} - \frac{1}{2}\right)(\omega_1 + \omega_2) \\ & \times \exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*) + \mathrm{i}(\xi(\eta^* - \alpha^*) + \xi^*(\eta - \alpha))] \\ = \ & \exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \int \mathrm{d}^2\alpha \exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)]\left(-\frac{1}{4}\frac{\partial^2}{\partial\alpha\partial\alpha^*} + \frac{|\alpha|^2}{4}\right) \\ & \times\delta^{(2)}(\eta - \alpha)(\omega_1 + \omega_2) - \frac{\omega_1 + \omega_2}{2} \\ = \ & -\frac{1}{4}(\omega_1 + \omega_2)\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)]\frac{\partial^2}{\partial\alpha\partial\alpha^*}\exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)]|_{\alpha=\eta,\alpha^*=\eta^*} + (\omega_1 + \omega_2)\left(\frac{|\eta|^2}{4} - \frac{1}{2}\right) \\ = \ & \frac{(\omega_1 + \omega_2)}{2}\left[\frac{1}{2}(|\chi|^2 + |\eta|^2) - 1\right]. \end{aligned} \tag{25}$$

Note that the meaning of $\delta^{(2)}(\eta-\alpha)$ is $\delta(\eta_1-\alpha_1)\delta(\eta_2-\alpha_2)$. In a similar manner we find

$$\begin{aligned}
h_{12} \equiv & \ 4\exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\iint\frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)]\langle 2\eta-\alpha|\xi\rangle\langle\xi|\mathcal{H}_{12}|\alpha\rangle \\
= & \ \exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\iint\frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}\exp\left[\frac{\mathrm{i}}{2}(\xi(2\eta-\alpha)^*+\xi^*(2\eta-\alpha))\right] \\
& \times\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)]\times\left(\frac{\omega_1-\omega_2}{2}\right)\left(\alpha\frac{\partial}{\partial\alpha}-\alpha^*\frac{\partial}{\partial\alpha^*}\right)\exp\left[-\frac{\mathrm{i}}{2}(\alpha\xi^*+\xi\alpha^*)\right] \\
= & \ \frac{\omega_1-\omega_2}{2}\exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\iint\frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}\exp\left[-\frac{\mathrm{i}}{2}(\alpha\xi^*-\xi\alpha^*)\right. \\
& \left.\times\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)+\mathrm{i}(\xi(\eta^*-\alpha^*)+\xi^*(\eta-\alpha))\right] \\
= & \ \frac{\omega_1-\omega_2}{4}\exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\int\mathrm{d}^2\alpha\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)]\times\left(\frac{\alpha}{2}\frac{\partial}{\partial\alpha}-\frac{\alpha^*}{2}\frac{\partial}{\partial\alpha^*}\right)\delta^{(2)}(\eta-\alpha) \\
= & \ -\frac{\omega_1-\omega_2}{4}\exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\left[\frac{\partial}{\partial\alpha}\alpha\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)]|_{\alpha=\eta}-\frac{\partial}{\partial\alpha^*}\alpha^*\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)]|_{\alpha^*=\eta^*}\right] \\
= & \ -\frac{\mathrm{i}}{4}(\omega_1-\omega_2)(\eta\chi^*-\chi\eta^*), \qquad (26)
\end{aligned}$$

so

$$h_1(\chi,\eta)=\frac{1}{2}(\omega_1+\omega_2)\left[\frac{1}{2}(|\chi|^2+|\eta|^2)-1\right]-\frac{\mathrm{i}}{4}(\omega_1-\omega_2)(\eta\chi^*-\eta^*\chi). \qquad (27)$$

Next we calculate

$$\begin{aligned}
h_2 \equiv & \int\frac{\mathrm{d}^2\sigma}{\pi}\exp\left[-\frac{\mathrm{i}}{2}(\chi^*\sigma-\chi\sigma^*)\right]\left\langle\eta+\frac{\sigma}{2}\middle|H_2\middle|\eta-\frac{\sigma}{2}\right\rangle \\
= & \ 4\exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\int\frac{\mathrm{d}^2\alpha}{\pi}\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)]\langle 2\eta-\alpha|g(ab+a^+b^+)|\alpha\rangle. \qquad (28)
\end{aligned}$$

According to Eq.(22) we have

$$(ab+a^+b^+)|\alpha\rangle=\left(-2\frac{\partial^2}{\partial\alpha\partial\alpha^*}-\frac{|\alpha|^2}{2}\right)|\alpha\rangle. \qquad (29)$$

Inserting the complete relation of $|\chi\rangle|_{\chi=\xi}$ into Eq.(27) and using Eq.(28) we have

$$\begin{aligned}
h_2 = & \ 4g\exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\iint\frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)]\langle 2\eta-\alpha|\xi\rangle\left\langle\xi\middle|\left(-2\frac{\partial^2}{\partial\alpha\partial\alpha^*}-\frac{|\alpha|^2}{2}\right)\middle|\alpha\right\rangle \\
= & \ g\exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\iint\frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}\exp\left(\frac{|\xi|^2}{2}-\frac{|\alpha|^2}{2}\right)\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)+\mathrm{i}(\xi(\eta^*-\alpha^*)+\xi^*(\eta-\alpha))] \\
= & \ g\exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\int\mathrm{d}^2\alpha\exp[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)]\left(-\frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}-\frac{|\alpha|^2}{2}\right)\delta^{(2)}(\eta-\alpha) \\
= & \ g\left[-\frac{1}{2}\exp[-\mathrm{i}(\eta\chi^*+\chi\eta^*)]\frac{\partial^2}{\partial\alpha\partial\alpha^*}\exp\left[\mathrm{i}(\alpha\chi^*+\chi\alpha^*)|_{\alpha=\eta,\alpha^*=\eta^*}-\frac{|\eta|^2}{2}\right]\right. \\
= & \ \frac{g}{2}(|\chi|^2-|\eta|^2). \qquad (30)
\end{aligned}$$

Now we calculate H_3. From Eq.(22) we obtain

$$(a^2+a^{+2})|\alpha\rangle=\left(\frac{\partial^2}{\partial\alpha^{*2}}+\frac{\partial^2}{\partial\alpha^2}-\alpha\frac{\partial}{\partial\alpha^*}+\alpha^*\frac{\partial}{\partial\alpha}+\frac{\alpha^2}{4}+\frac{\alpha^{*2}}{4}\right)|\alpha\rangle\equiv\mathcal{H}_3|\alpha\rangle. \qquad (31)$$

We can split $\mathcal{H}_3$ into three parts

$$\mathcal{H}_{31}=f\left(\frac{\partial^2}{\partial\alpha^{*2}}+\frac{\partial^2}{\partial\alpha^2}\right),\quad \mathcal{H}_{32}=f\left(-\alpha\frac{\partial}{\partial\alpha^*}+\alpha^*\frac{\partial}{\partial\alpha}\right),\quad \mathcal{H}_{33}=\left(\frac{\alpha^2}{4}+\frac{\alpha^{*2}}{4}\right). \qquad (32)$$

Considering the similarity between $\mathcal{H}_{12}$ and $\mathcal{H}_{32}$, with the help of Eq.(25), we can easily obtain

$$\begin{aligned} h_{32} = & \ 4f\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \iint \frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}\exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)]\langle 2\eta - \alpha|\xi\rangle\langle\xi| \left(-\alpha\frac{\partial}{\partial\alpha^*} + \alpha^*\frac{\partial}{\partial\alpha}\right)|\alpha\rangle \\ = & \ -\frac{\mathrm{i}}{2}f(\eta^*\chi^* - \chi\eta). \end{aligned} \tag{33}$$

Now we calculate h_{31} and h_{33},

$$\begin{aligned} h_{31} = & \ 4f\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \iint \frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}\exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)]\langle 2\eta - \alpha|\xi\rangle\langle\xi| \left(\frac{\partial^2}{\partial\alpha^{*2}} + \frac{\partial^2}{\partial\alpha^2}\right)|\alpha\rangle \\ = & \ \frac{f}{4}\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \iint \frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}\exp(-\xi^2 - \xi^{*2})\exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*) + \mathrm{i}(\xi(\eta^* - \alpha^*) + \xi^*(\eta - \alpha))] \\ = & \ \frac{f}{4}\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \int \mathrm{d}^2\alpha\exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)] \left(\frac{\partial^2}{\partial\alpha^{*2}} + \frac{\partial^2}{\partial\alpha^2}\right)\delta^{(2)}(\eta - \alpha) \\ = & \ \frac{f}{4}\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \left[\frac{\partial^2}{\partial\alpha^{*2}}\exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)]|_{\alpha=\eta} + \frac{\partial^2}{\partial\alpha^2}\exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)]|_{\alpha^*=\eta^*}\right] \\ = & \ -\frac{f}{4}(\chi^2 + \chi^{*2}), \end{aligned} \tag{34}$$

and

$$\begin{aligned} h_{33} = & \ \frac{f}{4}\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \iint \frac{\mathrm{d}^2\alpha\mathrm{d}^2\xi}{\pi^2}(\alpha^2 + \alpha^{*2})\exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*) + \mathrm{i}(\xi(\eta^* - \alpha^*) + \xi^*(\eta - \alpha))] \\ = & \ \frac{f}{4}\exp[-\mathrm{i}(\eta\chi^* + \chi\eta^*)] \int \mathrm{d}^2\alpha\exp[\mathrm{i}(\alpha\chi^* + \chi\alpha^*)](\alpha^2 + \alpha^{*2})\delta^{(2)}(\eta - \alpha) \\ = & \ \frac{f}{4}(\eta^2 + \eta^{*2}). \end{aligned} \tag{35}$$

Thus we obtain

$$h_3 = -\frac{f}{4}(\chi^2 + \chi^{*2}) - \frac{\mathrm{i}}{2}f(\eta^*\chi^* - \eta\chi) + \frac{f}{4}(\eta^2 + \eta^{*2}). \tag{36}$$

Finally we calculate H_4. From Eq.(22), we have

$$(b^2 + b^{+2})|\alpha\rangle = \left(\frac{\partial^2}{\partial\alpha^{*2}} + \frac{\partial^2}{\partial\alpha^{*2}} + \alpha\frac{\partial}{\partial\alpha^*} - \alpha^*\frac{\partial}{\partial\alpha} + \frac{\alpha^2}{4} + \frac{\alpha^{*2}}{4}\right)|\alpha\rangle \equiv \mathcal{H}_4|\alpha\rangle, \tag{37}$$

then divide $\mathcal{H}_4$ into three parts,

$$\begin{aligned} \mathcal{H}_{41} &= k\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\alpha^{*2}}\right) = \mathcal{H}_{31}, \\ \mathcal{H}_{42} &= k\left(\alpha\frac{\partial}{\partial\alpha^*} - \alpha^*\frac{\partial}{\partial\alpha}\right) = -\mathcal{H}_{32}, \\ \mathcal{H}_{43} &= k\left(\frac{\alpha^2}{4} + \frac{\alpha^{*2}}{4}\right) = \mathcal{H}_{33}. \end{aligned} \tag{38}$$

Similar to h_3, we obtain h_4,

$$h_4 = -\frac{k}{4}(\chi^2 + \chi^{*2}) + \frac{\mathrm{i}}{2}k(\eta^*\chi^* - \eta\chi) + \frac{k}{4}(\eta^2 + \eta^{*2}). \tag{39}$$

Combining Eqs.(27), (30), (36) and (39), we finally obtain $h(\chi,\eta) = h_1 + h_2 + h_3 + h_4$,

$$\begin{aligned} h(\chi,\eta) = & \ \frac{1}{2}(\omega_1 + \omega_2)\left[\frac{1}{2}(|\chi|^2 + |\eta|^2) - 1\right] - \frac{\mathrm{i}(\omega_1 - \omega_2)}{4}(\eta\chi^* - \eta\chi^*) + \frac{g}{2}(|\chi|^2 - |\eta|^2) \\ & -\frac{(f+k)}{4}(\chi^2 + \chi^{*2}) - \frac{\mathrm{i}}{2}(f-k)(\eta^*\chi^* - \eta\chi) + \frac{(f+k)}{4}(\eta^2 + \eta^{*2}). \end{aligned} \tag{40}$$

5. Lagrangian in the path integral formalism

Using Eq.(40) we can directly write down $h\left(\frac{\chi_j+\chi_{j-1}}{2}, 2\eta_j\right)$, and then substituting it into Eq.(17), we have

$$\begin{aligned}\langle\chi_j|\mathrm{e}^{-\mathrm{i}\varepsilon H}|\chi_{j-1}\rangle = \ & \Theta(\chi_j,\chi_{j-1})\int\frac{\mathrm{d}^2\eta_j}{\pi}\exp\left\{-\mathrm{i}\varepsilon(\omega_1+\omega_2-2g)|\eta_j|^2-\eta_j\left[\mathrm{i}(\chi_j^*-\chi_{j-1}^*)+\frac{\varepsilon}{2}(\omega_1-\omega_2)\right.\right.\\ & \times\left(\frac{\chi_j+\chi_{j-1}}{2}\right)^*-\varepsilon(f-k)\left(\frac{\chi_j+\chi_{j-1}}{2}\right)\Big]+\eta_j^*\left[-\mathrm{i}(\chi_j-\chi_{j-1})+\frac{\varepsilon}{2}(\omega_1-\omega_2)\right.\\ & \left.\left.\times\left(\frac{\chi_j+\chi_{j-1}}{2}\right)-\varepsilon(f-k)\left(\frac{\chi_j+\chi_{j-1}}{2}\right)^*\right]-\mathrm{i}\varepsilon(f+k)\eta_j^2-\mathrm{i}\varepsilon(f+k)\eta_j^{*2}\right\},\end{aligned}\tag{41}$$

where $\Theta(\chi_j,\chi_{j-1})$ is given by

$$\begin{aligned}\Theta(\chi_j,\chi_{j-1}) = \ & \exp\left(-\mathrm{i}\varepsilon\left\{-\frac{1}{4}(\omega_1+\omega_2-2g)\left|\frac{\chi_j+\chi_{j-1}}{2}\right|^2+\frac{1}{4}(f+k)\left[\left(\frac{\chi_j+\chi_{j-1}}{2}\right)^2\right.\right.\right.\\ & \left.\left.\left.+\left(\frac{\chi_j+\chi_{j-1}}{2}\right)^{*2}\right]+\frac{1}{2}(\omega_1+\omega_2)\right\}\right).\end{aligned}\tag{42}$$

Using the integral formula

$$\int\frac{\mathrm{d}^2z}{\pi}\exp\left[\lambda|z|^2+fz^2+gz^{*2}+\xi z+\eta z^*\right]=\frac{1}{\sqrt{\lambda^2-4fg}}\exp\left[\frac{-\lambda\xi\eta+\xi^2g+\eta^2f}{\lambda^2-4fg}\right],\tag{43}$$

we further calculate

$$\begin{aligned}\langle\chi_j|\mathrm{e}^{-\mathrm{i}\varepsilon H}|\chi_{j-1}\rangle =& \Theta(\chi_j,\chi_{j-1})\frac{1}{\sqrt{-(\omega_1+\omega_2-2g)^2+4(f+k)^2}}\exp\left\{\frac{-\mathrm{i}\varepsilon}{-(\omega_1+\omega_2-2g)^2+4(f+k)^2}\right.\\ & \times\left[(\omega_1+\omega_2-2g)|\frac{(\omega_1-\omega_2)}{2}\left(\frac{\chi_j+\chi_{j-1}}{2}\right)-(f-k)\left(\frac{\chi_j+\chi_{j-1}}{2}\right)^*-\mathrm{i}\frac{\chi_j-\chi_{j-1}}{\varepsilon}|^2\right.\\ & -\left[\mathrm{i}\left(\frac{\chi_j-\chi_{j-1}}{\varepsilon}\right)^*+\frac{(\omega_1-\omega_2)}{2}\left(\frac{\chi_j+\chi_{j-1}}{2}\right)^*-(f-k)\left(\frac{\chi_j+\chi_{j-1}}{2}\right)\right](f+k)\\ & \left.+\left[-\mathrm{i}\left(\frac{\chi_j-\chi_{j-1}}{\varepsilon}\right)^*+\frac{(\omega_1-\omega_2)}{2}\left(\frac{\chi_j+\chi_{j-1}}{2}\right)-(f-k)\left(\frac{\chi_j+\chi_{j-1}}{2}\right)^*\right](f+k)\right\}.\end{aligned}\tag{44}$$

By denoting

$$A=(\omega_1+\omega_2-2g),\quad B=-(\omega_1+\omega_2-2g)^2+4(f+k)^2,\tag{45}$$

and calculating the Feynman transition matrix element

$$\begin{aligned}\langle\chi_j|\mathrm{e}^{-\mathrm{i}\varepsilon H}|\chi_{j-1}\rangle = \ & \frac{1}{\sqrt{B}}\left\{\mathrm{i}\varepsilon\left[-\frac{\omega_1+\omega_2+2g}{4}+\frac{A(\omega_1-\omega_2)^2}{4B}+\frac{A}{B}(f-k)^2+\frac{2(\omega_1-\omega_2)(f^2-k^2)}{B}\right]\right.\\ & \times\left|\frac{\chi_j-\chi_{j-1}}{2}\right|^2-\left[-\frac{f+k}{4}+\frac{A(\omega_1-\omega_2)(f-k)}{2B}+\frac{(f^2-k^2)}{B}+\frac{(\omega_1-\omega_2)(f+k)}{4B}\right]\\ & \times\left[\left(\frac{\chi_j+\chi_{j-1}}{2}\right)^2+\left(\frac{\chi_j+\chi_{j-1}}{2}\right)^{*2}\right]-\mathrm{i}\left[\frac{A}{B}(f-k)+\frac{(f+k)}{B}(\omega_1-\omega_2)\right.\\ & \left.+\frac{2(f^2-k^2)}{B}\right]\left(\frac{\chi+\chi_{j-1}}{2}\right)^*\left(\frac{\chi-\chi_{j-1}}{\varepsilon}\right)^*+\mathrm{i}\left[\frac{A}{2B}(\omega_1-\omega_2)+\frac{(f+k)}{B}(\omega_1-\omega_2)\right.\\ & \left.+\frac{2(f^2-k^2)}{B}\right]\left(\frac{\chi+\chi_{j-1}}{2}\right)\left(\frac{\chi-\chi_{j-1}}{\varepsilon}\right)^*+\frac{\mathrm{i}}{B}(f-k)\left(\frac{\chi+\chi_{j-1}}{2}\right)\left(\frac{\chi-\chi_{j-1}}{\varepsilon}\right)\\ & -\frac{\mathrm{i}}{2B}(\omega_1-\omega_2)\left(\frac{\chi+\chi_{j-1}}{2}\right)^*\left(\frac{\chi-\chi_{j-1}}{\varepsilon}\right)+\frac{1}{2}(\omega_1+\omega_2)\\ & \left.+\frac{2(f+k)}{B}\left(\frac{\chi_j-\chi_{j-1}}{\varepsilon}\right)^{*2}+\frac{A}{B}\left|\frac{\chi_j-\chi_{j-1}}{\varepsilon}\right|^2\right\}.\end{aligned}\tag{46}$$

Therefore the transition amplitude is obtained

$$\langle\chi'',t''|\chi',t'\rangle=\int\prod_t\left[\frac{\mathrm{d}^2\chi(t)}{\pi^2}\right]\exp\left\{\mathrm{i}\int_{t'}^{t''}\mathrm{d}tL\right\},\tag{47}$$

where the Lagrangian is

$$\begin{aligned}L=&\left[-\frac{\omega_1+\omega_2+2g}{4}+\frac{A(\omega_1-\omega_2)^2}{4B}+\frac{A}{B}(f-k)^2+\frac{2(\omega_1-\omega_2)(f^2-k^2)}{B}\right]|\chi(t)|^2\\&-\left[-\frac{f+k}{4}+\frac{A(\omega_1-\omega_2)(f-k)}{2B}+\frac{(f^2-k^2)}{B}+\frac{(\omega_1-\omega_2)(f+k)}{4B}\right][\chi(t)^2+\chi(t)^{*2}]\\&-\mathrm{i}\left[\frac{A}{B}(f-k)+\frac{(f+k)}{B}(\omega_1-\omega_2)+\frac{2(f^2-k^2)}{B}\right]\chi(t)^*\dot{\chi}(t)^*\\&+\mathrm{i}\left[\frac{A}{2B}(\omega_1-\omega_2)+\frac{(f+k)}{B}(\omega_1-\omega_2)+\frac{2(f^2-k^2)}{B}\right]\chi(t)\dot{\chi}(t)^*+\frac{\mathrm{i}}{\mathrm{B}}(f-k)\chi(t)\dot{\chi}(t)\\&-\frac{\mathrm{i}(\omega_1-\omega_2)}{2B}\chi(t)^*\dot{\chi}(t)+\frac{2(f+k)}{B}\dot{\chi}(t)^{*2}+\frac{A}{B}|\dot{\chi}(t)|^2+\frac{(\omega_1+\omega_2)}{2}.\end{aligned}\tag{48}$$

As a concrete example of the above formalism, we consider the path integral for nondegenerate parametric amplifiers in entangled state representations. The following is a Hamiltonian describing a nondegenerate parametric amplifier,

$$H=\omega(a^+a+b^+b)+g(ab+a^+b^+).\tag{49}$$

Comparing Eq.(49) with Eq.(19), here $\omega_1=\omega_2=\omega$ and $f=k=0$, so the Lagrangian for a nondegenerate parametric amplifier is

$$L=\omega-\frac{1}{2}(\omega+g)\chi(t)\chi(t)^*-\frac{1}{2(g-\omega)}\dot{\chi}(t)\dot{\chi}(t)^*.\tag{50}$$

The classical Euler–Lagrange equations are

$$\ddot{\chi}(t)^*-(\omega^2-g^2)\chi(t)^*=0,\tag{51}$$

$$\ddot{\chi}(t)-(\omega^2-g^2)\chi(t)=0.\tag{52}$$

To confirm the correctness of our approach, we use the Heisenberg equation and Eq.(19) to derive ($\hbar=1$)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(a^++b)=[a^++b,H]=(g-\omega)(a^+-b),\tag{53}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(a^++b)=(g-\omega)\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}(a^+-b)=(\omega^2-g^2)(a^++b).\tag{54}$$

Since $|\chi\rangle$ is the eigenstate of (a^++b), Eq.(54) just corresponds to (52). This correspondence confirms the feasibility of our path integral approach.

In summary, we have established a path integral formalism for quantum mechanical Hamiltonians. It seems to us that for entangled systems, the best way for constructing path integral is to work in the entangled state representation.

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