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Group-theoretical method for physical property tensors of quasicrystals

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In addition to the phonon variable there is the phason variable in hydrodynamics for quasicrystals. These two kinds of hydrodynamic variables have different transformation properties. The phonon variable transforms under the vector representation, whereas the phason variable transforms under another related representation. Thus, a basis (or a set of basis functions) in the representation space should include such two kinds of variables. This makes it more difficult to determine the physical property tensors of quasicrystals. In this paper the group-theoretical method is given to determine the physical property tensors of quasicrystals. As an illustration of this method we calculate the third-order elasticity tensors of quasicrystals with five-fold symmetry by means of basis functions. It follows that the linear phonon elasticity is isotropic, but the nonlinear phonon elasticity is anisotropic for pentagonal quasicrystals. Meanwhile, the basis functions are constructed for all noncrystallographic point groups of quasicrystals.

Keywords: quasicrystals, elastic constants, basis functions
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1. Introduction

The discovery of icosahedral quasicrystals (QCs) made by Schechtman in 1984 challenges long-held beliefs in the nature of translational order, and motivated a reassessment of conventional crystallography. Since then, a vast literature has been published in this field and significant progress has been made in both experiment and theory.[1−12] While more and more studies show the distinctions between crystal and quasicrystal properties, the research methods of QCs are considerably borrowed from crystallography.[6,7,13,14] The symmetry and group theory is of great importance to scientists who want to reveal the characteristic features of physical properties of QCs just from their symmetries but not from the details of atomic position and interatomic interaction.[15]

According to the higher-dimensional description of QCs, a quasiperiodic structure in a d-dimensional (dD) physical space ($V_{\rm E}$) can be obtained by intersecting a lattice–periodic structure in an n-dimensional (nD) embedding space V with this subspace, where V is the direct sum of $V_{\rm E}$ and $V_{\rm I}$ ($V = V_{\rm E} + V_{\rm I}$), and $V_{\rm I}$ is the orthogonal complement of $V_{\rm E}$. A vector in $V_{\rm E}$ transforms under the vector representation of the point group considered, whereas a vector in $V_{\rm I}$ transforms under another irreducible representation. In elastic properties the most prominent feature that distinguishes QCs from ordinary crystals is that in QCs there are two types of hydrodynamic variables: phonon variable $\boldsymbol{u}$ (belonging to $V_{\rm E}$) and phason variable $\boldsymbol{w}$ (belonging to $V_{\rm I}$). As mentioned above, they have different transformation properties. For this reason a basis (or a set of basis functions) in the representation space should include such two kinds of vectors.

There has been a growing interest in the anisotropy of QCs in recent years.[16,17] Since QCs possess positional and orientational long-range order with noncrystallographic rotational symmetries, they are fundamentally anisotropic and at the macroscopic level there should be some anisotropic physical properties. However, the commonly observed properties are isotropic due to their low tensorial rank and do not reveal the anisotropic structure of QCs.[18−21] To detect the QC anisotropy some people have investigated nonlinear elasticity of the QC.[22−24] Obviously, to determine the nonlinear elasticity tensor of QCs is very complicated because on the one hand the tensor is of higher rank, and on the other hand it comprises several parts associated with the phonon field, the phason field and their coupling.

Since the discovery of QCs, great effort has been exerted to investigate their physical properties. We know today that QCs possess a number of really exotic physical properties. In view of this point the determination of the physical property tensors of QCs

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is highly important for both theory and practice. In this paper the group-theoretical method is used for determining the physical property tensors of QCs. Meanwhile, in 1996 a pentagonal QC was observed in an $Al_{70}Co_{15}Ni_{10}Tb$ alloy.[25,26] Since pentagonal QCs represent interesting immediate states between icosahedral and crystalline phases with anisotropic physical and mechanical properties, some researchers focused on pentagonal QCs to discuss the theory of structural QC-to-crystal transformation.[27] The linear elasticity of pentagonal QCs was also investigated.[28,29] As an example, we calculate the nonlinear elasticity tensor of quasicrystals with fivefold symmetry by means of basis functions. Other physical properties of QCs can be deduced in a similar manner.

The rest of this paper is organized as follows. The next section is devoted to symmetry groups and basis functions of QCs. The group-theoretical method of physical property tensors of QCs is presented in Section 3. In the last section some conclusions are drawn from the present study. The basis functions accompanied by the tables of most characters are listed in the Appendix.

2. Symmetry groups and basis functions

The classification of Laue classes and symmetry groups has been well established for QCs.[8] As mentioned in the previous section, to determine the transformation properties of basis functions, it should be noted that a vector in $V_{\rm E}$ transforms under the vector representation, according to which the position vector transforms, whereas a vector in $V_{\rm I}$ transforms under another irreducible representation which can be deduced by using the higher-dimensional projection method of QCs. It follows that with a rotation through an angle α about z-axis in $V_{\rm E}$ associated is a rotation through an angle β in $V_{\rm I}$ where $\alpha = \frac{2\pi}{n}(n = 5, 8, 10$ and $12)$ and $\beta = p\alpha(p = 3, 3, 3$ and $5)$ for QCs with noncrystallographic symmetries.[30] In elastic properties of QCs there are two types of hydrodynamic variables: phonon variable $\boldsymbol{u}$ (belonging to $V_{\rm E}$) and phason variable $\boldsymbol{w}$ (belonging to $V_{\rm I}$). For practical applications, what one is most interested in is the transformation properties of such basis functions as the polar vectors $x_i(u_i)$, the axial vectors R_i, the symmetric tensors of rank 2 $x_ix_j(E_{ij})$, the phason displacements w_i and the phason strains $W_{\alpha i}$. For example, from the table of characters and basis functions for $C_{5\rm v}$ (see Table 3 in the Appendix), we can see that the phonon displacements $u_{\rm i}$ transform under $E_1 + A_1$, the phonon strains E_{ij} transform under $E_1+E_2+2A_1$, the phason displacements $w_{\rm i}$ transform under E_2 and the phason strains $W_{\alpha i}$ transform under $E_1 + E_2$. Obviously, using the transformation properties of basis functions, we can immediately construct any physical property tensor of QCs. In the following section we will discuss this in detail.

The tables of characters and basis functions are given for all noncrystallographic point groups of QCs in the Appendix.

3. The group-theoretical method

As an example of the application of group theory we discuss how to calculate the third-order elastic constants in this section. The elastic energy density F that is a function of phonon strains ($E_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$) and phason strains ($W_{\alpha i} = \partial_i w_\alpha$) can be expanded into the Taylor series in the vicinity of $E_{ij} = 0$ and $W_{\alpha i} = 0$ (to the third order) as follows:

$$\begin{aligned} F = {} & \tfrac{1}{2}C_{ijkl}E_{ij}E_{kl} + \tfrac{1}{2}K_{\alpha i\beta j}W_{\alpha i}W_{\beta j} + R_{ij\alpha k}E_{ij}W_{\alpha k} \\ & + \tfrac{1}{6}C_{ijklmn}E_{ij}E_{kl}E_{mn} + \tfrac{1}{6}K_{\alpha i\beta j\gamma k}W_{\alpha i}W_{\beta j}W_{\gamma k} \\ & + \tfrac{1}{2}R^{(1)}_{ijkl\alpha m}E_{ij}E_{kl}W_{\alpha m} + \tfrac{1}{2}R^{(2)}_{ij\alpha k\beta l}E_{ij}W_{\alpha k}W_{\beta l}, \end{aligned} \tag{1}$$

where $C_{ijkl} = \left(\frac{\partial^2 F}{\partial E_{ij}\partial E_{kl}}\right)_0$, $K_{\alpha i\beta j} = \left(\frac{\partial^2 F}{\partial W_{\alpha i}\partial W_{\beta j}}\right)_0$ and $R_{ij\alpha k} = \left(\frac{\partial^2 F}{\partial E_{ij}\partial W_{\alpha k}}\right)_0$ are the linear(second-order) elastic constants of phonon field, phason field and phonon–phason coupling, respectively, and $C_{ijklmn} = \left(\frac{\partial^3 F}{\partial E_{ij}\partial E_{kl}\partial E_{mn}}\right)_0$, $K_{\alpha i\beta j\gamma k} = \left(\frac{\partial^3 F}{\partial W_{\alpha i}\partial W_{\beta j}\partial W_{\gamma k}}\right)_0$, $R^{(1)}_{ijkl\alpha m} = \left(\frac{\partial^3 F}{\partial E_{ij}\partial E_{kl}\partial W_{\alpha m}}\right)_0$ and $R^{(2)}_{ij\alpha k\beta l} = \left(\frac{\partial^3 F}{\partial E_{ij}\partial W_{\alpha k}\partial W_{\beta l}}\right)_0$ are the nonlinear

(third-order) elastic constants of phonon field, phason field, phonon–phonon–phason coupling and phonon–phason–phason coupling, respectively. Since F is a scalar, it can be expressed as the combinations of polynomials in gradients $\boldsymbol{u}$ and $\boldsymbol{w}$, which are form-invariant under all of the point group operations of the structure considered, i.e. they transform according to the identity representation of the group (A_1). If we construct all invariants, we can eventually determine the elastic constant tensors.

As we know, the number of invariants (or independent tensor components) is equal to the number of times that the identity representation (A_1) is contained in this tensor representation, e.g.

$$n = \frac{1}{|G|}\sum_{g\in F}\bar{x}(g), \tag{2}$$

where $|G|$ is the order of G, $\bar{x}$ is the character of the tensor representation. Once the characters of C_{ijklmn}, $K_{\alpha i\beta j\gamma k}$, $R^{(1)}_{ijkl\alpha m}$, and $R^{(2)}_{ij\alpha k\beta l}$ are calculated for any $g \in G$, the number of independent components of all nonlinear elastic tensors are obtained from Eq.(2). For example, in the case of QCs with the C_{5v} symmetry there are 43 independent components of the nonlinear elastic tensor (10 components are for the phonon field, 5 for the phason field, 12 for the phonon–phonon–phason coupling, and 16 for the phonon–phason–phason coupling).

Now we determine the form of the third-order elastic constant tensors. Clearly, the determination of explicit forms for these independent components is much more complicated than counting their number. As an illustration of this procedure, we consider $K_{\alpha i\beta j\gamma k}$ of QCs with the C_{5v} symmetry.

By the definition of $K_{\alpha i\beta j\gamma k}$, their transformation property can be deduced directly from $\{W\times W\times W\}_{\rm s}$, where $\{\}_{\rm s}$ denotes the completely symmetric part. From Table 3 it follows that six components of the phason strain tensor span three irreducible representations $E_1+E_2+E_2$. The corresponding basis functions are

$$(W_{21}+W_{12}, W_{11}-W_{22}),\ (W_{21}-W_{12}, W_{11}+W_{22}),\ (W_{13}, W_{23}). \tag{3}$$

Bearing in mind that

1) There are five invariants hence, five independent components of $K_{\alpha i\beta j\gamma k}$,
2) $A_1 \subset E_i \times E_j$ if and only if $i = j$,
3) $K_{\alpha i\beta j\gamma k}$ is completely symmetric,

one can easily construct the five invariants according to

$$\begin{aligned} &E_2^a \times E_2^b \times E_1,\ E_2^a \times E_2^a \times E_1,\ E_2^b \times E_2^b \times E_1,\\ &E_1 \times E_1 \times E_2^a,\ E_1 \times E_1 \times E_2^b, \end{aligned} \tag{4}$$

where the representation space of $E_2^{\rm a}$ is formed by the basis functions $(W_{21}-W_{12}, W_{11}+W_{22})$, and $E_2^{\rm b}$ formed by (W_{13}, W_{23}). For the first formula of Eq.(4) the direct product of representations $E_2^a \times E_2^b$ decomposes into three irreducible representations, i.e. $A_1 + A_2 + E_1$. It is obvious to see that the products

$$\Big[(W_{21}-W_{12})W_{13}+(W_{11}+W_{22})W_{23}\Big],$$
$$\Big[(W_{21}-W_{12})W_{23}-(W_{11}+W_{22})W_{13}\Big],$$

and

$$\begin{aligned} &\Big[(W_{11}+W_{22})W_{13}+(W_{21}-W_{12})W_{23},\\ &(W_{11}+W_{22})W_{23}-(W_{21}-W_{12})W_{13}\Big], \end{aligned} \tag{5}$$

transform under A_1, A_2 and E_1, respectively. Since the last formula of Eq.(5) transforms according to the same representation E_1 as $(W_{21}+W_{12}, W_{11}-W_{22})$, this produces one invariant as follows:

$$\begin{aligned} &\Big[(W_{11}+W_{22})W_{13}+(W_{21}-W_{12})W_{23}\Big](W_{21}+W_{12})\\ &+\Big[(W_{11}+W_{22})W_{23}-(W_{21}-W_{12})W_{13}\Big](W_{11}-W_{22}). \end{aligned} \tag{6}$$

Similarly, the four other cubic invariants can be obtained from the four other formulae of Eq.(4). They are

$$\begin{aligned} &\Big[(W_{11}+W_{22})^2-(W_{21}-W_{12})^2\Big](W_{11}-W_{22})\\ &+2(W_{11}+W_{22})(W_{21}-W_{12})(W_{21}+W_{12}),\\ &(W_{13}^2-W_{23}^2)(W_{21}+W_{12})+2W_{13}W_{23}(W_{11}-W_{22}),\\ &\Big[(W_{11}-W_{22})^2-(W_{21}+W_{12})^2\Big](W_{21}-W_{12})\\ &+2(W_{11}-W_{22})(W_{21}+W_{12})(W_{11}+W_{22}),\\ &\Big[(W_{11}-W_{22})^2-(W_{21}+W_{12})^2\Big]W_{13}\\ &+2(W_{11}-W_{22})(W_{21}+W_{12})W_{23}. \end{aligned} \tag{7}$$

Consequently, the corresponding nonvanishing components are

$$
\begin{gathered}
K_{113}=K_{145}=-K_{223}=K_{256}=-K_{344}=K_{366},\\
-K_{135}=K_{235}=K_{334}=K_{336}=-K_{455}=-K_{556},\\
-K_{111}/3=-K_{112}=K_{122}=K_{144}/3=-K_{146}=-K_{166}=K_{222}/3=K_{244}=K_{246}=-K_{266}/3,\\
-K_{114}=-K_{116}/3=-K_{124}=K_{126}=K_{224}/3=K_{226}=K_{444}/3=-K_{446}=K_{466}=K_{666}/3,\\
-K_{115}=K_{125}=-K_{134}=-K_{136}=-K_{225}=K_{234}=K_{236}=K_{445}=K_{456}=K_{566}.
\end{gathered}
\tag{8}
$$

Similarly, we can calculate other third-order elastic constants as follows:

$$
\begin{gathered}
C_{144}=C_{255},C_{155}=C_{244},C_{456}=(C_{155}-C_{144})/2,\\
C_{113}=C_{223},C_{123},C_{366}=(C_{113}-C_{123})/2,\\
C_{111}=C_{222},C_{112}=C_{122},C_{166}=C_{266}=(C_{111}-C_{112})/4,\\
C_{133}=C_{233},C_{333},C_{344}=C_{355},\\
-C_{114}=C_{124}=-C_{156}=-C_{224}=C_{256}=C_{466},
\end{gathered}
\tag{9}
$$

for phonon field;

$$
\begin{gathered}
-R^{(1)}_{166}=R^{(1)}_{264},-R^{(1)}_{111}=R^{(1)}_{222}=(R^{(1)}_{164}-3R^{(1)}_{166})/2,\\
-R^{(1)}_{164}=R^{(1)}_{266},R^{(1)}_{112}=-R^{(1)}_{221}=(3R^{(1)}_{164}-R^{(1)}_{166})/2,\\
R^{(1)}_{121}=-R^{(1)}_{122}=R^{(1)}_{661}=-R^{(1)}_{662}=(R^{(1)}_{164}+R^{(1)}_{166})/2,\\
R^{(1)}_{141}=R^{(1)}_{156})=-R^{(1)}_{242}=R^{(1)}_{254},\\
-R^{(1)}_{142}=R^{(1)}_{154}=R^{(1)}_{241}=R^{(1)}_{256},\\
R^{(1)}_{464}=-R^{(1)}_{466}=R^{(1)}_{561}=R^{(1)}_{562}=(R^{(1)}_{141}+R^{(1)}_{142})/2,\\
R^{(1)}_{113}=2R^{(1)}_{165}=-R^{(1)}_{223}=2R^{(1)}_{265},-R^{(1)}_{443}=R^{(1)}_{455}=R^{(1)}_{553},\\
R^{(1)}_{341}=-R^{(1)}_{342}=R^{(1)}_{354}=R^{(1)}_{356},R^{(1)}_{133}=-R^{(1)}_{233}=R^{(1)}_{365},\\
-R^{(1)}_{131}=-R^{(1)}_{132}=R^{(1)}_{231}=R^{(1)}_{232}=R^{(1)}_{364}=-R^{(1)}_{366},\\
R^{(1)}_{441}=R^{(1)}_{442}=R^{(1)}_{454}=-R^{(1)}_{456}=-R^{(1)}_{551}=-R^{(1)}_{552},\\
-R^{(1)}_{143}=-R^{(1)}_{155}=R^{(1)}_{243}=R^{(1)}_{255}=R^{(1)}_{465})=-R^{(1)}_{563},\\
-R^{(1)}_{144}=-R^{(1)}_{146}=-R^{(1)}_{151}=R^{(1)}_{152}=R^{(1)}_{244}=R^{(1)}_{246}=R^{(1)}_{251}=-R^{(1)}_{252}=-R^{(1)}_{461}=R^{(1)}_{462}=R^{(1)}_{564}=R^{(1)}_{566},
\end{gathered}
\tag{10}
$$

for phonon–phonon–phason coupling;

$$
\begin{gathered}
R^{(2)}_{111}=R^{(2)}_{244}=R^{(2)}_{166}=R^{(2)}_{222},\\
R^{(2)}_{122}=R^{(2)}_{144}=R^{(2)}_{211}=R^{(2)}_{266}=R^{(2)}_{111}+2R^{(2)}_{614},\\
R^{(2)}_{614}=R^{(2)}_{626},\quad -R^{(2)}_{112}=R^{(2)}_{146}=-R^{(2)}_{212}=R^{(2)}_{246},\\
-R^{(2)}_{312}=R^{(2)}_{346},\quad R^{(2)}_{311}=R^{(2)}_{322}=R^{(2)}_{344}=R^{(2)}_{366},\\
R^{(2)}_{413}=-R^{(2)}_{456}=R^{(2)}_{525}=-R^{(2)}_{534},\\
R^{(2)}_{423}=R^{(2)}_{445}=R^{(2)}_{515}=R^{(2)}_{536},\\
R^{(2)}_{133}=R^{(2)}_{155}=R^{(2)}_{233}=R^{(2)}_{255},R^{(2)}_{333}=R^{(2)}_{355},\\
R^{(2)}_{313}=R^{(2)}_{323}=-R^{(2)}_{345}=R^{(2)}_{356},\quad R^{(2)}_{435}=-R^{(2)}_{533}=R^{(2)}_{555},\\
-R^{(2)}_{114}=-R^{(2)}_{116}=R^{(2)}_{124}=R^{(2)}_{126}=R^{(2)}_{214}=R^{(2)}_{216}=-R^{(2)}_{224}=-R^{(2)}_{226}\\
=-R^{(2)}_{611}=R^{(2)}_{612}=-R^{(2)}_{622}=R^{(2)}_{644}=R^{(2)}_{646}=R^{(2)}_{666},\\
R^{(2)}_{115}=-R^{(2)}_{125}=-R^{(2)}_{134}=-R^{(2)}_{136}=-R^{(2)}_{215}=R^{(2)}_{225}=R^{(2)}_{234}=R^{(2)}_{236}=R^{(2)}_{613}=-R^{(2)}_{623}=R^{(2)}_{645}=R^{(2)}_{656},\\
R^{(2)}_{113}=R^{(2)}_{123}=-R^{(2)}_{145}=R^{(2)}_{156}=R^{(2)}_{213}=R^{(2)}_{223}=-R^{(2)}_{245}=R^{(2)}_{256},\\
R^{(2)}_{416}=R^{(2)}_{424}=R^{(2)}_{511}=-R^{(2)}_{522}=R^{(2)}_{544}=-R^{(2)}_{566},\\
R^{(2)}_{411}=R^{(2)}_{412}=R^{(2)}_{422}=-R^{(2)}_{444}=R^{(2)}_{446}=-R^{(2)}_{466}=-R^{(2)}_{514}=-R^{(2)}_{524}=R^{(2)}_{526}=R^{(2)}_{516},
\end{gathered}
\tag{11}
$$

for phonon–phason–phason coupling.

In the same way any physical property tensors can be constructed for QCs.

4. Conclusion

In contrast to conventional crystals, there are two types of hydrodynamic variables in the hydrodynamics for QCs: phonon and phason variables. These two types of hydrodynamic variables have different transformation properties. The phonon variable transforms under vector representation, whereas the phason variable transforms under another irreducible representation. This unique feature makes it more difficult to determine the physical property tensors of QCs. In this paper the group-theoretical method is presented to determine the physical property tensors of QCs. Since they play an important role in this method, the basis functions for all noncrystallographic point groups of QCs are given in the Appendix.

In this paper the third-order elasticity tensors of QCs with $C_{5\mathrm{v}}$ symmetry are calculated with the help of basis functions. The results show that there are 43 independent components of the third-order elastic constant where 10 components are for phonon field, 5 for phason field, 12 and 16 for phonon–phonon–phason coupling and phonon–phason–phason coupling, respectively. Other physical property tensors can be readily determined for all QCs in a similar manner.

Moreover, since the phason displacement of pentagonal QCs is a tensor of rank 2, it is easy to find out the constants C_{ijkl}, $K_{\alpha i\beta j}$, $R_{ij\alpha k}$, C_{ijklmn}, $K_{\alpha i\beta j\gamma k}$, $R^{(1)}_{ijkl\alpha m}$, and $R^{(2)}_{ij\alpha k\beta l}$ are tensors of ranks 4, 6, 5, 6, 9, 7 and 8, respectively. According to Hermann's theorem, the macroscopic properties are anisotropic if they are described by a tensor with rank $N \geq 5$ (the order of the symmetry rotation of pentagonal QCs is 5).[18,19] Therefore, the third-order nonlinear phonon elastic properties and linear elastic properties at high temperatures due to phonon–phason coupling of pentagonal QCs are expected to be anisotropic. This situation is the same as that for icosahedral QCs.

Appendix. Tables of the characters and basis functions for noncrystallographic point groups of QCs (Tables 1–28).

Table 1. $C_5(5)$ (Note: $\rho = \mathrm{e}^{2\pi \mathrm{i}/5}$).

	E	c_5	c_5^2	c_5^3	c_5^4				
A	1	1	1	1	1	x_3, R_{x_3}	$x_1^2+x_2^2, x_3^2$		
E_1	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	(x_1, x_2),	(x_1x_3, x_2x_3)	$(W_{21}+W_{12}, W_{11}-W_{22})$	
	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	(R_{x_1}, R_{x_2})			
E_2	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{21}-W_{12}, W_{11}+W_{22})$	(w_1, w_2)
	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2			(W_{13}, W_{23})	

Table 2. $C_{5\mathrm{i}}(\bar{5})$ (Note: $\rho = \mathrm{e}^{2\pi \mathrm{i}/5}$).

	E	c_5	c_5^2	c_5^3	c_5^4	I	s_{10}^7	s_{10}^9	s_{10}	s_{10}^3				
A_g	1	1	1	1	1	1	1	1	1	1	R_{x_3}	$x_1^2+x_2^2, x_3^2$		
E_{1g}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)	$(W_{21}+W_{12}, W_{11}-W_{22})$	
	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ				
E_{2g}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{21}-W_{12}, W_{11}+W_{22})$	
	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2			(W_{13}, W_{23})	
A_u	1	1	1	1	1	-1	-1	-1	-1	-1	x_3			
E_{1u}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	-1	$-\rho$	$-\rho^2$	$-\rho^{-2}$	$-\rho^{-1}$	(x_1, x_2)			
	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^2$	$-\rho$				
E_{2u}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	-1	$-\rho^2$	$-\rho^{-1}$	$-\rho$	$-\rho^{-2}$				(w_1, w_2)
	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	-1	$-\rho^{-2}$	$-\rho$	$-\rho^{-1}$	$-\rho^2$				

Table 3. $C_{5\mathrm{v}}(5\mathrm{m})$ (Note: $1/\tau = 2\cos 72^\circ, -\tau = 2\cos 144^\circ$).

	E	$2c_5$	$2c_5^2$	$5\sigma_v$				
A_1	1	1	1	1	x_3	$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	-1	R_{x_3}			
E_1	2	$1/\tau$	$-\tau$	0	$(x_1,x_2),(R_{x_1},R_{x_2})$	(x_1x_3, x_2x_3)	$(W_{21}+W_{12}, W_{11}-W_{22})$	
E_2	2	$-\tau$	$1/\tau$	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{21}-W_{12}, W_{11}+W_{22}),(W_{13}, W_{23})$	(w_1, w_2)

Table 4. $D_5(52)$ (Note: $1/\tau = 2\cos 72^\circ, -\tau = 2\cos 144^\circ$).

	E	$2c_5$	$2c_5^2$	$5c_2'$				
A_1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	-1	x_3, R_{x_3}			
E_1	2	$1/\tau$	$-\tau$	0	$(x_1,x_2),(R_{x_1},R_{x_2})$	(x_1x_3, x_2x_3)	$(W_{21}+W_{12}, W_{11}-W_{22})$	
E_2	2	$-\tau$	$1/\tau$	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{21}-W_{12}, W_{11}+W_{22}),(W_{13}, W_{23})$	(w_1, w_2)

Table 5. $D_{5\mathrm{d}}(\bar{5}\mathrm{m})$ (Note: $1/\tau = 2\cos 72^\circ, -\tau = 2\cos 144^\circ$).

	E	$2c_5$	$2c_5^2$	$5c_2'$	I	$2s_{10}^3$	$2s_{10}$	$5\sigma_d$			
A_{1g}	1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$	
A_{2g}	1	1	1	-1	1	1	1	-1	R_{x_3}		
E_{1g}	2	$1/\tau$	$-\tau$	0	2	$1/\tau$	$-\tau$	0	(R_{x_1},R_{x_2})	(x_1x_3, x_2x_3)	$(W_{21}+W_{12}, W_{11}-W_{22})$
E_{2g}	2	$-\tau$	$1/\tau$	0	2	$-\tau$	$1/\tau$	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{21}-W_{12}, W_{11}+W_{22}),(W_{13}, W_{23})$
A_{1u}	1	1	1	1	-1	-1	-1	-1			
A_{2u}	1	1	1	-1	-1	-1	-1	1	x_3		
E_{1u}	2	$1/\tau$	$-\tau$	0	-2	$-1/\tau$	τ	0	(x_1,x_2)		
E_{2u}	2	$-\tau$	$1/\tau$	0	-2	τ	$-1/\tau$	0			(w_1, w_2)

Table 6. $C_{10}(10)$ (Note: $\rho = \mathrm{e}^{2\pi\mathrm{i}/5}$).

	E	c_{10}	c_5	c_{10}^3	c_5^2	c_2	c_5^3	c_{10}^7	c_5^4	c_{10}^9			
A	1	1	1	1	1	1	1	1	1	1	x_3, R_{x_3}	$x_1^2+x_2^2, x_3^2$	
B	1	-1	1	-1	1	-1	1	-1	1	-1			
E_1	1	$-\rho^2$	ρ^{-1}	$-\rho$	ρ^{-2}	-1	ρ^2	$-\rho^{-1}$	ρ	$-\rho^{-2}$	$(x_1,x_2),$	(x_1x_3, x_2x_3)	
	1	$-\rho^{-2}$	ρ	$-\rho^{-1}$	ρ^2	-1	ρ^{-2}	$-\rho$	ρ^{-1}	$-\rho^2$	(R_{x_1},R_{x_2})		
E_2	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$
	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ			
E_3	1	$-\rho$	ρ^2	$-\rho^{-2}$	ρ^{-1}	-1	ρ	$-\rho^2$	ρ^{-2}	$-\rho^{-1}$		(W_{13}, W_{23})	(w_1, w_2)
	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	ρ^2	$-\rho$			
E_4	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}		$(W_{11}-W_{22}, W_{21}+W_{12})$	
	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2			

Table 7. $C_{5\mathrm{h}}(\overline{10})$ (Note: $\rho = \mathrm{e}^{2\pi\mathrm{i}/5}$).

	E	c_5	c_5^2	c_5^3	c_5^4	$5\sigma_h$	s_5	s_5^7	s_5^8	s_5^9				
A'	1	1	1	1	1	1	1	1	1	1	R_{x_3}	$x_1^2+x_2^2, x_3^2$		
E_1'	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	(x_1,x_2)		$(W_{21}+W_{12}, W_{11}-W_{22})$	
	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ				
E_2'	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{21}-W_{12}, W_{11}+W_{22})$	(w_1, w_2)
	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2				
A''	1	1	1	1	1	-1	-1	-1	-1	-1	x_3			
E_1''	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	-1	$-\rho$	$-\rho^2$	$-\rho^{-2}$	$-\rho^{-1}$	(R_{x_1},R_{x_2})	(x_1x_3, x_2x_3)		
	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^2$	$-\rho$				
E_2''	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	-1	$-\rho^2$	$-\rho^{-1}$	$-\rho$	$-\rho^{-2}$			(W_{13}, W_{23})	
	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	-1	$-\rho^{-2}$	$-\rho$	$-\rho^{-1}$	$-\rho^2$				

Table 8. $C_{10\mathrm{h}}(10/m)$ (Note: $\rho = \mathrm{e}^{2\pi \mathrm{i}/5}$).

	E	c_{10}	c_5	c_{10}^3	c_5^2	c_2	c_5^3	c_{10}^7	c_5^4	c_{10}^9	I	s_5^3	s_{10}^7	s_5^9	s_{10}^9	σ_h	s_{10}	s_5	s_{10}^3	s_5^7			
A_g	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	R_{x_3}	$x_1^2+x_2^2, x_3^2$	
B_g	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1			
E_{1g}	1	$-\rho^2$	ρ^{-1}	$-\rho$	ρ^{-2}	-1	ρ^2	$-\rho^{-1}$	ρ	$-\rho^{-2}$	1	$-\rho^2$	ρ^{-1}	$-\rho$	ρ^{-2}	-1	ρ^2	$-\rho^{-1}$	ρ	$-\rho^{-2}$	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)	
	1	$-\rho^{-2}$	ρ	$-\rho^{-1}$	ρ^2	-1	ρ^{-2}	$-\rho$	ρ^{-1}	$-\rho^2$	1	$-\rho^{-2}$	ρ	$-\rho^{-1}$	ρ^2	-1	ρ^{-2}	$-\rho$	ρ^{-1}	$-\rho^2$			
E_{2g}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$
	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ			
E_{3g}	1	$-\rho$	ρ^2	$-\rho^{-2}$	ρ^{-1}	-1	ρ	$-\rho^2$	ρ^{-2}	$-\rho^{-1}$	1	$-\rho$	ρ^2	$-\rho^{-2}$	ρ^{-1}	-1	ρ	$-\rho^2$	ρ^{-2}	$-\rho^{-1}$			(W_{13}, W_{23})
	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	ρ^2	$-\rho$	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	ρ^2	$-\rho$			
E_{4g}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}			$(W_{11}-W_{22}, W_{21}+W_{12})$
	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2			
A_u	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	x_3		
B_u	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1			
E_{1u}	1	$-\rho^2$	ρ^{-1}	$-\rho$	ρ^{-2}	-1	ρ^2	$-\rho^{-1}$	ρ	$-\rho^{-2}$	-1	ρ^2	$-\rho^{-1}$	ρ	$-\rho^{-2}$	1	$-\rho^2$	ρ^{-1}	$-\rho$	ρ^{-2}	(x_1, x_2)		
	1	$-\rho^{-2}$	ρ	$-\rho^{-1}$	ρ^2	-1	ρ^{-2}	$-\rho$	ρ^{-1}	$-\rho^2$	-1	ρ^{-2}	$-\rho$	ρ^{-1}	$-\rho^2$	1	$-\rho^{-2}$	ρ	$-\rho^{-1}$	ρ^2			
E_{2u}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^{-2}	ρ^{-1}	-1	$-\rho$	$-\rho^2$	$-\rho^{-2}$	$-\rho^{-1}$	-1	$-\rho$	$-\rho^2$	$-\rho^{-2}$	$-\rho^{-1}$			
	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	ρ^2	ρ	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^2$	$-\rho$	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^2$	$-\rho$			
E_{3u}	1	$-\rho$	ρ^2	$-\rho^{-2}$	ρ^{-1}	-1	ρ	$-\rho^2$	ρ^{-2}	$-\rho^{-1}$	-1	ρ	$-\rho^2$	ρ^{-2}	$-\rho^{-1}$	1	$-\rho$	ρ^2	$-\rho^{-2}$	ρ^{-1}			(w_1, w_2)
	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	ρ^2	$-\rho$	-1	ρ^{-1}	$-\rho^{-2}$	ρ^2	$-\rho$	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^2$	ρ			
E_{4u}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	1	ρ^2	ρ^{-1}	ρ	ρ^{-2}	-1	$-\rho^2$	$-\rho^{-1}$	-ρ	$-\rho^{-2}$	-1	$-\rho^2$	$-\rho^{-1}$	$-\rho$	$-\rho^{-2}$			
	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	1	ρ^{-2}	ρ	ρ^{-1}	ρ^2	-1	$-\rho^{-2}$	$-\rho$	$-\rho^{-1}$	$-\rho^2$	-1	$-\rho^{-2}$	$-\rho$	$-\rho^{-1}$	$-\rho^2$			

Table 9. $C_{10\mathrm{v}}(10mm)$ (Note: $1/\tau = 2\cos 72^\circ, -\tau = 2\cos 144^\circ$).

	E	$2c_{10}$	$2c_5$	$2c_{10}^3$	$2c_5^2$	c_2	$5\sigma_v$	$5\sigma_d$				
A_1	1	1	1	1	1	1	1	1	x_3	$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	1	1	1	-1	-1	R_{x_3}			
B_1	1	-1	1	-1	1	-1	1	-1				
B_2	1	-1	1	-1	1	-1	-1	1				
E_1	2	τ	$1/\tau$	$-1/\tau$	$-\tau$	-2	0	0	$(x_1, x_2), (R_{x_1}, R_{x_2})$	(x_1x_3, x_2x_3)		
E_2	2	$1/\tau$	$-\tau$	$-\tau$	$1/\tau$	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_3	2	$-1/\tau$	$-\tau$	τ	$1/\tau$	-2	0	0			(W_{13}, W_{23})	(w_1, w_2)
E_4	2	$-\tau$	$1/\tau$	$1/\tau$	$-\tau$	2	0	0			$(W_{11}-W_{22}, W_{21}+W_{12})$	

Table 10. $D_{10}(1022)$ (Note: $1/\tau = 2\cos 72^\circ, -\tau = 2\cos 144^\circ$).

	E	$2c_{10}$	$2c_5$	$2c_{10}^3$	$2c_5^2$	c_2	$5c_2'$	$5c_2''$				
A_1	1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	1	1	1	-1	-1	x_3, R_{x_3}			
B_1	1	-1	1	-1	1	-1	1	-1				
B_2	1	-1	1	-1	1	-1	-1	1				
E_1	2	τ	$1/\tau$	$-1/\tau$	$-\tau$	-2	0	0	$(x_1, x_2), (R_{x_1}, R_{x_2})$	(x_1x_3, x_2x_3)		
E_2	2	$1/\tau$	$-\tau$	$-\tau$	$1/\tau$	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_3	2	$-1/\tau$	$-\tau$	τ	$1/\tau$	-2	0	0			(W_{13}, W_{23})	(w_1, w_2)
E_4	2	$-\tau$	$1/\tau$	$1/\tau$	$-\tau$	2	0	0			$(W_{11}-W_{22}, W_{21}+W_{12})$	

Table 11. $D_{10\mathrm{h}}(10/mmm)$ (Note: $1/\tau = 2\cos 72^\circ, -\tau = 2\cos 144^\circ$).

	E	$2c_{10}$	$2c_5$	$2c_{10}^3$	$2c_5^2$	c_2	$5c_2'$	$5c_2''$	I	$2s_5^7$	$2s_{10}^3$	$2s_5$	$2s_{10}$	σ_h	$5\sigma_v$	$5\sigma_d$				
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_{2g}	1	1	1	1	1	1	-1	-1	1	1	1	1	1	1	-1	-1	R_{x_3}			
B_{1g}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1				
B_{2g}	1	-1	1	-1	1	-1	-1	1	1	-1	1	-1	1	-1	-1	1				
E_{1g}	2	τ	$1/\tau$	$-1/\tau$	$-\tau$	-2	0	0	2	τ	$1/\tau$	$-1/\tau$	$-\tau$	-2	0	0	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)		
E_{2g}	2	$1/\tau$	$-\tau$	$-\tau$	$1/\tau$	2	0	0	2	$1/\tau$	$-\tau$	$-\tau$	$1/\tau$	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_{3g}	2	$-1/\tau$	$-\tau$	τ	$1/\tau$	-2	0	0	2	$-1/\tau$	$-\tau$	τ	$1/\tau$	-2	0	0			(W_{13}, W_{23})	
E_{4g}	2	$-\tau$	$1/\tau$	$1/\tau$	$-\tau$	2	0	0	2	$-\tau$	$1/\tau$	$1/\tau$	$-\tau$	2	0	0			$(W_{11}-W_{22}, W_{21}+W_{12})$	
A_{1u}	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	x_3			
A_{2u}	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1				
B_{1u}	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1				
B_{2u}	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	1	-1				
E_{1u}	2	τ	$1/\tau$	$-1/\tau$	$-\tau$	-2	0	0	-2	$-\tau$	$-1/\tau$	$1/\tau$	τ	2	0	0	(x_1, x_2)			
E_{2u}	2	$1/\tau$	$-\tau$	$-\tau$	$1/\tau$	2	0	0	-2	$-1/\tau$	τ	τ	$-1/\tau$	-2	0	0				
E_{3u}	2	$-1/\tau$	$-\tau$	τ	$1/\tau$	-2	0	0	-2	$1/\tau$	τ	$-\tau$	$-1/\tau$	2	0	0				(w_1, w_2)
E_{4u}	2	$-\tau$	$1/\tau$	$1/\tau$	$-\tau$	2	0	0	-2	τ	$-1/\tau$	$-1/\tau$	τ	-2	0	0				

Table 12. $D_{5h}(\overline{10}m2)$ (Note: $1/\tau = 2\cos 72°, -\tau = 2\cos 144°$).

	E	$2c_5$	$2c_5^2$	$5c_2'$	σ_h	$2s_5$	$2s_5^3$	$5\sigma_v$				
A_1'	1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_2'	1	1	1	-1	1	1	1	-1	R_{x_3}			
E_1'	2	l/τ	$-\tau$	0	2	l/τ	$-\tau$	0	(x_1, x_2)		$(W_{21}+W_{12}, W_{11}-W_{22})$	
E_2'	2	$-\tau$	l/τ	0	2	$-\tau$	l/τ	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{21}-W_{12}, W_{11}+W_{22})$	(w_1, w_2)
A_1''	1	1	1	1	-1	-1	-1	-1				
A_2''	1	1	1	-1	-1	-1	-1	1	x_3			
E_1''	2	l/τ	$-\tau$	0	-2	$-l/\tau$	τ	0	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)		
E_2''	2	$-\tau$	l/τ	0	-2	τ	$-1/\tau$	0			(W_{13}, W_{23})	

Table 13. $C_8(8)$ (Note: $\rho = \mathrm{e}^{2\pi \mathrm{i}/8}$).

	E	c_8	c_4	c_8^3	c_2	c_8^5	c_4^3	c_8^7				
A	1	1	1	1	1	1	1	1	x_3, R_{x_3}	$x_1^2+x_2^2, x_3^2$		
B	1	-1	1	-1	1	-1	1	-1			$W_{11}-W_{22}, W_{21}+W_{12}$	
E_1	1	ρ	ρ^2	$-\rho^{-1}$	-1	$-\rho$	ρ^{-2}	ρ^{-1}	(x_1, x_2),	(x_1x_3, x_2x_3)		
	1	ρ^{-1}	ρ^{-2}	$-\rho$	-1	$-\rho^{-1}$	ρ^2	ρ	(R_{x_1}, R_{x_2})			
E_2	1	ρ^2	-1	ρ^{-2}	1	ρ^2	-1	ρ^{-2}		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$	
	1	ρ^{-2}	-1	ρ^2	1	ρ^{-2}	-1	ρ^2				
E_3	1	$-\rho^{-1}$	ρ^{-2}	ρ	-1	ρ^{-1}	ρ^2	$-\rho$			(W_{13}, W_{23})	(w_1, w_2)
	1	$-\rho$	ρ^{-2}	ρ^{-1}	-1	ρ	ρ^{-2}	$-\rho^{-1}$				

Table 14. $S_8(\bar{8})$ (Note: $\rho = \mathrm{e}^{2\pi \mathrm{i}/8}$).

	E	s_8	c_4	s_8^3	c_2	s_8^5	c_4^3	s_8^7				
A	1	1	1	1	1	1	1	1	R_{x_3}	$x_1^2+x_2^2, x_3^2$		
B	1	-1	1	-1	1	-1	1	-1	x_3		$W_{11}-W_{22}, W_{21}+W_{12}$	
E_1	1	ρ	ρ^2	$-\rho^{-1}$	-1	$-\rho$	ρ^{-2}	ρ^{-1}	(x_1, x_2)			
	1	ρ^{-1}	ρ^{-2}	$-\rho$	-1	$-\rho^{-1}$	ρ^2	ρ				
E_2	1	ρ^2	-1	ρ^{-2}	1	ρ^2	-1	ρ^{-2}		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$	
	1	ρ^{-2}	-1	ρ^2	1	ρ^{-2}	-1	ρ^2				
E_3	1	$-\rho^{-1}$	ρ^{-2}	ρ	-1	ρ^{-1}	ρ^2	$-\rho$	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)	(W_{13}, W_{23})	(w_1, w_2)
	1	$-\rho$	ρ^2	ρ^{-1}	-1	ρ	ρ^{-2}	$-\rho^{-1}$				

Table 15. $C_{8\mathrm{v}}(8mm)$.

	E	$2c_8$	$2c_4$	$2c_8^3$	c_2	$4\sigma_v$	$4\sigma_d$				
A_1	1	1	1	1	1	1	1	x_3	$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	1	1	-1	-1	R_{x_3}			
B_1	1	-1	1	-1	1	1	-1			$W_{11}-W_{22}$	
B_2	1	-1	1	-1	1	-1	1			$W_{21}+W_{12}$	
E_1	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	0	0	$(x_1, x_2), (R_{x_1}, R_{x_2})$	(x_1x_3, x_2x_3)		
E_2	2	0	-2	0	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_3	2	$-\sqrt{2}$	0	$\sqrt{2}$	-2	0	0			(W_{13}, W_{23})	(w_1, w_2)

Table 16. $D_8(822)$.

	E	$2c_8$	$2c_4$	$2c_8^3$	c_2	$4c_2'$	$4c_2''$				
A_1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	1	1	-1	-1	x_3, R_{x_3}			
B_1	1	-1	1	-1	1	1	-1			$W_{11}-W_{22}$	
B_2	1	-1	1	-1	1	-1	1			$W_{21}+W_{12}$	
E_1	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	0	0	$(x_1, x_2)(R_{x_1}, R_{x_2})$	(x_1x_3, x_2x_3)		
E_2	2	0	-2	0	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_3	2	$-\sqrt{2}$	0	$\sqrt{2}$	-2	0	0			(W_{13}, W_{23})	(w_1, w_2)

Table 17. $C_{8h}(8/m)$ (Note: $\rho = e^{2\pi i/8}$).

	E	c_8	c_4	c_8^3	c_2	c_8^5	c_4^3	c_8^7	I	s_8^5	s_4^3	s_8^7	σ_h	s_8	s_4	s_8^3			
A_g	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	R_{x_3}	$x_1^2+x_2^2, x_3^2$	
B_g	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1			W_{11}- W_{22}, W_{21}+ W_{12}
E_{1g}	1	ρ	ρ^2	$-\rho^{-1}$	-1	$-\rho$	ρ^{-2}	ρ^{-1}	1	ρ	$-\rho^{-2}$	$-\rho^{-1}$	-1	$-\rho$	$-\rho^2$	ρ^{-1}	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)	
	1	ρ^{-1}	ρ^{-2}	$-\rho$	-1	$-\rho^{-1}$	ρ^2	ρ	1	ρ^{-1}	$-\rho^2$	$-\rho$	-1	$-\rho^{-1}$	$-\rho^{-2}$	ρ			
E_{2g}	1	ρ^2	-1	ρ^{-2}	1	ρ^2	-1	ρ^{-2}	1	ρ^2	-1	ρ^{-2}	1	ρ^2	-1	ρ^{-2}		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}$- $W_{12})$
	1	ρ^{-2}	-1	ρ^2	1	ρ^{-2}	-1	ρ^2	1	ρ^{-2}	-1	ρ^2	1	ρ^{-2}	-1	ρ^2			
E_{3g}	1	$-\rho^{-1}$	ρ^{-2}	ρ	-1	ρ^{-1}	ρ^2	$-\rho$	1	$-\rho^{-1}$	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	$-\rho$			(W_{13}, W_{23})
	1	$-\rho$	ρ^2	ρ^{-1}	-1	ρ	ρ^{-2}	$-\rho$	1	$-\rho$	ρ^2	ρ^{-1}	-1	ρ	ρ^{-2}	$-\rho$			
A_u	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	x_3		
B_u	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1			
E_{1u}	1	ρ	ρ^2	$-\rho^{-1}$	-1	$-\rho$	ρ^{-2}	ρ^{-1}	-1	$-\rho$	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	$-\rho^{-1}$	(x_1, x_2)		
	1	ρ^{-1}	ρ^{-2}	$-\rho$	-1	$-\rho^{-1}$	ρ^2	ρ	-1	$-\rho^{-1}$	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	$-\rho$			
E_{2u}	1	ρ^2	-1	ρ^{-2}	1	ρ^2	-1	ρ^{-2}	-1	$-\rho^2$	1	$-\rho^{-2}$	-1	$-\rho^2$	1	$-\rho^{-2}$			
	1	ρ^{-2}	-1	ρ^2	1	ρ^{-2}	-1	ρ^2	-1	$-\rho^{-2}$	1	$-\rho^2$	-1	$-\rho^{-2}$	1	$-\rho^2$			
E_{3g}	1	$-\rho^{-1}$	ρ^{-2}	ρ	-1	ρ^{-1}	ρ^2	$-\rho$	-1	ρ^{-1}	ρ^2	$-\rho$	1	$-\rho^{-1}$	ρ^{-2}	ρ			(w_1, w_2)
	1	$-\rho$	ρ^2	ρ^{-1}	-1	ρ	ρ^{-2}	$-\rho^{-1}$	-1	ρ	ρ^{-2}	$-\rho^{-1}$	1	$-\rho$	ρ^2	ρ^{-1}			

Table 18. $D_{4d}(\bar{8}m2)$.

	E	$2s_8$	$2s_4$	$2s_8^3$	c_2	$4c_2'$	$4\sigma_d$				
A_1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	1	1	-1	-1	R_{x_3}			
B_1	1	-1	1	-1	1	1	-1			$W_{11}-W_{22}$	
B_2	1	-1	1	-1	1	-1	1	x_3		$W_{21}+W_{12}$	
E_1	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	0	0	(x_1, x_2)			
E_2	2	0	-2	0	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_3	2	$-\sqrt{2}$	0	$\sqrt{2}$	-2	0	0	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)	(W_{13}, W_{23})	(w_1, w_2)

Table 19. $D_{8h}(8/mmm)$.

	E	$2c_8$	$2c_4$	$2c_8^3$	c_2	$4c_2'$	$4c_2''$	I	$2s_8^3$	$2s_4$	$2s_8$	σ_h	$4\sigma_v$	$4\sigma_d$				
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_{2g}	1	1	1	1	1	-1	-1	1	1	1	1	1	-1	-1	R_{x_3}			
B_{1g}	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	1	-1			$W_{11}-W_{22}$	
B_{2g}	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1			$W_{21}+W_{12}$	
E_{1g}	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	0	0	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	0	0	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)		
E_{2g}	2	0	-2	0	2	0	0	2	0	-2	0	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$	$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_{3g}	2	$-\sqrt{2}$	0	$\sqrt{2}$	-2	0	0	2	$-\sqrt{2}$	0	$\sqrt{2}$	-2	0	0			(W_{13}, W_{23})	
A_{1u}	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1				
A_{2u}	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	1	1	x_3			
B_{1u}	1	-1	1	-1	-1	1	-1	-1	1	-1	1	-1	-1	1				
B_{2u}	1	-1	1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1				
E_{1u}	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	0	0	-2	$-\sqrt{2}$	0	$\sqrt{2}$	2	0	0	(x_1, x_2)			
E_{2u}	2	0	-2	0	2	0	0	-2	0	2	0	-2	0	0				
E_{3u}	2	$-\sqrt{2}$	0	$\sqrt{2}$	-2	0	0	-2	$\sqrt{2}$	0	$-\sqrt{2}$	2	0	0				(w_1, w_2)

Table 20. $C_{12}(12)$ (Note: $\rho = e^{2\pi i/12}$).

	E	c_{12}	c_6	c_4	c_3	c_{12}^5	c_2	c_{12}^7	c_3^2	c_4^3	c_6^5	c_{12}^{11}				
A	1	1	1	1	1	1	1	1	1	1	1	1	x_3, R_{x_3}	$x_1^2+x_2^2, x_3^2$		
B	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1			$W_{11}-W_{22}, W_{21}+W_{12}$	
E_1	1	ρ	ρ^2	ρ^3	$-\rho^{-2}$	$-\rho^{-1}$	-1	$-\rho$	$-\rho^2$	$-\rho^3$	ρ^{-2}	ρ^{-1}	(x_1, x_2),	(x_1x_3, x_2x_3)		
	1	ρ^{-1}	ρ^{-2}	ρ^{-3}	$-\rho^2$	$-\rho$	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^{-3}$	ρ^2	ρ	(R_{x_1}, R_{x_2})			
E_2	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}		$(x_1^2-x_2^2, 2x_1x_2)$		
	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2				
E_3	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$				
	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$				
E_4	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$			$(W_{11}+W_{22}, W_{21}-W_{12})$	
	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$				
E_5	1	$-\rho$	ρ^2	$-\rho^3$	$-\rho^{-2}$	ρ^{-1}	-1	ρ	$-\rho^2$	ρ^3	ρ^{-2}	$-\rho^{-1}$			(W_{13}, W_{23})	(w_1, w_2)
	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^{-3}$	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	ρ^{-3}	ρ^2	$-\rho$				

Table 21. $C_{12h}(12/m)$ (Note: $\rho = e^{2\pi i/12}$).

	E	c_{12}	c_6	c_4	c_3	c_{12}^5	c_2	c_{12}^7	c_3^2	c_4^3	c_6^5	c_{12}^{11}	I	s_{12}^7	s_3^5	s_4^3	s_6^5	s_{12}^{11}	σ_h	s_{12}	s_6	s_4	s_3	s_{12}^5			
A_g	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	R_{x_3}	$x_1^2+x_2^2, x_3^2$	
B_g	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1			W_{11}- $W_{22}, W_{21}+W_{12}$
E_{1g}	1	ρ	ρ^2	ρ^3	$-\rho^{-2}$	$-\rho^{-1}$	-1	$-\rho$	$-\rho^2$	$-\rho^3$	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^3	$-\rho^{-2}$	$-\rho^{-1}$	-1	$-\rho$	$-\rho^2$	$-\rho^3$	ρ^{-2}	ρ^{-1}	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)	
	1	ρ^{-1}	ρ^{-2}	ρ^{-3}	$-\rho^2$	$-\rho$	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^{-3}$	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	ρ^{-3}	$-\rho^2$	$-\rho$	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^{-3}$	ρ^2	ρ			
E_{2g}	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}		$(x_1^2-x_2^2, 2x_1x_2)$	
	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2			
E_{3g}	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$			
	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$			
E_{4g}	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$			$(W_{11}+W_{22}, W_{21}$- $W_{12})$
	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$			
E_{5g}	1	$-\rho$	ρ^2	$-\rho^3$	$-\rho^{-2}$	ρ^{-1}	-1	ρ	$-\rho^2$	ρ^3	ρ^{-2}	$-\rho^{-1}$	1	$-\rho$	ρ^2	$-\rho^3$	$-\rho^{-2}$	ρ^{-1}	-1	ρ	$-\rho^2$	ρ^3	ρ^{-2}	$-\rho^{-1}$			(W_{13}, W_{23})
	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^{-3}$	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	ρ^{-3}	ρ^2	$-\rho$	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^{-3}$	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	ρ^{-3}	ρ^2	$-\rho$			
A_u	1	1	1	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	x_3		
B_u	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1			
E_{1u}	1	ρ	ρ^2	ρ^3	$-\rho^{-2}$	$-\rho^{-1}$	-1	$-\rho$	$-\rho^2$	$-\rho^3$	ρ^{-2}	ρ^{-1}	-1	$-\rho$	$-\rho^2$	$-\rho^3$	ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^3	$-\rho^{-2}$	$-\rho^{-1}$	(x_1, x_2)		
	1	ρ^{-1}	ρ^{-2}	ρ^{-3}	$-\rho^2$	$-\rho$	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^{-3}$	ρ^2	ρ	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^{-3}$	ρ^2	ρ	1	ρ^{-1}	ρ^{-2}	ρ^{-3}	$-\rho^2$	$-\rho$			
E_{2u}	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}	-1	$-\rho^2$	ρ^{-2}	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}	1	ρ^2	$-\rho^{-2}$			
	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2	-1	$-\rho^{-2}$	ρ^2	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2	1	ρ^{-2}	$-\rho^2$			
E_{3u}	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	1	ρ^3			
	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}			
E_{4u}	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	-1	ρ^2	ρ^{-2}	-1	ρ^2	ρ^{-2}	-1	ρ^2	ρ^{-2}	-1	ρ^2	ρ^{-2}			
	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	-1	ρ^{-2}	ρ^2	-1	ρ^{-2}	ρ^2	-1	ρ^{-2}	ρ^2	-1	ρ^{-2}	ρ^2			
E_{5u}	1	$-\rho$	ρ^2	$-\rho^3$	$-\rho^{-2}$	ρ^{-1}	-1	ρ	$-\rho^2$	ρ^3	ρ^{-2}	$-\rho^{-1}$	-1	ρ	$-\rho^2$	ρ^3	ρ^{-2}	$-\rho^{-1}$	1	$-\rho$	ρ^2	$-\rho^3$	$-\rho^{-2}$	ρ^{-1}			(w_1, w_2)
	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^{-3}$	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	ρ^{-3}	ρ^2	$-\rho$	-1	ρ^{-1}	$-\rho^{-2}$	ρ^{-3}	ρ^2	$-\rho$	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^{-3}$	$-\rho^2$	ρ			

Table 22. $S_{12}(\overline{12})$ (Note: $\rho = e^{2\pi i/12}$).

	E	s_{12}	c_6	s_4	c_3	s_{12}^5	c_2	s_{12}^7	c_3^2	s_4^3	c_6^5	s_{12}^{11}				
A	1	1	1	1	1	1	1	1	1	1	1	1	R_{x_3}	$x_1^2+x_2^2, x_3^2$		
B	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	x_3		$W_{11}-W_{22}, W_{21}+W_{12}$	
E_1	1	ρ	ρ^2	ρ^3	$-\rho^{-2}$	$-\rho^{-1}$	-1	$-\rho$	$-\rho^2$	$-\rho^3$	ρ^{-2}	ρ^{-1}	(x_1, x_2)			
	1	ρ^{-1}	ρ^{-2}	ρ^{-3}	$-\rho^2$	$-\rho$	-1	$-\rho^{-1}$	$-\rho^{-2}$	$-\rho^{-3}$	ρ^2	ρ				
E_2	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}	1	ρ^2	$-\rho^{-2}$	-1	$-\rho^2$	ρ^{-2}		$(x_1^2-x_2^2, 2x_1x_2)$		
	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2	1	ρ^{-2}	$-\rho^2$	-1	$-\rho^{-2}$	ρ^2				
E_3	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$	1	ρ^3	-1	$-\rho^3$				
	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$	1	ρ^{-3}	-1	$-\rho^{-3}$				
E_4	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$	1	$-\rho^2$	$-\rho^{-2}$			$(W_{11}+W_{22}, W_{21}-W_{12})$	
	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$	1	$-\rho^{-2}$	$-\rho^2$				
E_5	1	$-\rho$	ρ^2	$-\rho^3$	$-\rho^{-2}$	ρ^{-1}	-1	ρ	$-\rho^2$	ρ^3	ρ^{-2}	$-\rho^{-1}$	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)	(W_{13}, W_{23})	(w_1, w_2)
	1	$-\rho^{-1}$	ρ^{-2}	$-\rho^{-3}$	$-\rho^2$	ρ	-1	ρ^{-1}	$-\rho^{-2}$	ρ^{-3}	ρ^2	$-\rho$				

Table 23. $D_{6\mathrm{d}}(\overline{12}m2)$.

	E	$2s_{12}$	$2c_6$	$2s_4$	$2c_3$	$2s_{12}^5$	c_2	$6c_2'$	$6\sigma_d$				
A_1	1	1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	1	1	1	1	-1	-1	R_{x_3}			
B_1	1	-1	1	-1	1	-1	1	1	-1			$W_{11}-W_{22}$	
B_2	1	-1	1	-1	1	-1	1	-1	1	x_3		$W_{21}+W_{12}$	
E_1	2	$\sqrt{3}$	1	0	-1	$-\sqrt{3}$	-2	0	0	(x_1, x_2)			
E_2	2	1	-1	-2	-1	1	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$		
E_3	2	0	-2	0	2	0	-2	0	0				
E_4	2	-1	-1	2	-1	-1	2	0	0			$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_5	2	$-\sqrt{3}$	1	0	-1	$\sqrt{3}$	-2	0	0	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)	(W_{13}, W_{23})	(w_1, w_2)

Table 24. $D_{12\mathrm{h}}(12/mmm)$.

	E	$2c_{12}$	$2c_6$	$2c_4$	$2c_3$	$2c_{12}^5$	c_2	$6c_2'$	$6c_2''$	I	$2s_{12}^5$	$2s_3$	$2s_4$	$2s_6$	$2s_{12}$	σ_h	$4\sigma_v$	$4\sigma_d$				
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_{2g}	1	1	1	1	1	1	1	-1	-1	1	1	1	1	1	1	1	-1	-1	R_{x_3}			
B_{1g}	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1	1	-1			$W_{11}-W_{22}$	
B_{2g}	1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1			$W_{21}+W_{12}$	
E_{1g}	2	$\sqrt{3}$	1	0	-1	$-\sqrt{3}$	-2	0	0	2	$\sqrt{3}$	1	0	-1	$-\sqrt{3}$	-2	0	0	(R_{x_1}, R_{x_2})	(x_1x_3, x_2x_3)		
E_{2g}	2	1	-1	-2	-1	1	2	0	0	2	1	-1	-2	-1	1	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$		
E_{3g}	2	0	-2	0	2	0	-2	0	0	2	0	-2	0	2	0	-2	0	0				
E_{4g}	2	-1	-1	2	-1	-1	2	0	0	2	-1	-1	2	-1	-1	2	0	0			$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_{5g}	2	$-\sqrt{3}$	1	0	-1	$\sqrt{3}$	-2	0	0	2	$-\sqrt{3}$	1	0	-1	$\sqrt{3}$	-2	0	0			(W_{13}, W_{23})	
A_{1u}	1	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1				
A_{2u}	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	x_3			
B_{1u}	1	-1	1	-1	1	-1	1	1	-1	-1	1	-1	1	-1	1	-1	-1	1				
B_{2u}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1				
E_{1u}	2	$\sqrt{3}$	1	0	-1	$-\sqrt{3}$	-2	0	0	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2	0	0	(x_1, x_2)			
E_{2u}	2	1	-1	-2	-1	1	2	0	0	-2	-1	1	2	1	-1	-2	0	0				
E_{3u}	2	0	-2	0	2	0	-2	0	0	-2	0	2	0	-2	0	2	0	0				
E_{4u}	2	-1	-1	2	-1	-1	2	0	0	-2	1	1	-2	1	1	-2	0	0				
E_{5u}	2	$-\sqrt{3}$	1	0	-1	$\sqrt{3}$	-2	0	0	-2	$\sqrt{3}$	-1	0	1	$-\sqrt{3}$	2	0	0				(w_1, w_2)

Table 25. $D_{12}(1222)$.

	E	$2c_{12}$	$2c_6$	$2c_4$	$2c_3$	$2c_{12}^5$	c_2	$6c_2'$	$6c_2''$				
A_1	1	1	1	1	1	1	1	1	1		$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	1	1	1	1	-1	-1	x_3, R_{x_3}			
B_1	1	-1	1	-1	1	-1	1	1	-1			$W_{11}-W_{22}$	
B_2	1	-1	1	-1	1	-1	1	-1	1			$W_{21}+W_{12}$	
E_1	2	$\sqrt{3}$	1	0	-1	$-\sqrt{3}$	-2	0	0	$(x_1,x_2),(R_{x_1},R_{x_2})$	(x_1x_3, x_2x_3)		
E_2	2	1	-1	-2	-1	1	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$		
E_3	2	0	-2	0	2	0	-2	0	0				
E_4	2	-1	-1	2	-1	-1	2	0	0			$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_5	2	$-\sqrt{3}$	1	0	-1	$\sqrt{3}$	-2	0	0			(W_{13}, W_{23})	(w_1,w_2)

Table 26. $C_{12\mathrm{v}}(12mm)$.

	E	$2c_{12}$	$2c_6$	$2c_4$	$2c_3$	$2c_{12}^5$	c_2	$6\sigma_v$	$6\sigma_d$				
A_1	1	1	1	1	1	1	1	1	1	x_3	$x_1^2+x_2^2, x_3^2$		
A_2	1	1	1	1	1	1	1	-1	-1	R_{x_3}			
B_1	1	-1	1	-1	1	-1	1	1	-1			$W_{11}-W_{22}$	
B_2	1	-1	1	-1	1	-1	1	-1	1			$W_{21}+W_{12}$	
E_1	2	$\sqrt{3}$	1	0	-1	$-\sqrt{3}$	-2	0	0	$(x_1,x_2),(R_{x_1},R_{x_2})$	(x_1x_3, x_2x_3)		
E_2	2	1	-1	-2	-1	1	2	0	0		$(x_1^2-x_2^2, 2x_1x_2)$		
E_3	2	0	-2	0	2	0	-2	0	0				
E_4	2	-1	-1	2	-1	-1	2	0	0			$(W_{11}+W_{22}, W_{21}-W_{12})$	
E_5	2	$-\sqrt{3}$	1	0	-1	$\sqrt{3}$	-2	0	0			(W_{13}, W_{23})	(w_1,w_2)

Table 27. $I(235)$ (Note: $\mathrm{l}/\tau = 2\cos 72°, -\tau = 2cos144°$, $^*x_1^2+x_2^2+x_3^2$, $^{**}(x_1,x_2,x_3)$, $^{***}(R_{x_1},R_{x_2},R_{x_3})$, $^{****}(w_1,w_2,w_3)$, $^{*****}(\frac{1}{\sqrt{3}}(W_{11}-W_{31}-W_{13}), \frac{1}{\sqrt{3}}(W_{21}+W_{32}+W_{23}), \frac{1}{\sqrt{3}}(W_{31}+W_{22}-W_{13}), \frac{1}{\sqrt{3}}(W_{12}+W_{32}-W_{23}))$, $^{******}(\frac{1}{\sqrt{6}}(x_1^2+x_2^2-2x_3^2), \frac{1}{\sqrt{2}}(x_1^2-x_2^2), \sqrt{2}x_1x_2, \sqrt{2}x_1x_3, \sqrt{2}x_2x_3)$, $^{*******}(W_{33}, \frac{1}{\sqrt{3}}(W_{11}+W_{22}+W_{33}), \frac{1}{\sqrt{3}}(W_{21}-W_{12}-W_{23}), \frac{1}{\sqrt{3}}(W_{11}+W_{31}-W_{22}), \frac{1}{\sqrt{3}}(W_{32}-W_{21}-W_{12}))$.

	E	$12c_5$	$12c_5^2$	$20c_3$	$15c_2$				
A	1	1	1	1	1		*		
T_1	3	τ	$-1/\tau$	0	-1	**, ***			
T_2	3	$-1/\tau$	τ	0	-1				****
G	4	-1	-1	1	0			*****	
H	5	0	0	-1	1		******		

Table 28. $I_{\mathrm{h}}(\mathrm{m}\bar{3}\bar{5})$ (Note: $\mathrm{l}/\tau = 2\cos 72°, -\tau = 2\cos 144°$, $^*x_1^2+x_2^2+x_3^2$, $^{**}(x_1,x_2,x_3)$, $^{***}(R_{x_1},R_{x_2},R_{x_3})$, $^{****}(w_1,w_2,w_3)$, $^{*****}(\frac{1}{\sqrt{3}}(W_{11}-W_{31}-W_{13}), \frac{1}{\sqrt{3}}(W_{21}+W_{32}+W_{23}), \frac{1}{\sqrt{3}}(W_{31}+W_{22}-W_{13}), \frac{1}{\sqrt{3}}(W_{12}+W_{32}-W_{23}))$, $^{******}(\frac{1}{\sqrt{6}}(x_1^2+x_2^2-2x_3^2), \frac{1}{\sqrt{2}}(x_1^2-x_2^2), \sqrt{2}x_1x_2, \sqrt{2}x_1x_3, \sqrt{2}x_2x_3)$, $^{*******}(W_{33}, \frac{1}{\sqrt{3}}(W_{11}+W_{22}+W_{33}), \frac{1}{\sqrt{3}}(W_{21}-W_{12}-W_{23}), \frac{1}{\sqrt{3}}(W_{11}+W_{31}-W_{22}), \frac{1}{\sqrt{3}}(W_{32}-W_{21}-W_{12}))$.

	E	$12c_5$	$12c_5^2$	$20c_3$	$15c_2$	I	$12s_{10}$	$12s_{10}^3$	$20s_6$	15σ	$l/\tau = 2\cos 72°$,	$-\tau = 2\cos 144°$		
A_g	1	1	1	1	1	1	1	1	1	1		*		
T_{1g}	3	τ	$-1/\tau$	0	-1	3	$-1/\tau$	τ	0	-1	***			
T_{2g}	3	$-1/\tau$	τ	0	-1	3	τ	$-1/\tau$	0	-1				
G_g	4	-1	-1	1	0	4	-1	-1	1	0			*****	
H_g	5	0	0	-1	1	5	0	0	-1	1		******	*******	
A_u	1	1	1	1	1	-1	-1	-1	-1	-1				
T_{1u}	3	τ	$-1/\tau$	0	-1	-3	$1/\tau$	$-\tau$	0	1	**			
T_{2u}	3	$-1/\tau$	τ	0	-1	-3	$-\tau$	$1/\tau$	0	1				****
G_u	4	-1	-1	1	0	-4	1	1	-1	0				
H_u	5	0	0	-1	1	-5	0	0	1	-1				

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