Vol 15 No 9, September 2006
1009-1963/2006/15(09)/2012-06

Chinese Physics

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Entanglement in a system of two two-level atoms interacting with a single-mode field*

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(Received 30 June 2005; revised manuscript received 22 May 2006)

We investigate the entanglement in a system of two coupling atoms interacting with a single-mode field by means of quantum information entropy theory. The quantum entanglement between the two atoms and the coherent field is discussed by using the quantum reduced entropy, and the entanglement between the two coupling atoms is also investigated by using the quantum relative entropy. In addition, the influences of the atomic dipole–dipole interaction intensity and the average photon number of the coherent field on the degree of the entanglement is examined. The results show that the evolution of the degree of entanglement between the two atoms and the field is just opposite to that of the degree of entanglement between the two atoms. And the properties of the quantum entanglement in the system rely on the atomic dipole–dipole interaction and the average photon number of the coherent field.

Keywords: quantum entanglement, reduced entropy, relative entropy
PACC: 4250

1. Introduction

Quantum entanglement is possessed of some unique features that make quantum mechanics distinct from classical theory. It was found that the entanglement formed a fundamental resource for quantum information processing,[1] such as quantum teleportation,[2] quantum encoding and error-correction,[3] cryptography[4] and computing.[5] It was noted that the preparation of an entangled state of two atoms is important in the study of entanglement, as the entangled state of photons cannot be stored for long.[6] Recently, Clark and Parkins proposed a scheme employing quantum-reservoir engineering to controllably entangle the internal states of two atoms trapped in a highly-finessed optical cavity.[7] Li *et al* studied the generation of an entangled state for a pair of two-level atoms embedded in a bad cavity injected with a squeezed vacuum by using the master equation, and found that the degree of entanglement between the two atoms strongly depended on the mean photon number, the strength of two-photon correlations with the squeezed vacuum, the cavity-induced decay rate of atoms and the atomic separation.[8] Kim *et al* investigated the interaction with the thermal field and a quantum system composed of two atoms (qubit) and found that such a chaotic field with minimal information could entangle atoms that had been prepared initially in a separable state.[9] However, little attention has been paid to the influence of the atomic dipole–dipole interaction on the degree of the entanglement in the system under consideration.[10] In fact, the atomic dipole–dipole interaction plays an important role in the dynamics, such as atomic inversion,[11] entropic dynamics[12] in a two-atom system.

In this paper, we investigate the entanglement properties in the system consisting of the two two-level atoms with dipole–dipole interaction interacting with a single mode coherent field. The quantum entanglement between two two-level atoms and the single-mode coherent field is studied by using the quantum reduced entropy, and the quantum entanglement between two atoms is also investigate by using the quantum relative entropy. In addition, the influences of the dipole–dipole interaction and the average photon number of the coherent field on the entanglement properties of the system are also examined.

2. The model and its solution

The system under consideration consists of two two-level atoms and a single-mode of quantized field,

*Project supported by the National Natural Science Foundation of China (Grant No 10374025).
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and the Hamiltonian of this system in the rotating-wave approximation is[11]

$$H = \omega_0 \sum_{i=1}^{2} S_3^{(i)} + \omega a^+ a + \sum_{i=1}^{2} g\left(a^+ S_-^{(i)} + a S_+^{(i)}\right) + \Omega\left(S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)}\right), \quad (1)$$

where a^+ and a are creation and annihilation operators for the single-mode field, $S_3^{(i)}$; $S_\pm^{(i)}$ are the pseudospin operators of the ith atom; ω_0 and ω are the frequencies of the atomic transition and the field; g is the atoms-field coupling constant; Ω is the atomic dipole–dipole interaction constant. Here we assume that initially the two two-level atoms are in the excited state $|\psi_A\rangle$

$$|\psi_A(0)\rangle = |+,+\rangle, \quad (2)$$

and the field is coherent

$$|\psi_F(0)\rangle = \sum_{n=0}^{\infty} f_n |n\rangle,$$

and

$$f_n = \exp(-N/2)\frac{N^{n/2}}{\sqrt{n!}}\exp(\mathrm{i}n\varphi), \quad (3)$$

where N is the average photon number of the coherent field, and φ is the phase angle of the coherent field(in this paper $\varphi = 0$). So the initial state of the system is

$$|\psi_{FA}(0)\rangle = |\psi_F(0)\rangle \otimes |\psi_A(0)\rangle. \quad (4)$$

In the case of resonance ($\omega_0 = \omega$), by solving the Schrödinger equation of the system in the interaction picture, we can obtain the system vector at time t

$$|\psi_{FA}(t)\rangle = |a\rangle|+,+\rangle + |b\rangle|+,-\rangle + |c\rangle|-,+\rangle + |d\rangle|-,-\rangle, \quad (5)$$

with

$$\begin{aligned} |a\rangle &= \sum_{n=0}^{\infty} a(n,t)|n\rangle, \\ |b\rangle &= \sum_{n=0}^{\infty} b(n,t)|n+1\rangle, \\ |c\rangle &= \sum_{n=0}^{\infty} c(n,t)|n+1\rangle, \\ |d\rangle &= \sum_{n=0}^{\infty} d(n,t)|n+2\rangle, \end{aligned} \quad (6)$$

where

$$a(n,t) = \beta_0(n) + \beta_1(n)\exp[\mathrm{i}\alpha_1(n)t\,] + \beta_2(n)\exp[\mathrm{i}\alpha_2(n)t\,], \quad (7)$$

$$b(n,t) = c(n,t) \quad (8)$$

$$= -\left\{\,\alpha_1(n)\beta_1(n)\exp[\mathrm{i}\alpha_1(n)t\,] + \alpha_2(n)\beta_2(n)\exp[\mathrm{i}\alpha_2(n)t\,]\right\}\Big/\left(2g\sqrt{n+1}\right), \quad (9)$$

$$d(n,t) = \Big\{ -2(n+1)g^2\beta_0(n) + [\,\alpha_1^2(n) - 2(n+1)g^2 + \alpha_1(n)\Omega\,]\,\beta_1(n)\exp[\mathrm{i}\alpha_1(n)t\,] + [\alpha_2^2(n) - 2(n+1)g^2 + \alpha_2(n)\Omega\,]\,\beta_2(n)\exp[\mathrm{i}\alpha_2(n)t\,]\Big\}\Big/\left(2\sqrt{(n+1)(n+2)}g^2\right), \quad (10)$$

and

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\left[-\Omega + \sqrt{24g^2 + 16ng^2 + \Omega^2}\right], \\ \alpha_2 &= \frac{1}{2}\left[-\Omega - \sqrt{24g^2 + 16ng^2 + \Omega^2}\right], \\ \beta_0 &= \Big[\alpha_1(n)\alpha_2(n) + 2(n+1)g^2\Big] f_n \Big/ \Big[\alpha_1(n)\alpha_2(n)\Big], \\ \beta_1 &= 2(n+1)g^2 f_n \Big/ \Big\{\alpha_1(n)[\,\alpha_1(n) - \alpha_2(n)\,]\Big\}, \\ \beta_2 &= -2(n+1)g^2 f_n \Big/ \Big\{\alpha_2(n)[\,\alpha_1(n) - \alpha_2(n)\,]\Big\}. \end{aligned} \quad (11)$$

At any time $t > 0$ the reduced two two-level atomic density operator for the system is given by

$$\rho_A(t) = \mathrm{Tr}_F|\psi_{AF}(t)\rangle\langle\psi_{AF}(t)| = \begin{bmatrix} \langle a|a\rangle & \langle b|a\rangle & \langle c|a\rangle & \langle d|a\rangle \\ \langle a|b\rangle & \langle b|b\rangle & \langle c|b\rangle & \langle d|b\rangle \\ \langle a|c\rangle & \langle b|c\rangle & \langle c|c\rangle & \langle d|c\rangle \\ \langle a|d\rangle & \langle b|d\rangle & \langle c|d\rangle & \langle d|d\rangle \end{bmatrix}. \quad (12)$$

The eigenvalues of the reduced two two-level atomic density operator $\rho_A(t)$ can be obtained as follows:

$$\begin{aligned}
\lambda_A^1 =&0,\\
\lambda_A^2 =&\frac{1}{6}\left[2-\frac{2\cdot 2^{1/3}(3A-1)}{\xi}+2^{2/3}\xi\right],\\
\lambda_A^3 =&\frac{1}{6}\left[2+\frac{2(-2)^{1/3}(3A-1)}{\xi}+(-2)^{2/3}\xi\right],\\
\lambda_A^4 =&\frac{1}{6}\left[2-\frac{2(-1)^{2/3}2^{1/3}(3A-1)}{\xi}-(-1)^{1/3}2^{2/3}\xi\right],
\end{aligned} \tag{13}$$

where

$$\xi=\left[2-9A-27B+\sqrt{4(3A-1)^3+(-2+9A+27B)^2}\right]^{1/3}, \tag{14}$$

$$\begin{aligned}
A =&\langle a|a\rangle\langle b|b\rangle+\langle a|a\rangle\langle c|c\rangle+\langle a|a\rangle\langle d|d\rangle+\langle b|b\rangle\langle c|c\rangle+\langle b|b\rangle\langle d|d\rangle+\langle c|c\rangle\langle d|d\rangle-(\langle b|a\rangle\langle a|b\rangle\\
&+\langle c|a\rangle\langle a|c\rangle+\langle d|a\rangle\langle a|d\rangle+\langle c|b\rangle\langle b|c\rangle+\langle d|b\rangle\langle b|d\rangle+\langle d|c\rangle\langle c|d\rangle),
\end{aligned} \tag{15}$$

$$\begin{aligned}
B =&-\langle a|a\rangle\langle b|b\rangle\langle c|c\rangle-\langle a|a\rangle\langle b|b\rangle\langle d|d\rangle-\langle a|a\rangle\langle c|c\rangle\langle d|d\rangle-\langle b|b\rangle\langle c|c\rangle\langle d|d\rangle+\langle a|a\rangle\langle c|b\rangle\langle b|c\rangle\\
&+\langle a|a\rangle\langle d|b\rangle\langle b|d\rangle+\langle a|a\rangle\langle d|c\rangle\langle c|d\rangle+\langle b|b\rangle\langle d|a\rangle\langle a|d\rangle+\langle b|b\rangle\langle d|a\rangle\langle a|d\rangle+\langle b|b\rangle\langle d|c\rangle\langle c|d\rangle\\
&+\langle c|c\rangle\langle b|a\rangle\langle a|b\rangle+\langle c|c\rangle\langle d|a\rangle\langle a|d\rangle+\langle c|c\rangle\langle d|b\rangle\langle b|d\rangle+\langle d|d\rangle\langle b|a\rangle\langle a|b\rangle+\langle d|d\rangle\langle c|b\rangle\langle b|c\rangle\\
&-\langle b|a\rangle\langle a|c\rangle\langle c|b\rangle-\langle b|a\rangle\langle a|d\rangle\langle d|b\rangle-\langle c|a\rangle\langle a|b\rangle\langle b|c\rangle-\langle c|a\rangle\langle a|b\rangle\langle b|c\rangle-\langle c|a\rangle\langle a|d\rangle\langle d|c\rangle\\
&-\langle d|a\rangle\langle a|b\rangle\langle b|d\rangle-\langle d|a\rangle\langle a|c\rangle\langle c|d\rangle-\langle c|b\rangle\langle b|d\rangle\langle d|c\rangle-\langle d|b\rangle\langle b|c\rangle\langle c|d\rangle.
\end{aligned} \tag{16}$$

By using the eigenvalues of the reduced two two-level atomic density operator, we can study the entanglement properties of the considered system in the next section.

3. The entanglement between the two two-level atoms and the coherent field

For a bicomponent system in a pure state, Phoenix and Knight[13] and Buzek *et al* [14] have shown that the reduced entropy is a very accurate measure of the degree of entanglement between two components. Based on that, the considered system can be referred to as a quantum system with two components (the two atoms and a single-mode field). In this paper we study the entanglement between the two atoms and a single-mode field by using the reduced entropy. The time evolution of the two atomic (a single-mode field) reduced entropy reflects the time evolution of the degree of the entanglement between the two atoms and a single-mode field (DEAF). The higher the reduced entropy, the greater the DEAF.

The entropies of the two atoms and a single-mode field can be defined through their respective reduced density operators as

$$S_i(t)=-\mathrm{Tr}_i(\rho_i\log_2\rho_i),\quad (i=A\ \text{ or }\ F). \tag{17}$$

Since we have assumed that the two atoms and a single-mode field are initially in a disentangled pure state, the total entropy $S_{AF}(t)$ of the two-atom-field system is zero. In terms of the triangle inequality of the entropy[15]

$$|S_A(t)-S_F(t)|\le S_{AF}\le S_A(t)+S_F(t), \tag{18}$$

we can see that the reduced entropies of the two subsystem are identical, namely $S(\rho_A)=S(\rho_F)$. In Eq.(18), $S_{AF}(t)=-\mathrm{Tr}\{\rho_{AF}^{\mathrm{I}}\log_2\rho_{AF}^{\mathrm{I}}\}$ is the total entropy of the two-atom-field system. Consequently we only need to calculate the atomic reduced entropy $S_A(t)$. We can express the reduced entropy $S(\rho_A)$ of the two atoms in terms of the eigenvalue of $\rho_A(t)$ given by Eq.(13),

$$\begin{aligned}
S(\rho_A)=&-\sum_{i=1}^{4}\lambda_A^i\log_2\lambda_A^i\\
=&-\lambda_A^2\log_2\lambda_A^2-\lambda_A^3\log_2\lambda_A^3\\
&-\lambda_A^4\log_2\lambda_A^4.
\end{aligned} \tag{19}$$

At first, we discuss the influence of the atomic dipole–dipole interaction on the entanglement between the coherent field and the two two-level atoms by using the above results.

Figure 1 shows the time evolution of the reduced entropy $S(\rho_A)$ of the entanglement (the solid lines) with the average photon number of the field $N = 5$ and the different atomic dipole–dipole interaction intensities. From Figs.1(a)–(d), it can been seen that when the atomic dipole–dipole interaction is weaker ($\Omega/g = 1.0$), the DEAF has an irregular oscillation in the course of the time evolution (shown in Fig.1(a)); with the intensity of the atomic dipole–dipole interaction increasing, the time evolution of the DEAF gradually becomes periodic with a certain period (see Figs.1(b)–(c)); when the atomic dipole–dipole interaction is strong enough (i.e. the Ω/g is large enough), the time evolution of the DEAF displays an approximate periodic oscillation (see Fig.1(d)). Meanwhile, the DEAF decreases with the increase of the intensity of the atomic dipole–dipole interaction.

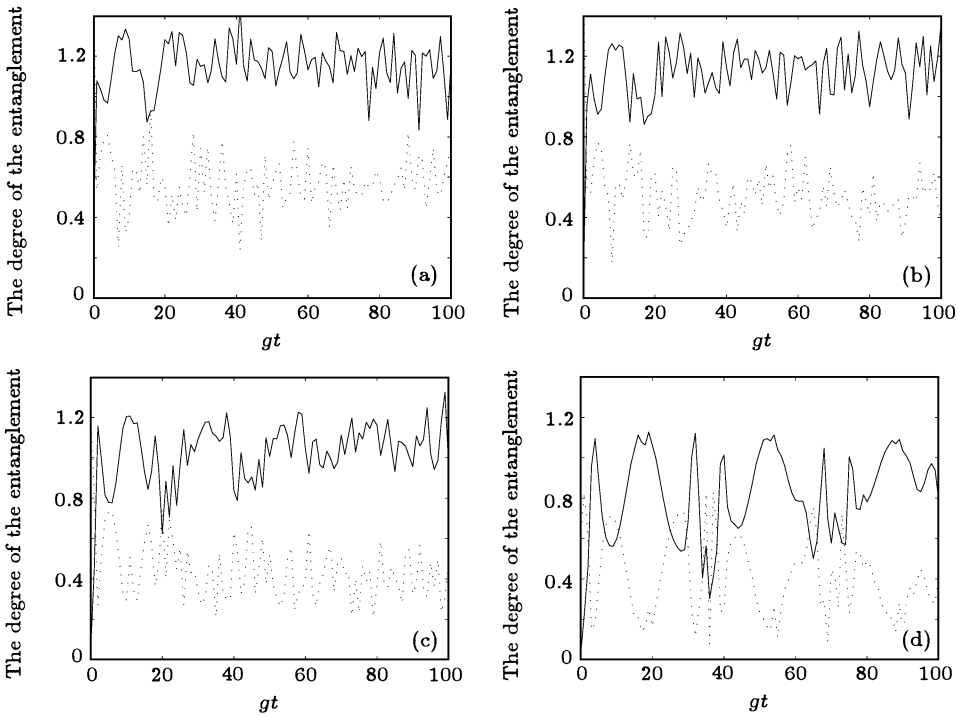


Fig.1. The time evolution of the degree of the entanglement. $S(\rho_A)$ is shown by the solid lines and $E_{\mathrm{R}}(\rho_A)$ by the dotted lines with $N = 5$ and (a) $\Omega/g = 1.0$, (b) $\Omega/g = 5.0$, (c) $\Omega/g = 10.0$, (d) $\Omega/g = 20.0$.

Furthermore, we discuss the effects of the average photon number of the field on the DEAF (the solid lines in Fig.2) with $\Omega/g = 20.0$ and different average photon numbers of the coherent field N. From Figs.2(a)–(d), it is obvious that the amplitude and the period of oscillation of the DEAF both increases monotonically with N. This result shows that if the two-atom-field entangled state is prepared, we should diminish the atomic dipole–dipole interaction intensity and enhance the average photon number of the field.

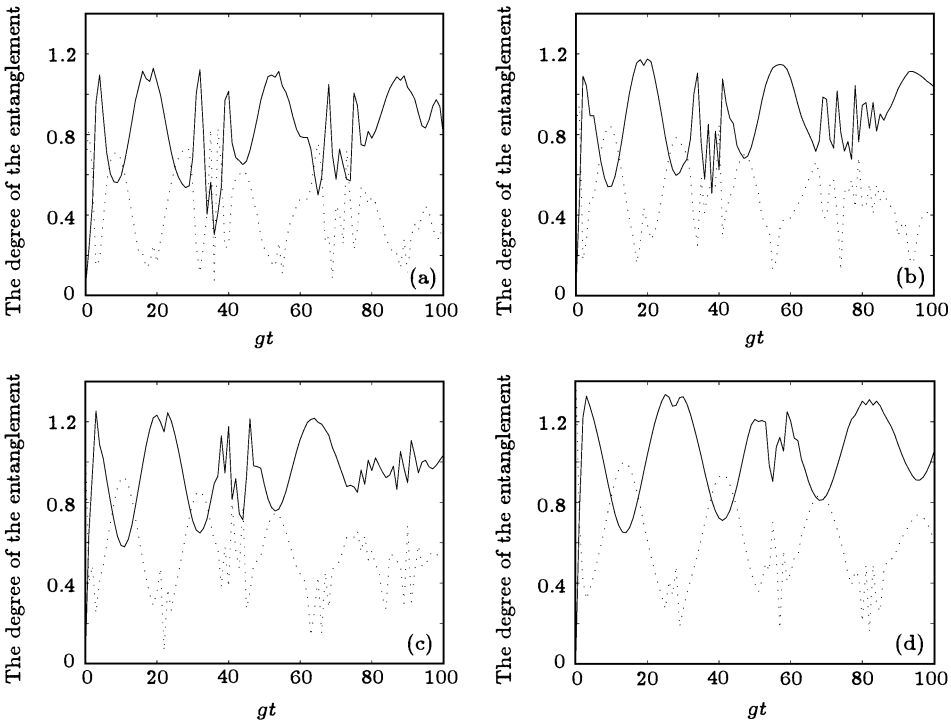


Fig.2. The time evolution of the degree of the entanglement. $S(\rho_A)$ is shown by the solid lines and $E_{\rm R}(\rho_A)$ by the dotted lines with $\Omega/g = 20.0$ and (a) $N = 5$, (b) $N = 10$, (c) $N = 20$, (d) $N = 50$.

4. The entanglement between two two-level atoms

The above discussion shows that Hamiltonian given by Eq.(1) leads to the entanglement between the two atoms and the coherent field, so the state of the two atoms evolves into a mixed state. In this case, the degree of the entanglement between the two atoms cannot be measured by the reduced entropy of the entanglement. However, Vedral *et al*[16] and Liu *et al*[17] have proved that the relative entropy of the entanglement is a good measure for the degree of entanglement of two-particle mixed states, which is defined as

$$E_{\rm R}(\rho) = \min_{\sigma\in D} S(\rho \parallel \sigma), \tag{20}$$

where $S(\rho \parallel \sigma) = {\rm Tr}[\rho(\log_2\rho - \log_2\sigma)]$. This is the quantum relative entropy, and its minimum is taken to be over that of D, the set of all disentangled states. The relative entropy of the entanglement is viewed as the minimal 'distance' between the state ρ and the disentangled state σ. For a pure state, the relative entropy of the entanglement reduces to the reduced entropy of the entanglement.

For the two-particle mixed states, it is difficult to calculate numerically the relative entropy of the entanglement. Recently, the following theorem on the relative entropy of the entanglement has been proved,[18] and it is quite suitable for the present analysis.

If a bipartite quantum state can be expressed as

$$\rho = \sum_{n_1,n_2} a_{n_1,n_2} |\Phi_{n_1}, \phi_{n_1}\rangle\langle\Phi_{n_2}, \phi_{n_2}|, \tag{21}$$

where $|\Phi_{n_1}\rangle$ and $|\phi_{n_1}\rangle$ are the orthonormal states of each subsystem, then the relative entropy of the entanglement is given by

$$E_{\rm R}(\rho) = -\sum_n a_{n,n}\log_2 a_{n,n} + {\rm Tr}(\rho\log_2\rho), \tag{22}$$

and the disentanglement state σ that minimizes the quantum relative entropy is given by

$$\sigma = \sum_n a_{n,n}\log_2 a_{n,n} + {\rm Tr}(\rho\log_2\rho). \tag{23}$$

It is clear that $\rho_A(t)$ given by Eq.(12) can be written in the same form as that in Eq.(21) as follows:

$$\rho_A(t) = \sum_{m_1,m_2} a_{m_1,m_2} |m_1, m_1\rangle\langle m_2, m_2|, \tag{24}$$

with

$$a_{m_1,m_2} = \langle m_1, m_1|\rho_A(t)|m_2, m_2\rangle, \tag{25}$$

where $|m_1\rangle = |\pm\rangle$ and $|m_2\rangle = |\pm\rangle$. Consequently, the relative entropy of the entanglement of the state

Eq.(12) is written as

$$\begin{aligned} E_{\mathrm{R}}(\rho_A) &= -\sum_m a_{m,m}\log_2 a_{m,m} - S(\rho_A) \\ &= -\sum_m a_{m,m}\log_2 a_{m,m} \\ &\quad + \sum_{i=1}^{4} \lambda_A^i \log_2 \lambda_A^i. \end{aligned} \tag{26}$$

Hence, it is possible to compute numerically the relative entropy of the entanglement given by Eq.(26). The time evolutions of the relative entropy of the entanglement $E_{\mathrm{R}}(\rho_A)$ are depicted by using the dotted lines in Figs.1 and 2, which reflect the time evolution of the degree of entanglement between the two two-level atoms (DEAA). The DEAA increases monotonically with the relative entropy. Comparing the dotted lines with the solid lines in Figs.1(a)–1(d), we can find that the time evolution of the DEAA is just opposite to that of the DEAF, that is, when the latter increases with time, the former decreases with time. This comes from the fact that the DEAF can impair the DEAA due to the two atoms interacting with the coherent field.

5. Conclusions

We have investigated the quantum entanglement in a system of two two-level atoms with dipole–dipole interaction interacting with a single-mode coherent field. We have discussed the quantum entanglement between two two-level atoms and the coherent field by using the reduced entropy and that between the two atoms by the relative entropy separately. Our results show that the evolution of the degree of entanglement between the two atoms and the field is just opposite to that of the degree of entanglement between the two atoms: that is, the degree of the entanglement between the two atoms and the field can impair the degree of entanglement between the two atoms. The properties of the quantum entanglement in the system rely on the atomic dipole–dipole interaction and the average photon number of the coherent field. An increase in the atomic dipole–dipole interaction leads to the periodic evolution of entanglement and a reduction in the degree of entanglement in the system; an increase in the average photon number of the single-mode field results in a prolongation of the evolution periodicity of entanglement dynamics and an increase in the degree of entanglement.

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