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Density matrix for an electron confined in quantum dots under uniform magnetic field and static electrical field

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Using unitary transformations, this paper obtains the eigenvalues and the common eigenvector of Hamiltonian and a new-defined generalized angular momentum (L_z) for an electron confined in quantum dots under a uniform magnetic field (UMF) and a static electric field (SEF). It finds that the eigenvalue of L_z just stands for the expectation value of a usual angular momentum l_z in the eigen-state. It first obtains the matrix density for this system via directly calculating a transfer matrix element of operator $\exp(-\beta H)$ in some representations with the technique of integral within an ordered products (IWOP) of operators, rather than via solving a Bloch equation. Because the quadratic homogeneity of potential energy is broken due to the existence of SEF, the virial theorem in statistical physics is not satisfactory for this system, which is confirmed through the calculation of thermal averages of physical quantities.

Keywords: unitary transformation, IWOP technique, density matrix in (x_1, p_2) representation, coherent rotational-entangled state

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1. Introduction

Feynman had pointed out that the concept of the density matrix plays an important role in the reformulation quantum mechanics because pure states are not general enough to describe a quantum-mechanical system.[1] The density matrix for a definite Hamiltonian H in an equilibrium state is defined as $\exp(-\beta H) = \rho$, where $\beta = (k_{\rm B}T)^{-1}$ and $k_{\rm B}$ is the Boltzmann constant. The density matrix in the coordinate representation can be obtained by solving a Bloch equation, which was used to derive the density matrix for a harmonic oscillator by Feynman.[1] However, as is well known, only for few physical systems can the density matrix be exactly obtained by solving the Bloch equation. In this work, we derive the density matrix via directly calculating a transfer matrix element of operator $\exp(-\beta H)$ in some representation. In the calculation, we use a powerful tool called the technique of integral within an ordered products ($IWOP$) of operators,[2] which makes the complicated incommutable operation of operators becoming a simple integral within an ordered products of operators.

Firstly we search for the unitary transformation (UT) diagonalizing the Hamiltonian of an electron confined in quantum dots under a uniform magnetic field (UMF) and a static electric field (SEF). Then we obtain the eigenvalues and the common eigenvector of Hamiltonian and a new-defined generalized angular momentum (L_z). We find that the eigenvalue of L_z just stands for the expectation value of a usual angular momentum l_z in the common eigenstate of H and L_z.

Using the UT method and the IWOP technique, we first obtain the matrix density for this system. For the need of calculating thermal averages of physical quantities, we derive respectively the density matrices in the coordinate representation, momentum representation and their mixed representation in particular. We find that the thermal average of each calculated physical quantity composes of two parts: one is dependent on the temperature and UMF, which is the same as the result[3] for an electron confined in quantum dots under UMF (without SEF); the other part is independent on the temperature, but dependent on both UMF and SEF, which is an additional item due to the existence of SEF. We also prove that the thermal average of total potential energy is unequal to the thermal average of total kinetic energy. Because the quadratic homogeneity of potential energy is broken due to the existence of SEF, the virial theorem in statistical physics is not satisfactory to our system.

This work is arranged as follows. Unitary trans-

formations, energy spectrum and generalized angular momentum of the system are given in Section 2. Density matrix in coordinate or momentum representation is derived in Section 3. Density matrix in coordinate-momentum mixed representation is obtained in Section 4. Thermal averages of physical quantities are derived in Section 5.

2. Unitary transformations, energy spectrum and generalized angular momentum

We know that, with an effective mass approximation, Hamiltonian of an electron confined in quantum dots under a UMF in z direction and a static electric field (SEF) in x direction is

$$H=\frac{1}{2m^*}(\boldsymbol{p}+e\boldsymbol{A})^2+\frac{1}{8}m^*\omega_0^2(x^2+y^2)+eEx, \quad (1)$$

where m^* is the effective mass of an electron and $\boldsymbol{A}$ is the vector potential of the magnetic field. Using the symmetrical gauge of $\boldsymbol{A}$, we can represent Hamiltonian as

$$H=\frac{p_x^2}{2m^*}+\frac{p_y^2}{2m^*}+\frac{m^*}{8}(\Omega^2+\omega_0^2)(x^2+y^2) +\frac{Be}{2m^*}(xp_y-yp_x)+eEx, \quad (2)$$

where $\Omega=Be/m^*$ and the electron motion in z direction is not included. For convenience, we set

$$\sqrt{\frac{m^*\Omega'}{\hbar}}x=x_1,\quad \sqrt{\frac{m^*\Omega'}{\hbar}}y=x_2,$$
$$\frac{p_x}{\sqrt{m^*\hbar\Omega'}}=p_1,\quad \frac{p_y}{\sqrt{m^*\hbar\Omega'}}=p_2, \quad (3)$$

where

$$\Omega'=\frac{1}{2}\sqrt{\Omega^2+\omega_0^2}$$

and x_1, x_2, p_1, p_2 are dimensionless position and momentum operators of a two dimensional (2D) electrons. They can be expressed as

$$x_1=\frac{1}{\sqrt{2}}(a+a^\dagger),\quad x_2=\frac{1}{\sqrt{2}}(b+b^\dagger),$$
$$p_1=\frac{1}{\sqrt{2}\mathrm{i}}(a-a^\dagger),\quad p_2=\frac{1}{\sqrt{2}\mathrm{i}}(b-b^\dagger), \quad (4)$$

where $(a,a^\dagger)$ and $(b,b^\dagger)$ are annihilation and creation operators of a 2D electron respectively on x and y dimensions. As shown later, the introduction of these dimensionless operators will bring us much convenience for following calculation. Using Eq.(3), we change Eq.(2) into

$$H=\hbar\Omega'\Big[\frac{1}{2}(x_1^2+p_1^2)+\frac{1}{2}(x_2^2+p_2^2)\Big] +\frac{\hbar\Omega}{2}(x_1p_2-x_2p_1)+2\hbar kx_1\,, \quad (5)$$

where

$$k=\frac{eE}{2\sqrt{m^*\hbar\Omega'}}. \quad (6)$$

This simplified form Eq.(5) of Hamiltonian will become the starting point of this work. Firstly we deal with the eigenstate problem of the system by using UT method. We can prove that

$$H=UV(H_0)V^{-1}U^{-1}, \quad (7)$$

where

$$H_0=\hbar\omega_1\Big(a^\dagger a+\frac{1}{2}\Big)+\hbar\omega_2\Big(b^\dagger b+\frac{1}{2}\Big) -\hbar k^2\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big), \quad (8)$$

$$\omega_1=\Omega'\Big(1-\frac{\Omega}{2\Omega'}\Big),\quad \omega_2=\Omega'\Big(1+\frac{\Omega}{2\Omega'}\Big) \quad (9)$$

and unitary transformations V and U are respectively defined as

$$V=\exp\Big[\frac{k}{\omega_1}(a^\dagger-a)+\mathrm{i}\frac{k}{\omega_2}(b^\dagger+b)\Big], \quad (10)$$

$$U=\int \mathrm{d}x_1\mathrm{d}p_2\left|\boldsymbol{u}\begin{pmatrix}x_1\\p_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}x_1\\p_2\end{pmatrix}\right|,$$
$$\boldsymbol{u}=\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}. \quad (11)$$

In Eq.(11),

$$\left|\begin{pmatrix}x_1\\p_2\end{pmatrix}\right\rangle=|x_1\rangle|p_2\rangle.$$

$|x_1\rangle$ and $|p_2\rangle$ are respectively the eigenvectors of operators x_1 and p_2. From Eqs.(10) and (11), we can prove that

$$VaV^{-1}=a-\frac{k}{\omega_1}\,,\quad VbV^{-1}=b-\mathrm{i}\frac{k}{\omega_2};$$
$$UaU^{-1}=\frac{-(a+\mathrm{i}b)}{\sqrt{2}},\quad UbU^{-1}=\frac{(a-\mathrm{i}b)}{\sqrt{2}\mathrm{i}}\,. \quad (12)$$

In terms of Eqs.(7) and (8), we immediately obtain the eigenvalue of Hamiltonian

$$E=\hbar\omega_1(n_1+\frac{1}{2})+\hbar\omega_2\Big(n_2+\frac{1}{2}\Big)-\hbar k^2\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big), \quad (13)$$

where $n_1,n_2=0,1,2,\cdots$, ω_1 and ω_2 are expressed by Eq.(9) and k is expressed by Eq.(6). When $k=0$, Eq.(13) reduces to the Fock–Darwin energy level.[3]

We find that the electric field E only produces a constant energy shift for eigenstates of the system. On the other hand, we can write the eigenvector $|E\rangle_{n_1,n_2}$ of Hamiltonian as

$$|E\rangle_{n_1,n_2} = UV|n_1, n_2\rangle, \tag{14}$$

where $|n_1, n_2\rangle$ is the eigenvector of a two-mode uncoupled oscillator system in Fock space

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}} a^{\dagger n_1} b^{\dagger n_2}|0, 0\rangle. \tag{15}$$

By the way, we can construct an operator as follows:

$$\begin{aligned} L_z =& UV\Big[\hbar(b^\dagger b - a^\dagger a) \\ & - \hbar k^2\Big(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\Big)\Big]V^{-1}U^{-1}. \end{aligned} \tag{16}$$

Considering Eqs.(10–12) and (4), we can rewrite Eq.(16) into

$$\begin{aligned} L_z =& \hbar(x_1p_2 - x_2p_1) + \hbar k p_2\Big(\frac{1}{\omega_2} + \frac{1}{\omega_1}\Big) \\ & + \hbar k x_1\Big(\frac{1}{\omega_2} - \frac{1}{\omega_1}\Big), \end{aligned} \tag{17}$$

where $\hbar(x_1p_2 - x_2p_1) = xp_y - yp_x \equiv l_z$ just stands for the angular momentum in z direction. Obviously, there is the commutation relation between L_z and H,

$$[L_z, H] = 0. \tag{18}$$

So there is a common eigenvector of L_z and H, which is just expressed by Eq.(14). The eigenvalue of L_z corresponding to Eq.(14) is

$$L_z = \hbar(n_2 - n_1) - \hbar k^2\Big(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\Big). \tag{19}$$

We call L_z a generalized angular momentum in z direction for this system. From Eq.(17), we can prove that the eigenvalue of operator L_z shown in Eq.(19) is just equal to the expectation value of the angular momentum l_z in eigenstate Eq.(14).

3. Density matrix in coordinate or momentum representation

We know that the density matrix for a system in coordinate representation is defined as

$$\rho(y_1, y_2; x_1, x_2; \beta) = \langle y_1, y_2| \exp(-\beta H)|x_1, x_2\rangle, \tag{20}$$

$|x_1, x_2\rangle = |x_1\rangle|x_2\rangle$, $|x_1\rangle$ and $|x_2\rangle$ are respectively the eigenvectors of coordinate operators x_1 and x_2 shown by Eq.(4). Generally, $\rho(y_1, y_2; x_1, x_2; \beta)$ can be obtained by solving a Bloch equation[1] with an initial condition

$$\rho(y_1, y_2; x_1, x_2; 0) = \delta(x_1 - y_1)\delta(x_2 - y_2). \tag{21}$$

Here it will be directly derived via calculating a transfer matrix element of $\exp(-\beta H)$ between two 2-mode coordinate eigenvectors. From Eq.(7), we rewrite Eq.(20) as

$$\rho(y_1, y_2; x_1, x_2; \beta) = \langle y'|\exp(-\beta H_0)|x'\rangle\,, \tag{22}$$

where H_0 is shown in Eq.(8) and

$$\langle y'| = \langle y_1, y_2|UV, \quad |x'\rangle = V^{-1}U^{-1}|x_1, x_2\rangle.$$

In Appendix, we prove that

$$\begin{aligned} \langle y'| =& (\pi)^{-1/2}\exp\Big[\mathrm{i}y_2k\Big(\frac{1}{\omega_1} - \frac{1}{\omega_2}\Big)\Big]\langle 0, 0| \\ & \times \exp\Big[-\frac{|y'|^2}{2} - ay'^* + \mathrm{i}by' + \mathrm{i}ab\Big], \\ y' =& y_1 + \mathrm{i}y_2 + \frac{k}{\omega_1} + \frac{k}{\omega_2}. \end{aligned} \tag{23}$$

Similarly we can obtain the expression of $|x'\rangle$. As represented in Appendix, $\langle y'|$ is a coherent rotational-entangled state. So, from Eqs.(20–23), we see that calculating the matrix element of $\exp(-\beta H)$ between two 2-mode coordinate eigenvectors is changed into calculating the matrix element of $\exp(-\beta H_0)$ between two coherent rotational-entangled states. The principle of the entangled state has been applied in the derivation of density matrix as well as studying quantum optics[4] and solving eigen-state problems.[5]

In the following calculation, we will use the IWOP technique presented by Fan, which makes the complicated incommutable of operation becoming a simple integral within an ordered products of operators. Considering the completeness condition[6] of coherent states

$$\int \frac{\mathrm{d}^2 z_l}{\pi}|z_l\rangle\langle z_l| = 1, \quad (l = 1, 2), \tag{24}$$

where z_l is a complex number, and the important operator formula[7]

$$\begin{aligned} &\exp(-\beta\hbar\omega_1 a^\dagger a) =: \exp[(\mathrm{e}^{-\beta\hbar\omega_1} - 1)a^\dagger a] :, \\ &\exp(-\beta\hbar\omega_2 b^\dagger b) =: \exp[(\mathrm{e}^{-\beta\hbar\omega_2} - 1)b^\dagger b] : . \end{aligned} \tag{25}$$

Substituting Eqs.(24) and (25) into Eq.(22), we have

$$\langle y'|\exp(-\beta H_0)|x'\rangle = \exp\Big[\mathrm{i}k(y_2-x_2)\Big(\frac{1}{\omega_1}-\frac{1}{\omega_2}\Big)\Big]\frac{1}{\pi}\int\int\int\int \frac{\mathrm{d}^2z_1}{\pi}\frac{\mathrm{d}^2z_2}{\pi}\frac{\mathrm{d}^2z_1'}{\pi}\frac{\mathrm{d}^2z_2'}{\pi}$$
$$\times \exp\Big[-\beta\hbar k^2\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big)\Big]\langle 0,0|\exp\Big[-\frac{1}{2}|y'|^2-y'^*a+\mathrm{i}y'b+\mathrm{i}ab\Big]|z_1\rangle\langle z_1|z_2\rangle\langle z_2|$$
$$:\exp\Big[(\mathrm{e}^{-\beta\hbar\omega_1}-1)a^\dagger a+(\mathrm{e}^{-\beta\hbar\omega_2}-1)b^\dagger b-\frac{\beta\hbar(\omega_1+\omega_2)}{2}\Big]:$$
$$\times |z_2'\rangle\langle z_2'|z_1'\rangle\langle z_1'|\exp\Big[-\frac{1}{2}|x'|^2-x'a^\dagger-\mathrm{i}x'^*b^\dagger-\mathrm{i}a^\dagger b^\dagger]|0,0\rangle, \tag{26}$$

where : : means an ordered product of operators. By further using the properties[6] of coherent states

$$a|z\rangle = z|z\rangle\,, \qquad \langle z|z'\rangle = \exp\Big[-\frac{1}{2}(|z|+|z'|)+z^*z'\Big], \tag{27}$$

and the formula of ordered operators[7]

$$|0,0\rangle\langle 0,0| =: \mathrm{e}^{-a^\dagger a-b^\dagger b}: \tag{28}$$

we perform the complex integral in Eq.(26) and obtain

$$\rho(y_1,y_2;x_1,x_2;\beta) = \frac{1}{2\pi\sinh\frac{\beta\hbar(\omega_1+\omega_2)}{2}}\exp\Big[\mathrm{i}k(y_2-x_2)\Big(\frac{1}{\omega_1}-\frac{1}{\omega_2}\Big)\Big]$$
$$\times\exp\Big[\frac{-1}{2\sinh\frac{\beta\hbar(\omega_1+\omega_2)}{2}}[(y_1'^2+y_2'^2)+(x_1'^2+x_2'^2)\Big)\Big]\cosh\frac{\beta\hbar(\omega_1+\omega_2)}{2}\Big]$$
$$\times\exp\Big[\frac{1}{2\sinh\frac{\beta\hbar(\omega_1+\omega_2)}{2}}\Big[2(y_1'x_1'+y_2'x_2')\cosh\frac{\beta\hbar(\omega_1-\omega_2)}{2}\Big]\Big]$$
$$\times\exp\Big[\frac{-1}{2\sinh\frac{\beta\hbar(\omega_1+\omega_2)}{2}}[2\mathrm{i}(y_1'x_2'-y_2'x_1')\sinh\frac{\beta\hbar(\omega_1-\omega_2)}{2}]\Big]\Big], \tag{29}$$

where

$$y_1' = y_1+\frac{k}{\omega_1}+\frac{k}{\omega_2}, \quad y_2'=y_2, \quad x_1'=x_1+\frac{k}{\omega_1}+\frac{k}{\omega_2}, \quad x_2'=x_2\,. \tag{30}$$

Equation (29) is just the density matrix satisfying the initial condition Eq.(21) in coordinate representation for this system. When $x_1=y_1$, $x_2=y_2$ and $\beta\neq 0$, we obtain a very compact expression of Eq.(28) as follows:

$$\rho(x_1,x_2;\beta)\equiv\rho(y_1,y_2;x_1,x_2;\beta)|_{x_1=y_1,x_2=y_2;\beta\neq 0} = \frac{1}{2\pi\sinh\frac{\beta\hbar(\omega_1+\omega_2)}{2}}\exp\Big[\frac{-2(x_1'^2+x_2'^2)}{\coth\frac{\beta\hbar\omega_1}{2}+\coth\frac{\beta\hbar\omega_2}{2}}\Big], \tag{31}$$

where x_1' and x_2' are expressed by Eq.(30). Similarly we can obtain the expression of density matrix in momentum representation as follows:

$$\rho(p_{x_1},p_{x_2};\beta)\equiv\rho(p_{y_1},p_{y_2};p_{x_1},p_{x_2};\beta)|_{p_{x_1}=p_{y_1},p_{x_2}=p_{y_2};\beta\neq 0} = \frac{1}{2\pi\sinh\frac{\beta\hbar(\omega_1+\omega_2)}{2}}$$
$$\times\exp\Big[-\beta\hbar k^2\Big(\frac{1}{\omega_1}+\frac{1}{\omega_2}\Big)\Big]\exp\Big[\frac{-2(p_{x_1}'^2+p_{x_2}'^2)}{\coth\frac{\beta\hbar\omega_1}{2}+\coth\frac{\beta\hbar\omega_2}{2}}\Big], \tag{32}$$

where

$$p_{x_1}' = p_{x_1}, p_{x_2}' = p_{x_2}+k\Big(\frac{1}{\omega_1}-\frac{1}{\omega_2}\Big). \tag{33}$$

4. Density matrix in coordinate-momentum mixed representation

For the calculation in next section, we continue the derivation of the expression of the density matrix in coordinate-momentum mixed representation. We define a density matrix in a coordinate-momentum mixed representation as follows:

$$\rho(y_1,p_{y_2};x_1,p_{x_2};\beta) = \langle y_1,p_{y_2}|\,\exp(-\beta H)|x_1,p_{x_2}\rangle, \tag{34}$$

where $\langle y_1, p_{y_2}| = \langle y_1|\langle p_{y_2}|$, $\langle y_1|$ and $\langle p_{y_2}|$ are eigenvectors of operators y_1 and p_{y_2}, which are shown by

$$\langle y_1| = (\pi)^{-1/4}\langle 0|\exp\Big(-\frac{y_1^2}{2} + \sqrt{2}y_1 a - \frac{a^2}{2}\Big), \quad \langle p_{y_2}| = (\pi)^{-1/4}\langle 0|\exp\Big(-\frac{p_{y_2}^2}{2} - \mathrm{i}\sqrt{2}p_{y_2} b + \frac{b^2}{2}\Big). \tag{35}$$

From Eqs.(7–11), we can easily prove that

$$\rho(y_1, p_{y_2}; x_1, p_{x_2}; \beta) = \langle y_1', p_{y_2}'|\exp(-\beta H_0)|x_1', p_{x_2}'\rangle, \tag{36}$$

where

$$\langle y_1', p_{y_2}'| = \langle y_1, p_{y_2}|UV, \quad |x_1', p_{x_2}'\rangle = V^{-1}U^{-1}|x_1, p_{x_2}\rangle, \tag{37}$$

and

$$\begin{aligned} y_1' &= -\frac{1}{\sqrt{2}}(y_1 - p_{y_2}) - \frac{\sqrt{2}k}{\omega_1}, \quad p_{y_2}' = -\frac{1}{\sqrt{2}}(y_1 + p_{y_2}) - \frac{\sqrt{2}k}{\omega_2}, \\ x_1' &= -\frac{1}{\sqrt{2}}(x_1 - p_{x_2}) - \frac{\sqrt{2}k}{\omega_1}, \quad p_{x_2}' = -\frac{1}{\sqrt{2}}(x_1 + p_{x_2}) - \frac{\sqrt{2}k}{\omega_2}. \end{aligned} \tag{38}$$

In Eqs.(36) and (37), $\langle y_1', p_{y_2}'| = \langle y_1'|\langle p_{y_2}'|$, $\langle y_1'|$ and $\langle p_{y_2}'|$ are also expressed by Eq.(35). They are still the eigenvector of coordinate and momentum operators, the only change is the argument of corresponding eigenvectors. This is a very different form from Eq.(23) obtained in a two-mode coordinate representation. Then the density matrix of the system can be separated into the product of density matrices for two independent oscillator as follows:

$$\begin{aligned} \rho(y_1, p_{y_2}; x_1, p_{x_2}; \beta) = \exp\Big[-\beta\hbar k^2\Big(\frac{1}{\omega_1} + \frac{1}{\omega_2}\Big)\Big]\langle y_1'|\exp\Big[-\beta\hbar\omega_1\Big(a^\dagger a + \frac{1}{2}\Big)\Big] \\ \times |x_1'\rangle \cdot \langle p_{y_2}'|\exp\Big[-\beta\hbar\omega_2(b^\dagger b + \frac{1}{2})\Big]\ |p_{x_2}'\rangle. \end{aligned} \tag{39}$$

By using the formula[1] about the density matrix for a single oscillator in coordinate or momentum representation, we immediately obtain

$$\begin{aligned} \rho(y_1, p_{y_2}; x_1, p_{x_2}; \beta) = \frac{1}{\sqrt{4\pi^2\sinh(\beta\hbar\omega_1)\sinh(\beta\hbar\omega_2)}}\exp\Big[-\beta\hbar k^2\Big(\frac{1}{\omega_1} + \frac{1}{\omega_2}\Big)\Big] \\ \times \exp\Big\{\frac{1}{2\sinh(\beta\hbar\omega_1)}[2y_1'x_1' - \coth(\beta\hbar\omega_1)(y_1'^2 + x_1'^2)]\Big\} \\ \times \exp\Big\{\frac{1}{2\sinh(\beta\hbar\omega_2)}[2p_{y_2}'p_{x_2}' - \coth(\beta\hbar\omega_2)(p_{y_2}'^2 + p_{x_2}'^2)]\Big\}, \end{aligned} \tag{40}$$

where x_1', y_1', p_{x_2}' and p_{y_2}' are expressed by Eq.(38). This is just the expression of density matrix in (x_1, p_{x_2}) mixed representation, which satisfies the initial condition $\rho(y_1, p_{y_2}; x_1, p_{x_2}; 0) = \delta(x_1 - y_1)\delta(p_{x_2} - p_{y_2})$. When $x_1 = y_1$, $p_{x_2} = p_{y_2}$ and $\beta \neq 0$, we have

$$\begin{aligned} \rho(x_1, p_{x_2}; \beta) \equiv \rho(y_1, p_{y_2}; x_1, p_{x_2}; \beta)|_{x_1=y_1, p_{x_2}=p_{y_2}; \beta\neq 0} = \frac{1}{\sqrt{4\pi^2\sinh(\beta\hbar\omega_1)\sinh(\beta\hbar\omega_2)}} \\ \times \exp\Big[-\beta\hbar k^2\Big(\frac{1}{\omega_1} + \frac{1}{\omega_2}\Big)\Big]\exp\Big(-\frac{x_1'^2}{\coth\big(\frac{\beta\hbar\omega_1}{2}\big)} - \frac{p_{x_2}'^2}{\coth\big(\frac{\beta\hbar\omega_2}{2}\big)}\Big), \end{aligned} \tag{41}$$

where x_1' and p_{x_2}' are expressed by Eq.(38). Similarly, we obtain the expression of density matrix in (p_{x_1}, x_2) mixed representation

$$\begin{aligned} \rho(p_{x_1}, x_2; \beta) = \rho(p_{y_1}, y_2, p_{x_1}, x_2; \beta)|_{x_2=y_2, p_{x_1}=p_{y_1}; \beta\neq 0} = \frac{1}{\sqrt{4\pi^2\sinh(\beta\hbar\omega_1)\sinh(\beta\hbar\omega_2)}} \\ \times \exp\Big[-\beta\hbar k^2\Big(\frac{1}{\omega_1} + \frac{1}{\omega_2}\Big)\Big]\exp\Big(-\frac{p_{x_1}'^2}{\coth\big(\frac{\beta\hbar\omega_1}{2}\big)} - \frac{x_2'^2}{\coth\big(\frac{\beta\hbar\omega_2}{2}\big)}\Big), \end{aligned} \tag{42}$$

where

$$p'_{x_1} = \frac{-(p_{x_1} + x_2)}{\sqrt{2}}, \qquad x'_2 = \frac{(p_{x_1} - x_2)}{\sqrt{2}} \tag{43}$$

In the comparison between Eqs.(41), we find that the arguments (p'_{x_1}, x'_2) in Eq.(42) are independent on ω_1 and ω_2.

So far we totally obtain four forms Eqs.(31), (32), (41) and (42) of density matrix for the same system, which respectively correspond to four different representations. In fact, they can be derived with the representation transformation of quantum mechanics[8] from one to another. We will see in the next section that, in order to calculate the thermal averages of physical quantities, a proper selection of the representation will make the calculation much more convenient.

5. Thermal averages of physical quantities

The thermal average of the square $(x^2 + y^2)$ of electron orbit radius is defined as

$$\langle x^2 + y^2 \rangle_\rho = \frac{\hbar}{m^* \Omega'} \langle x_1^2 + x_2^2 \rangle_\rho = \frac{\hbar}{m^* \Omega'} \frac{\mathrm{Tr}[(x_1^2 + x_2^2)\exp(-\beta H)]}{\mathrm{Tr}[\exp(-\beta H)]}, \tag{44}$$

where Eq.(3) is used. By using the density matrix Eq.(31) in coordinate representation, we obtain

$$\langle x^2 + y^2 \rangle_\rho = \frac{\hbar}{m^* \Omega'} \Big[\frac{1}{2} \Big(\coth \frac{\beta\hbar\omega_1}{2} + \coth \frac{\beta\hbar\omega_2}{2} \Big) + \Big(\frac{k}{\omega_1} + \frac{k}{\omega_2} \Big)^2 \Big]. \tag{45}$$

By using the density matrix Eq.(32) in momentum representation, we can calculate $\langle p_x^2/2m^* + p_y^2/2m^* \rangle_\rho$ as follows:

$$\langle \frac{p_x^2}{2m^*} + \frac{p_y^2}{2m^*} \rangle_\rho = \frac{\hbar\Omega'}{2} \langle p_1^2 + p_2^2 \rangle_\rho = \frac{\hbar\Omega'}{2} \Big[\frac{1}{2} \Big(\coth \frac{\beta\hbar\omega_1}{2} + \coth \frac{\beta\hbar\omega_2}{2} \Big) + \Big(\frac{k}{\omega_1} - \frac{k}{\omega_2} \Big)^2 \Big]. \tag{46}$$

By using the density matrix Eqs.(41) and (42) in coordinate-momentum mixed representation, we can easily obtain the thermal average of angular momentum l_z:

$$\langle l_z \rangle_\rho = \langle x p_y - y p_x \rangle_\rho = \hbar \langle x_1 p_{x_2} - x_2 p_{x_1} \rangle_\rho = \frac{\hbar}{2} \Big(\coth \frac{\beta\hbar\omega_2}{2} - \coth \frac{\beta\hbar\omega_1}{2} \Big) + \hbar k^2 \Big(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2} \Big). \tag{47}$$

By using Eq.(30) and the density matrix Eq.(31) in coordinate representation or the density matrix Eq.(41) in the mixed representation, we have

$$\langle eEx \rangle_\rho = -2\hbar k^2 \Big(\frac{1}{\omega_1} + \frac{1}{\omega_2} \Big). \tag{48}$$

Then, synthetically considering Eqs.(45–48), we obtain the thermal average of system Hamiltonian as follows :

$$\langle H \rangle_\rho = \frac{1}{2} \Big(\hbar\omega_1 \coth \frac{\beta\hbar\omega_1}{2} + \hbar\omega_2 \coth \frac{\beta\hbar\omega_2}{2} \Big) - \hbar k^2 \Big(\frac{1}{\omega_1} + \frac{1}{\omega_2} \Big), \tag{49}$$

where ω_1 and ω_2 are expressed by Eq.(9). We find that the thermal average of each calculated physical quantity, shown by Eqs.(45–47) and Eq.(49), composes of two parts: one is dependent on the temperature and UMF, which is the same as the result for an electron confined in quantum dots under UMF (without SEF); the other part is independent on the temperature, but dependent on SEF and UMF, which is an additional item due to the existence of SEF. We also find that

$$\Big\langle \frac{p_x^2}{2m^*} + \frac{p_y^2}{2m^*} \Big\rangle_\rho \neq \Big\langle \frac{m^*}{8} (\Omega^2 + \omega_0^2)(x^2 + y^2) \Big\rangle_\rho + \langle eEx \rangle_\rho, \tag{50}$$

which means that the thermal averages of total potential energy and total kinetic energy are not equal to each other. This is because eEx breaks the quadratic homogeneity of potential energy. Thus virial theorem in statistics physics is not satisfactory for our system.

Appendix

$\left\langle \binom{y_1}{y_2} \right| UV$ is a coherent rotational-entangled state

We firstly observe the state $\left\langle \binom{y_1}{y_2} \right| U$. From Eq.(11), we have

$$\left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right| U = \left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right| \int \mathrm{d}x_1 \mathrm{d}p_2 \left| \times \right| \boldsymbol{u} \begin{pmatrix} x_1 \\ p_2 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} x_1 \\ p_2 \end{pmatrix} \right|, \tag{A1}$$

where

$$\boldsymbol{u} = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

Because of

$$\left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right| \boldsymbol{u} \begin{pmatrix} x_1 \\ p_2 \end{pmatrix} \Big\rangle = \frac{1}{\sqrt{2\pi}} \delta(y_1 + \frac{x_1 + p_2}{\sqrt{2}}) \times \exp\Big(\frac{\mathrm{i}y_2(x_1 - p_2)}{\sqrt{2}}\Big), \tag{A2}$$

setting

$$\langle y_1, y_2| \, U \equiv \left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right| U$$

and performing the integral in Eq.(A1) with IWOP, we obtain

$$\langle y_1, y_2| \, U = \frac{1}{\sqrt{\pi}} \langle 0, 0| \exp\Big(-\frac{|y|^2}{2} - y^* a + \mathrm{i}yb + \mathrm{i}ab\Big), \quad y = y_1 + \mathrm{i}y_2 \,. \tag{A3}$$

We can prove that

$$\langle y_1, y_2|U = \frac{1}{\sqrt{\pi}} \langle 0, 0| \exp\Big(-\frac{|y|^2}{2} + y^* a + yb - ab\Big) R, \tag{A4}$$

where

$$R \equiv R(\theta_1, \theta_2) = \exp[\mathrm{i}(\theta_1 a^\dagger a + \theta_2 b^\dagger b)]; \quad \Big(\theta_1 = \pi, \quad \theta_2 = \frac{\pi}{2}\Big). \tag{A5}$$

Because of

$$R^{-1}(\theta_1, \theta_2) a R(\theta_1, \theta_2) = \mathrm{e}^{\mathrm{i}\theta_1} a, \quad R^{-1}(\theta_1, \theta_2) b R(\theta_1, \theta_2) \mathrm{e}^{\mathrm{i}\theta_2} b, \tag{A6}$$

we know that $R(\theta_1, \theta_2)$ is a rotation transformation to operators a and b. Then we have

$$\langle y_1, y_2| \, UV = \frac{1}{\sqrt{\pi}} \langle 0, 0| \times \exp\Big(-\frac{|y|^2}{2} + y^* a + yb - ab\Big) RV \,, \tag{A7}$$

where V is a typical 2-mode coherent transformation expressed by (10), which we know that

$$\frac{1}{\sqrt{\pi}} \langle 0, 0| \exp\Big(-\frac{|y|^2}{2} + y^* a + yb - ab\Big)$$

stands for an entangled state. So, in terms of Eq.(A7), $\langle y_1, y_2|UV$ is just a coherent rotational-entangled state which can be explicitly expressed as

$$\langle y_1, y_2|UV = (\pi)^{-1/2} \exp\Big[\mathrm{i}y_2 k \Big(\frac{1}{\omega_1} - \frac{1}{\omega_2}\Big)\Big] \langle 0, 0| \times \exp\Big[-\frac{|y'|^2}{2} - a y'^* + \mathrm{i}by' + \mathrm{i}ab\Big], \quad y' = y_1 + \mathrm{i}y_2 + \frac{k}{\omega_1} + \frac{k}{\omega_2}. \tag{A8}$$

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