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Dissipative dynamics of dark solitons in elongated Bose–Einstein condensates*

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The dissipative dynamic stability is investigated of dark solitons in elongated Bose–Einstein condensates that can be described by the Gross–Pitaevskii equation including an additional term. Based on the direct perturbation theory for the nonlinear Schrödinger equation, the dependence of the soliton velocity on time is explicitly given, and the shape of dark solitons remaining unchanged under the dissipative condition is confirmed theoretically for the first time. It is found that the dynamically stable dark solitons turn out to be thermodynamically unstable.

Keywords: dark solitons, kink states

PACC: 0375, 6740K, 1190, 0290

1. Introduction

The recent successful experiments on Bose–Einstein condensations (BECs) in trapped clouds of alkali atoms[1−3] have aroused the interest in the physics of ultracold gases. Dark solitons have been created before in single-component BECs using the phase-imprinting methods.[4] These experiments provide an unprecedented possibility of studying the dissipative dynamics of such macroscopically excited Bose-condensed states. The dynamically stable vortices decay only in the boundary of the condensate.

Dark solitons, also called 'kink-states', are expected to exist under repulsive interaction ($a > 0$), which have a density dip and a phase shift in a direction and are particular solutions of the Gross–Pitaevskii (GP) equation, as well as vortices. Dark solitons are prone to transverse instability,[5−7] which have been observed to bend via the 'snake' instability and ultimately decay into vortices, in both BECs[8] and nonlinear optics.[9] However, by imposing strong transverse confinement to freeze out the transverse degree of freedom, quasi-one-dimensional condensates can be created.[10,11] Meanwhile, the transverse instability of dark solitons can be suppressed by a strong radial confinement of their motion in elongate traps.[12] Several theoretical studies have discussed their dynamics and stability.[13,14] The evolution of the soliton velocity in cigar-shaped BECs has been realized in experiment,[15] which demonstrates that the moving soliton keeps its shape unchanged, however, velocity of the soliton increases due to the thermodynamic instability. Thermal instabilities, arising from the interaction with the thermal cloud, dissipate energy from the soliton, causing it to be accelerated toward the edge of the condensate and ultimately disappear.[16] In this paper, we study the dissipative dynamics of a moving dark soliton in 3D elongated Bose–Einstein condensates. We consider the case where a dark soliton is subjected to a strong radial confinement and also moves along the axial direction, following the quasi-one-dimensional (GP) equation[17] that is a standard nonlinear Schrödinger (NLS) equation with an additional term, but causes the dissipative dynamics of the soliton. The interaction between the axial and radial degrees of freedom is described by a dissipative term, which originates from the correction to the mean-field theory.[18] These effects make the NLS equation slightly different from the standard one by an additional term that can be considered as a perturbation. The direct perturbation theory for bright solitons has been developed.[19] We apply the direct perturbation theory[20] to a dark soliton based on a complete set of the squared Jost solutions to discuss the influence of these concrete parameters on the central velocity, depth and other relevant quantities of the dark soliton. The dark soliton shape remaining unchanged in elongated BECs is confirmed for the first time.

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2. NLS equation

We consider a condensate with repulsive interaction (the scattering length $a > 0$). The condensate wavefunction can be written as $\Psi(\boldsymbol{r},t)\exp(-\mu t)$,[17] where μ is the chemical potential. In an infinitely long cylindrical harmonic trap the function satisfies the GP equation

$$\mathrm{i}\hbar\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t}=\left\{-\frac{\hbar^2}{2m}\Delta+\frac{m}{2}\omega_\rho^2\rho^2+g\left|\Psi(\boldsymbol{r},t)\right|^2-\mu\right\}\Psi(\boldsymbol{r},t), \tag{1}$$

where ω_ρ is the frequency of the radial (ρ) confinement, $g=\frac{4\pi\hbar^2 a}{m}$ characterizes the interaction and is determined in terms of the ground-state scattering length a, m is the atomic mass. Dark solitons are the solutions of the 1D GP equation in free space. We apply an analytical approach to describing a moving soliton. We consider a limiting case, where the axial size L of the soliton notch greatly exceeds the radial size r of the condensate. And we will confirm that the soliton is dynamically stable. In an infinitely long cylindrical harmonic trap, the shape and velocity of the soliton remain unchanged in the absence of dissipation. The wavefunction Ψ depends on ρ and $x=z-v_0t$; we transform $\Psi(\rho,x)=\psi(\rho,x)f(x)$, where $\psi(\rho,x)$ is referred to as a slow variable along the x direction, but $f(x)$ as a fast variable along the x direction. These functions $\psi(\rho,x)$ and $f(x)$ satisfy the equations

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t}=\left\{-\frac{\hbar^2}{2m}\left[\Delta+2\left(\frac{\nabla_x f}{f}\right)\nabla_x\right]+\frac{m\omega_\rho^2\rho^2}{2}+g|f|^2|\psi|^2-\widetilde{\mu}(|f|)\right\}\psi, \tag{2}$$

$$\mathrm{i}\hbar\frac{\partial f}{\partial t}=-\frac{\hbar^2}{2m}\triangle_x f+[\widetilde{\mu}(f)-\mu]f. \tag{3}$$

Here $\widetilde{\mu}$ is a function of f and can be found self-consistently from Eqs.(2) and (3). The function $\psi(\rho,x)$ is selected such that at infinite x it becomes the wavefunction of ground-state condensate, $\psi_0(\rho)$. Hence, for $|x|\to\infty$ we have $|f|\to 1$ and $\widetilde{\mu}\to\mu$. Under the condition $r\ll L$ the radial distribution of particles is close to that for the ground-state condensate. Thus $\widetilde{\mu}(f)$ is close to μ and can be expressed as

$$\widetilde{\mu}(f)=\mu+(|f|^2-1)g\frac{\partial\mu}{\partial g}+\delta\widetilde{\mu}, \tag{4}$$

where $\delta\widetilde{\mu}$ is a correction of higher order in $\dfrac{r}{L}$.

$$\delta\widetilde{\mu}=-\frac{\gamma_0 n_{0m}g}{2}(|f|^2-1)^2,\ \gamma_0=\frac{3n_{0m}g\ln(\frac{4}{3})}{\hbar\omega_\rho}\ll 1. \tag{5}$$

Equations (2) − (5) come from Ref.[17]. We have $g\dfrac{\partial\mu}{\partial g}=m\bar{c_{\mathrm{s}}}^2$ and $\bar{c_{\mathrm{s}}}=\sqrt{\dfrac{n_{0m}g}{2m}}$, where $\bar{c_{\mathrm{s}}}$ is the speed of axially propagating sound in the ground-state condensate.

We substitute $\widetilde{\mu}(f)$ including the higher order correction $\delta\mu$ into Eq.(3) where the function $f(x)$ satisfies an ordinary 1D GP equation with a dissipative term. Introducing dimensionless variables $t\to\dfrac{m\bar{c_s}^2}{2\hbar}t$ and $x\to\dfrac{m\bar{c_s}}{\hbar}x$, Eq.(3) becomes

$$\mathrm{i}\frac{\partial f}{\partial t}-\frac{d^2f}{dx^2}+2(|f|^2-1)f=\gamma_0 2(|f|^2-1)^2 f. \tag{6}$$

The γ_0 is so small that the term on the right-hand side of Eq.(6) can be regarded as a perturbation term. The term on the right-hand side represents the interaction between the axial and the radial degrees of freedom. A dark soliton can be treated as the like of a heavy classical particle with negative mass. This term means the energy and momentum exchanges between the soliton and the thermal cloud. In the absence of this small term, there are no energy and momentum exchanges between the soliton and thermal cloud. We apply a systematic method to investigating the effect

of perturbation on the depth and width, phase and velocity of the soliton.

3. Direct approach and soliton parameters

We consider an ordinary NLS equation with a perturbation term as follows:

$$w_t - w_{xx} + 2(|w|^2 - \rho^2)w = \epsilon q[w], \tag{7}$$

where ϵ is a small positive constant measuring the degree of perturbation ($0 < \epsilon \ll 1$). The perturbation term $q[w]$ is a known functional of w. We introduce a slow time variable $\tau = \epsilon t$ and denote the time derivative ∂_t by $\partial_t + \epsilon\partial_\tau$. At the same time, we rewrite $w(x,t) = u(x,t) + \epsilon v(x,t)$. Applying the denotation $\partial_t + \epsilon\partial_\tau$ and the expression of $w(x,t)$ to Eq.(7), and equating the coefficients of each power of ϵ on both sides of the resulting equation, we obtain the following approximation equations:

$$\mathrm{i}u_t - u_{xx} + 2(|u|^2 - \rho^2)u = 0, \tag{8}$$

$$\begin{aligned}\mathrm{i}v_t - v_{xx} + 2(2|v|^2 - \rho^2)v + 2u^2\overline{v} &= (q[u] - \mathrm{i}v_\tau)\\ &= R[u],\end{aligned} \tag{9}$$

where a overbar denotes complex conjugation. The appropriate initial conditions are

$$u(x,0) = \mathrm{e}^{-\mathrm{i}\theta}[\alpha + \mathrm{i}\beta(x - x_0)], \tag{10}$$

$$v(x,0) = 0. \tag{11}$$

Introducing a set of coordinates moving together with the soliton

$$\begin{aligned}&t, \quad x \longrightarrow t, \quad z = x - x_0 - 2\alpha t;\\ &\partial t, \quad \partial_x \longrightarrow \partial t - 2\alpha t, \quad \partial_z,\end{aligned} \tag{12}$$

and combining Eq.(9) with its complex conjugate equation, we can obtain

$$\{\partial_t - \boldsymbol{L}(\boldsymbol{z})\}\boldsymbol{v} = \boldsymbol{R}, \tag{13}$$

in which

$$\boldsymbol{L}(\boldsymbol{z}) = \begin{pmatrix} \partial_z^2 + 2\mathrm{i}\alpha\partial_z - 2(2|u|^2 - \rho^2) & -2u^2 \\ 2\bar{u}^2 & -\partial_z^2 + 2\mathrm{i}\alpha\partial_z + 2(2|u|^2 - \rho^2) \end{pmatrix}, \tag{14}$$

$$\boldsymbol{v} = \begin{pmatrix} v \\ \bar{v} \end{pmatrix}, \qquad \boldsymbol{R} = \begin{pmatrix} R \\ \bar{R} \end{pmatrix}. \tag{15}$$

Equation (8) has a single soliton solution that can be rewritten as

$$u = \mathrm{e}^{-\mathrm{i}\theta}(\alpha + \mathrm{i}\beta\tanh z), \quad z = \beta(x - \xi), \quad \xi_t = 2\alpha, \quad \rho\mathrm{e}^{\mathrm{i}\theta} = \alpha + \mathrm{i}\beta. \tag{16}$$

Due to perturbation, the soliton parameters α, β, and ξ should be functions of the slow time variable τ, but α, β and θ are all independent of t, and the t-dependence of ξ is given by $\xi_t = 2\alpha$. $h = \sqrt{\alpha^2 + \beta^2} - \alpha$, $\delta = \dfrac{1}{\beta}$ which represent the soliton depth and width respectively.

The key to direct method of solving Eq.(13) is the basis vectors of a complete set constructed by using the squared Jost solutions. The basic vectors contain continuous spectrum and the discrete spectrum basis vectors, which obey the specified orthogonality and completeness relationship. Using secular conditions, we obtain the evolution equation of single-soliton parameters and a formula for calculating the first-order correction (see Ref.[20]) as follows:

$$\alpha_\tau = 0, \ \beta_\tau = 0, \quad \theta_\tau = 0, \quad \xi_\tau = \frac{1}{\beta^2}(\alpha^5 - 2\alpha^3 + \alpha - \frac{1}{9}\alpha\beta^4). \tag{17}$$

Considering the dependence of the phase θ on the soliton parameters α and β, we can derive

$$\xi(\theta)_\tau = \cos^3\theta\cot^2\theta - 2\cot^2\theta\cos\theta + \cot\theta\csc\theta - \frac{1}{9}\cos\theta\sin^2\theta. \tag{18}$$

We obtain the parameters of the soliton evolution

$$\alpha = \alpha(0), \beta = \beta(0), \theta = \theta(0), v_0 = 2\alpha(0) + \epsilon t\gamma[\gamma\alpha(0)^2 - 2\gamma\alpha(0) + \frac{1}{2\beta(0)} - \frac{\beta(0)^3}{9}], \tag{19}$$

where $\alpha(0)$, $\beta(0)$ and $\theta(0)$ are the initial soliton parameters. $\gamma = \frac{\alpha}{\beta}$, We can clearly see that the depth (as well as the width) of the dark soliton is unrelated to t. The term on the right-hand side of Eq.(6) is a dissipative term which takes into account the interaction between the axial and radial degrees of freedom. Consider a strong radial confinement that is conditional on a non-Thomas–Fermi (TF) regime having the radial size of the condensate $r \ll L$. Under this condition, the dynamic stability has been predicted in Ref.[17] and confirmed by our theory. The soliton shape remains unchanged in the limiting case $L \gg r$. The interaction between the soliton and thermal cloud leads to the reflection of the soliton, which provides a momentum transfer from the thermal cloud to the soliton. Hence, there is a friction force acting on the soliton. Dark soliton in BEC is characterized by a local density minimum and a sharp phase gradient of the wavefunction at the position of the minimum. In the process, the repulsion of interparticle tends to reduce the minimum but the phase gradient tries to enhance it. When they are balanced, the friction force cannot change the soliton shape, which has been proved in experiment.[15] We have applied a theory to pursuing there to be no shape change, although the perturbative treatment of noncondensed part at zero temperature exists already. The effect of this term on the soliton shape does not completely exhibit; however, it enhances the integrability of Eq.(8) and leads to the reflection of excitation from the soliton. The friction force gives an additional mean-velocity to the soliton and makes the soliton move faster.

Figure 1 shows the evolution of the additional velocity of the soliton ξ_τ versus initial phase. From $x = z - v_0 t$ we can obtain the practical velocity

$$v_0 = 2\alpha + \xi_\tau. \tag{20}$$

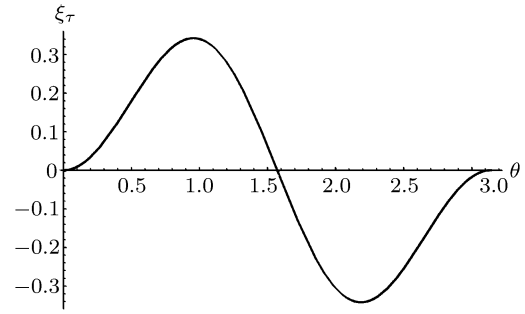


Fig.1. The additional velocity of the soliton ξ_τ versus initial phase θ.

The dark soliton will remain the constant velocity $v_0 = 2\alpha = 2\cos\theta$ in the absence of the dissipative term of Eq.(6). We discuss the relations between the soliton phase and velocity v_0 in three different cases: (i) $\theta = 0$, the soliton moves towards the boundary keeping the initial velocity of v_0 unchanged, which corresponds to the 'black' soliton; (ii) $0 < \theta < \frac{\pi}{2}$, and $\xi_\tau > 0$, where the soliton centre velocity v_0 increases, compared with the soliton velocity without dissipation. The soliton energy can be written in the form of $H = \frac{M\bar{c}_s^2}{3}(1 - \frac{v_0^2}{\bar{c}_s^2})^2$, with M being the effective mass of the soliton. As the soliton velocity v_0 increases, the soliton energy decreases; (iii) $\frac{\pi}{2} < \theta < \pi$, and we obtain $\xi_\tau < 0$, the soliton velocity v_0 increases in the process moving from the condensate centre to the boundary, which also leads to the soliton energy decreasing as well. The presence of dissipation provides an additional velocity and reduces its energy, which is in agreement with the result in Ref.[17].

4. First-order correction

On the basis of direct approach in Ref.[19], we can acquire the first-order correction of the solution w to Eq.(2):

$$V(z,t) = \frac{\mathrm{i}}{\beta^2} P \int_{-\infty}^{\infty} \mathrm{d}k \left[g^{(+)}(k) \frac{1}{\lambda^{(+)}} \left(\mathrm{e}^{-\mathrm{i}\beta^2\lambda^{(+)}t} - 1 \right) \phi^{(+)}(z,k) \right.$$
$$\left. + g^{(-)}(k) \frac{1}{\lambda^{(-)}} \left(\mathrm{e}^{-\mathrm{i}\beta^2\lambda^{(-)}t} - 1 \right) \phi^{(-)}(z,k) \right], \tag{21}$$

where

$$g^{(\pm)}(k) = \frac{\mathrm{i}\pi\mathrm{csch}(\frac{k\pi}{2})}{\sqrt{2\pi k\nu}}\left[-2\gamma a_1 + \frac{1}{4}u_{\pm}k^2 a_3 + \frac{1}{36}a_5 k^2(k^2+4)(u_{\pm}+\gamma) - 2b_0 + \frac{1}{12}b_4 k^2(4+k^2)\right] \tag{22}$$

with

$$\begin{aligned} a_1 &= 2\beta\alpha^4 - 4\alpha^2\beta + 2\beta + 4\alpha^2\beta^3 + 2\beta^5, \\ a_3 &= 4\beta^3 - 4\alpha^2\beta^3 - 4\beta^5, \quad a_5 = 2\beta^2, \\ b_0 &= -2\alpha^5 - 2\alpha + 4\alpha^3 + 4\beta^2\alpha - 4\alpha^3\beta^2 - 2\alpha\beta^4, \\ b_2 &= 4\alpha^3\beta^2 - 4\beta^2\alpha + 4\alpha\beta^4, \quad \text{and} \quad b_4 = -2\alpha\beta^4; \end{aligned} \tag{23}$$

and

$$\phi^{(\pm)}(z,k) = \frac{\mathrm{e}^{\mathrm{i}kz}}{\sqrt{2\pi k\nu}\mu^{(\pm)}}\begin{pmatrix} -\left[\frac{\mathrm{i}}{2}\left(\mu^{(\pm)}-k\right)+\tanh z\right]^2 \mathrm{e}^{-\mathrm{i}\theta} \\ \left[\frac{\mathrm{i}}{2}\left(\mu^{(\pm)}+k\right)-\tanh z\right]^2 \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \tag{24}$$

with

$$u^{(\pm)} = -2\gamma + \nu, \quad \text{and} \quad \nu = \sqrt{k^2+4}. \tag{25}$$

Owing to the analytically unsolvable integration included in Eq.(21), we can make a numerical analysis about the first-order solution. Select a small positive constant $\gamma_0 = 0.1$, $t = -2$, and parameters α and β that are different in three sections, and substitute these numerical values into Eqs.(21)–(25), and we will be able to display $|v|^2$ versus z as shown in Fig.2.

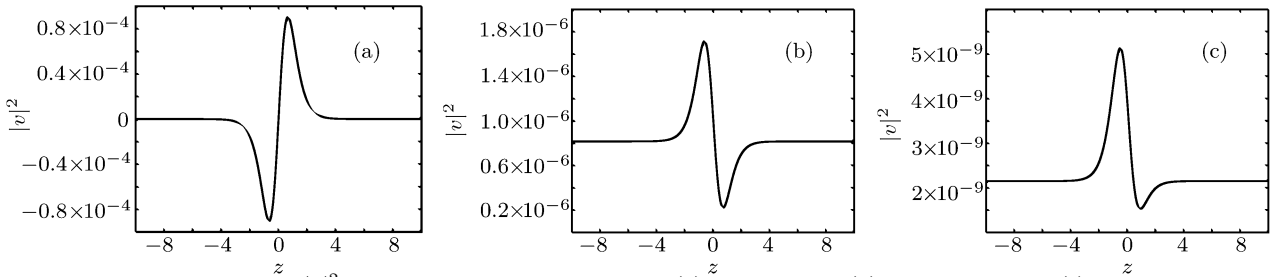


Fig.2. The first-order solution $|v|^2$ versus z, where $\gamma_0 = 0.1$, $t = -2$ (a) $\alpha = 0$, $\beta = 1$, (b) $\alpha = 0.8$, $\beta = 0.6$, (c) $\alpha = 0.95$, $\beta = 0.32$.

From Fig.2 we can see clearly that the first-order correction of the central location of dark soliton changes slightly, which corresponds to the soliton moving in the axial direction. The $|v|^2$ remains constant along the axis direction except the vicinity of the central location. On the basis of our analysis, the axial size L of the soliton notch greatly exceeds the radial size r, which corresponds to the size δ of the soliton width that greatly exceeds the size h of the soliton depth. The dark soliton is dynamically stable and not affected by the dissipative term. Considering two cases where the radial size $r > L$ and $r \sim L$, corresponding respectively to the soliton width $\delta < h$ and $\delta \sim h$, the first-order-correction wave shapes of the dark soliton are plotted in Figs.2(a) and 2(b) respectively, from which we can see that the dark soliton is unstable dynamically. From Fig.2 we can draw a conclusion that the correction to wave shape of the dark soliton becomes less and less important with the parameter r/L decreasing, that is, the ratio δ/h increasing. Once the radial size $r \ll L$, that is, the soliton width $\delta \gg h$, the correction to the soliton shape is completely absent. Based on our numerical analysis, the stability of the dark soliton in the limiting case $L \gg r$ can be expected. Figure 3 shows the evolution $|w|^2$ including the first-order-correction with z.

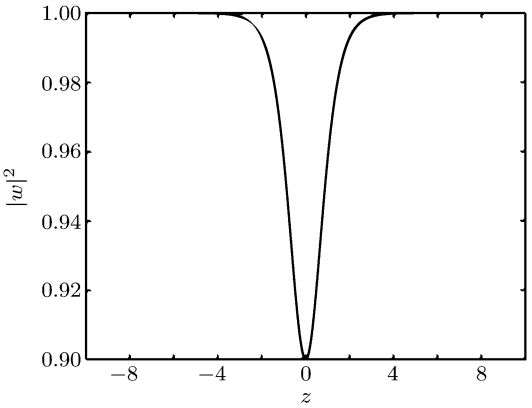


Fig.3. $|w|^2$ versus z with the parameters the same as those in Fig.2(c).

5. Conclusions

In this paper, we apply the direct method to the study of the effects of dissipation on the dark soliton. Based on our theory, it is confirmed that the dissipative term has no correction to the soliton shape in the limiting case, but give an additional effect on the soliton velocity. The soliton velocity increases and the energy decreases when the soliton moves toward the condensate boundary. We give the first-order correction to a stationary solution. Our theory is helpful for a understanding of experimental results about a soliton under dissipative condition.

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