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A new type of adiabatic invariants for nonconservative systems of generalized classical mechanics*

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The perturbations to symmetries and adiabatic invariants for nonconservative systems of generalized classical mechanics are studied. The exact invariant in the form of Hojman from a particular Lie symmetry for an undisturbed system of generalized mechanics is given. Based on the concept of high-order adiabatic invariant in generalized mechanics, the perturbation to Lie symmetry for the system under the action of small disturbance is investigated, and a new adiabatic invariant for the nonconservative system of generalized classical mechanics is obtained, which can be called the Hojman adiabatic invariant. An example is also given to illustrate the application of the results.

Keywords: generalized classical mechanics, adiabatic invariant, Lie symmetry, perturbation
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1. Introduction

In a mechanical system there exists an intimate relation between the integrability of the system and the variations in its symmetry and invariant under the action of small disturbance, and therefore research on the perturbation to symmetries and adiabatic invariants is of great significance. A classical adiabatic invariant is a certain physical quantity that changes more slowly than some parameter of the system when the parameter varies very slowly.[1,2] In fact, the parameter varying very slowly is equivalent to the action of small disturbance. In recent years, some important results of the study on perturbation to symmetries and adiabatic invariants for constrained mechanical systems have been obtained.[3−12] Zhao and Mei[3] gave the exact invariant and the adiabatic invariant of a general dynamical system. Chen *et al* studied the perturbation to symmetries and adiabatic invariants for Birkhoffian systems,[4] variable mass systems,[5,6] first-order Lagrange systems,[7] dynamical systems of relative motion[8] and holonomic systems in terms of quasi-coordinates.[9] Zhang obtained the adiabatic invariants for Birkhoffian systems with unilateral constraints[10] and generalized classical mechanics.[11] Qiao *et al* gave the exact invariants and the adiabatic invariants of Raitzin's canonical equations of motion[12] and so on. But the adiabatic invariants obtained by all these studies belong to the type of Noether. In this paper, we further study the perturbations to Lie symmetries for nonconservative systems of generalized classical mechanics under the action of small disturbance and will present a new type of adiabatic invariants, and we refer to the new adiabatic invariants as Hojman adiabatic invariants.

2. Lie symmetries and exact invariants of generalized mechanical systems

Consider a nonconservative system of generalized classical mechanics. Suppose that the configuration of the system is determined by n generalized coordinates $q^i (i = 1, \ldots, n)$, and the Lagrangian L is[13,14]

$$L = L(t, q^i_{(0)}, q^i_{(1)}, \ldots, q^i_{(\omega)}), \quad (i = 1, \ldots, n). \tag{1}$$

In a higher-dimensional space, the generalized nonpotential forces can be written as[15]

$$Q''_j = Q''_j(t, q^i_{(0)}, q^i_{(1)}, \ldots, q^i_{(\omega)}), \quad (j = 1, \ldots, n). \tag{2}$$

Then the Lagrangian equations of the generalized nonconservative system are[15]

$$\sum_{k=0}^{\omega} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q^i_{(k)}} + Q''_i = 0, \quad (i = 1, \ldots, n). \tag{3}$$

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where

$$q_{(k)}^{i}=\frac{\mathrm{d}^{k}}{\mathrm{d}t^{k}}q^{i}(t),\quad (i=1,\ldots,n;\quad k=0,1,\ldots,\omega). \tag{4}$$

Introducing the generalized momenta and the generalized Hamiltonian[14]

$$p_{i}^{(s)}=p_{i/(s+1)}=\sum_{k=0}^{\omega-s-1}(-1)^{k}\frac{\mathrm{d}^{k}}{\mathrm{d}t^{k}}\frac{\partial L}{\partial q_{(k+s+1)}^{i}},\quad (i=1,\ldots,n;\quad s=0,1,\ldots,\omega-1), \tag{5}$$

$$H\left(t,\boldsymbol{q}_{(s)},\boldsymbol{p}^{(s)}\right)=p_{i}^{(s)}q_{(s+1)}^{i}-L,\quad (i=1,\ldots,n;\quad s=0,1,\ldots,\omega-1), \tag{6}$$

and supposing the system to be non-singular, Eqs.(3) can be written in the canonical form[16]

$$\begin{aligned}\dot{q}_{(s)}^{i}&=\frac{\partial H}{\partial p_{i}^{(s)}},\quad (i=1,\ldots,n;\quad s=0,1,\ldots,\omega-1),\\ \dot{p}_{i}^{(0)}&=-\frac{\partial H}{\partial q_{(0)}^{i}}+Q_{i}'',\quad (i=1,\ldots,n),\\ \dot{p}_{i}^{(l)}&=-\frac{\partial H}{\partial q_{(l)}^{i}},\quad (i=1,\ldots,n;\quad l=1,\ldots,\omega-1).\end{aligned} \tag{7}$$

Expanding Eqs.(7), we have

$$\dot{q}_{(s)}^{i}=g_{(s)}^{i}\left(t,\boldsymbol{q}_{(k)},\boldsymbol{p}^{(k)}\right),\dot{p}_{i}^{(s)}=h_{i}^{(s)}\left(t,\boldsymbol{q}_{(k)},\boldsymbol{p}^{(k)}\right),\quad (i=1,\ldots,n;\quad s=0,1,\ldots,\omega-1). \tag{8}$$

In the high-dimensional extended phase space, take the infinitesimal transformations

$$t^{*}=t,\quad q_{(s)}^{i^{*}}(t^{*})=q_{(s)}^{i}(t)+\Delta q_{(s)}^{i},\quad p_{i}^{(s)^{*}}(t^{*})=p_{i}^{(s)}(t)+\Delta p_{i}^{(s)},\quad (i=1,\ldots,n;\quad s=0,1,\ldots,\omega-1), \tag{9}$$

and their expanded formulae will be

$$t^{*}=t,\quad q_{(s)}^{i^{*}}=q_{(s)}^{i}+\varepsilon\zeta_{(s)0}^{i}\left(t,\boldsymbol{q}_{(k)},\boldsymbol{p}^{(k)}\right),\quad p_{i}^{(i)^{*}}=p_{i}^{(s)}+\varepsilon\eta_{i}^{(s)0}\left(t,\boldsymbol{q}_{(k)},\boldsymbol{p}^{(k)}\right), \tag{10}$$

where ε is an infinitesimal parameter, $\zeta_{(s)0}^{i}$ and $\eta_{i}^{(s)0}$ are generators of the infinitesimal transformations. Introduce the infinitesimal generator vector

$$X_{0}^{(0)}=\zeta_{(s)0}^{i}\frac{\partial}{\partial q_{(s)}^{i}}+\eta_{i}^{(s)0}\frac{\partial}{\partial p_{i}^{(s)}}, \tag{11}$$

and the invariance of Eqs.(8) under the infinitesimal transformations (10) will lead to the following determining equations of Lie symmetry:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta_{(s)0}^{i}=X_{0}^{(0)}\left(g_{(s)}^{i}\right),\quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta_{i}^{(s)0}=X_{0}^{(0)}\left(h_{i}^{(s)}\right),\quad (i=1,\ldots,n;\quad s=0,1,\ldots,\omega-1), \tag{12}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+g_{(k)}^{j}\frac{\partial}{\partial q_{(k)}^{j}}+h_{j}^{(k)}\frac{\partial}{\partial p_{j}^{(k)}}. \tag{13}$$

Theorem 1 If the generators $\zeta_{(s)0}^{i}$ and $\eta_{i}^{(s)0}$ of the infinitesimal transformations satisfy the determining equations (12) of Lie symmetry, and there exists a function $\mu_{0}=\mu_{0}\left(t,\boldsymbol{q}_{(s)},\boldsymbol{p}^{(s)}\right)$ satisfying the following condition:

$$\frac{\partial g_{(s)}^{i}}{\partial q_{(s)}^{i}}+\frac{\partial h_{i}^{(s)}}{\partial p_{i}^{(s)}}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu_{0}=0, \tag{14}$$

then the nonconservative system of generalized classical mechanics has a conserved quantity of Lie symmetry as follows:

$$I_0 = \frac{1}{\mu_0}\frac{\partial}{\partial q^i_{(s)}}\left(\mu_0\zeta^i_{(s)0}\right) + \frac{1}{\mu_0}\frac{\partial}{\partial p^{(s)}_i}\left(\mu_0\eta^{(s)0}_i\right) = \text{const.} \tag{15}$$

The conserved quantity (15) is an exact invariant, it reveals the intrinsic relation between the invariant and the Lie symmetry of the undisturbed nonconservative system of generalized classical mechanics.

3. Perturbation to the symmetries and Hojman adiabatic invariants

Definition 1 If $I_z(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}, \varepsilon)$ is a physical quantity of a generalized mechanical system including ε whose highest power is z, and its derivative with respect to time t is directly proportional to ε^{z+1}, then I_z is called a zth-order adiabatic invariant of the generalized mechanical system.

Suppose that the generalized nonconservative system given by Eqs.(8) is disturbed by small forces of perturbation $\varepsilon\tilde{Q}^{(s)}_i\left(t, \boldsymbol{q}_{(k)}, \boldsymbol{p}^{(k)}\right)$, and the equations of motion of the system will become

$$\dot{q}^i_{(s)} = g^i_{(s)}\left(t, \boldsymbol{q}_{(k)}, \boldsymbol{p}^{(k)}\right), \quad \dot{p}^{(s)}_i = h^{(s)}_i\left(t, \boldsymbol{q}_{(k)}, \boldsymbol{p}^{(k)}\right) + \varepsilon\tilde{Q}^{(s)}_i\left(t, \boldsymbol{q}_{(k)}, \boldsymbol{p}^{(k)}\right), \quad (i = 1, \ldots, n; \quad s = 0, 1, \ldots, \omega - 1). \tag{16}$$

Due to the action of small forces of perturbation $\varepsilon\tilde{Q}^{(s)}_i$, the original symmetry and invariant of the system may change. Assume that the disturbed generators $\zeta^i_{(s)}$ and $\eta^{(s)}_i$ of infinitesimal transformation are a small perturbation to the generators of symmetrical transformations of undisturbed system, and we will have

$$\begin{aligned} \zeta^i_{(s)} &= \zeta^i_{(s)0} + \varepsilon\zeta^i_{(s)1} + \varepsilon^2\zeta^i_{(s)2} + \cdots \\ \eta^{(s)}_i &= \eta^{(s)0}_i + \varepsilon\eta^{(s)1}_i + \varepsilon^2\eta^{(s)2}_i + \cdots \end{aligned} \tag{17}$$

The infinitesimal generator vector is

$$X^{(0)} = \zeta^i_{(s)}\frac{\partial}{\partial q^i_{(s)}} + \eta^{(s)}_i\frac{\partial}{\partial p^{(s)}_i}. \tag{18}$$

Substituting Eqs.(17) into the vector (18), we have

$$X^{(0)} = \varepsilon^m X^{(0)}_m, \quad (m = 0, 1, 2, \ldots), \tag{19}$$

where

$$X^{(0)}_m = \zeta^i_{(s)m}\frac{\partial}{\partial q^i_{(s)}} + \eta^{(s)m}_i\frac{\partial}{\partial p^{(s)}_i}, \quad (m = 0, 1, 2, \ldots). \tag{20}$$

The invariance of Eqs.(16) under the infinitesimal transformations leads to the following determining equations of Lie symmetry

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta^i_{(s)} = X^{(0)}\left(g^i_{(s)}\right), \quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^{(s)}_i = X^{(0)}\left(h^{(s)}_i\right) + \varepsilon X^{(0)}\left(\tilde{Q}^{(s)}_i\right), \quad (i = 1, \ldots, n; \quad s = 0, 1, \ldots, \omega - 1), \tag{21}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + g^i_{(k)}\frac{\partial}{\partial q^j_{(k)}} + \left(h^{(k)}_j + \varepsilon\tilde{Q}^{(k)}_j\right)\frac{\partial}{\partial p^{(k)}_j}. \tag{22}$$

When the system is not disturbed by small forces of perturbation, then the expression (22) becomes the expression (13). Substituting Eqs.(17) and Eq.(19) into Eqs.(21), and comparing the coefficients of ε^m on the left-hand side of the resulting equality with those on the right-hand side, we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta^i_{(s)m} = X^{(0)}_m\left(g^i_{(s)}\right), \quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^{(s)m}_i = X^{(0)}_m\left(h^{(s)}_i\right) + X^{(0)}_{m-1}\left(\tilde{Q}^{(s)}_i\right),$$
$$(i = 1, \ldots, n; \quad s = 0, 1, \ldots, \omega - 1; \quad m = 0, 1, 2, \ldots), \tag{23}$$

where, we assume that $\zeta^i_{(s),m-1} = 0$ and $\eta^{(s),m-1}_i$=0 when $m = 0$.

Theorem 2 For the nonconservative system of generalized classical mechanics given by Eqs.(16), which is disturbed by small forces of perturbation $\varepsilon\tilde{Q}^{(s)}_i$, if the generators $\zeta^i_{(s)m}$ and $\eta^{(s)m}_i$ of infinitesimal transformations satisfy the determining Eqs.(23), and there exists a function $\mu = \mu\left(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}\right)$ satisfying the following condition:

$$\frac{\partial g^i_{(s)}}{\partial q^i_{(s)}} + \frac{\partial h^{(s)}_i}{\partial p^{(s)}_i} + \varepsilon\frac{\partial \tilde{Q}^{(s)}_i}{\partial p^{(s)}_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{24}$$

then the generalized nonconservative system has a zth-order adiabatic invariant as follows:

$$I_z = \sum_{m=0}^{z}\varepsilon^m\left[\frac{1}{\mu}\frac{\partial}{\partial q^i_{(s)}}\left(\mu\zeta^i_{(s)m}\right) + \frac{1}{\mu}\frac{\partial}{\partial p^{(s)}_i}\left(\mu\eta^{(s)m}_i\right)\right], \tag{25}$$

where we assume that $\mu = \mu_0$ when $m = 0$.

Proof

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_z = \sum_{m=0}^{z}\varepsilon^m\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\zeta^i_{(s)m}}{\partial q^i_{(s)}} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\eta^{(s)m}_i}{\partial p^{(s)}_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}X^{(0)}_m(\ln\mu)\right]. \tag{26}$$

For an arbitrary function $\phi\left(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}\right)$, if the infinitesimal generators $\zeta^i_{(s)}$ and $\eta^{(s)}_i$ satisfy the determining Eqs.(21) of Lie symmetry of the system, we can easily verify that there exists a relation along the dynamical trajectory of the generalized nonconservative system as follows:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}X^{(0)}(\phi) = X^{(0)}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\phi\right). \tag{27}$$

Hence, we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}X^{(0)}_m(\ln\mu) = X^{(0)}_m\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu\right). \tag{28}$$

By way of simple deduction, we can obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial\zeta^i_{(s)}}{\partial q^i_{(s)}} + \frac{\partial\eta^{(s)}_i}{\partial p^{(s)}_i}\right) - X^{(0)}\left(\frac{\partial g^i_{(s)}}{\partial q^i_{(s)}} + \frac{\partial h^{(s)}_i}{\partial p^{(s)}_i}\right) - \varepsilon X^{(0)}\left(\frac{\partial\tilde{Q}^{(s)}_i}{\partial p^{(s)}_i}\right)$$
$$= \frac{\partial}{\partial q^i_{(s)}}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta^i_{(s)} - X^{(0)}\left(g^i_{(s)}\right)\right] + \frac{\partial}{\partial p^{(s)}_i}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^{(s)}_i - X^{(0)}\left(h^{(s)}_i\right) - \varepsilon X^{(0)}\left(\tilde{Q}^{(s)}_i\right)\right]. \tag{29}$$

Substituting Eqs.(17) and Eq.(19) into Eq.(29) and equalizing the coefficients of ε^m on the left-hand side and right-hand side of the resulting equality, we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial\zeta^i_{(s)m}}{\partial q^i_{(s)}} + \frac{\partial\eta^{(s)m}_i}{\partial p^{(s)}_i}\right) - X^{(0)}_m\left(\frac{\partial g^i_{(s)}}{\partial q^i_{(s)}} + \frac{\partial h^{(s)}_i}{\partial p^{(s)}_i}\right) - X^{(0)}_{m-1}\left(\frac{\partial\tilde{Q}^{(s)}_i}{\partial p^{(s)}_i}\right)$$
$$= \frac{\partial}{\partial q^i_{(s)}}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta^i_{(s)m} - X^{(0)}_m\left(g^i_{(s)}\right)\right] + \frac{\partial}{\partial p^{(s)}_i}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^{(s)m}_i - X^{(0)}_m\left(h^{(s)}_i\right) - X^{(0)}_{m-1}\left(\tilde{Q}^{(s)}_i\right)\right]. \tag{30}$$

Substituting Eq.(30) and Eqs.(28) into Eq.(26), and using Eq.(24) and the determining Eqs.(23), we obtain

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_z &= \sum_{m=0}^{z}\varepsilon^m\left\{\frac{\partial}{\partial q^i_{(s)}}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta^i_{(s)m}-X^{(0)}_m\left(g^i_{(s)}\right)\right]+\frac{\partial}{\partial p^{(s)}_i}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^{(s)m}_i-X^{(0)}_m\left(h^{(s)}_i\right)-X^{(0)}_{m-1}\left(\tilde{Q}^{(s)}_i\right)\right]\right.\\ &\left.+X^{(0)}_m\left(\frac{\partial g^i_{(s)}}{\partial q^i_{(s)}}+\frac{\partial h^{(s)}_i}{\partial p^{(s)}_i}\right)+X^{(0)}_{m-1}\left(\frac{\partial\tilde{Q}^{(s)}_i}{\partial p^{(s)}_i}\right)+X^{(0)}_m\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu\right)\right\}\\ &=\sum_{m=0}^{z}\varepsilon^m\left[X^{(0)}_{m-1}\left(\frac{\partial\tilde{Q}^{(s)}_i}{\partial p^{(s)}_i}\right)-\varepsilon X^{(0)}_m\left(\frac{\partial\tilde{Q}^{(s)}_i}{\partial p^{(s)}_i}\right)\right]\\ &=-\varepsilon^{z+1}X^{(0)}_z\left(\frac{\partial\tilde{Q}^{(s)}_i}{\partial p^{(s)}_i}\right),\end{aligned}\tag{31}$$

which shows that the $\dfrac{\bar{\mathrm{d}}}{\mathrm{d}t}I_z$ is directly proportional to ε^{z+1}, so I_z is a zth-order adiabatic invariant of the generalized nonconservative mechanical system from Definition 1.

The expression (25) is a new adiabatic invariant, which is different from the adiabatic invariant in the form of Noether given in Ref.[11], and it can be called a Hojman adiabatic invariant. In terms of Theorem 2, we can find the Hojman adiabatic invariant directly from a particular Lie symmetry for a nonconservative system of generalized classical mechanics.

4. Example

Suppose that the Lagrangian of a generalized mechanical system[15] is

$$L=\frac{1}{2}\left[\left(q^1_{(1)}\right)^2+\left(q^1_{(2)}\right)^2\right],\tag{32}$$

and a generalized nonpotential force is

$$\tilde{Q}_1=-q^1_{(2)}.\tag{33}$$

First let us seek the exact invariant of the system. The generalized momenta and the generalized Hamiltonian respectively are

$$p^{(0)}_1=q^1_{(1)}-\dot{q}^1_{(2)},\quad p^{(1)}_1=q^1_{(2)},\tag{34}$$

$$H=p^{(0)}_1q^1_{(1)}+\frac{1}{2}\left(p^{(1)}_1\right)^2-\frac{1}{2}\left(q^1_{(1)}\right)^2.\tag{35}$$

The differential equations of motion of the system are

$$\dot{q}^1_{(0)}=q^1_{(1)},\quad \dot{q}^1_{(1)}=p^{(1)}_1,$$

$$\dot{p}^{(0)}_1=-p^{(1)}_1,\quad \dot{p}^{(1)}_1=-p^{(0)}_1+q^1_{(1)}.\tag{36}$$

The determining Eqs.(12) give

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta^1_{(0)0}=\zeta^1_{(1)0},\quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta^1_{(1)0}=\eta^{(1)0}_1,\\ &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^{(0)0}_1=-\eta^{(1)0}_1,\quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^{(1)0}_1=\zeta^1_{(1)0}-\eta^{(0)0}_1\end{aligned}\tag{37}$$

with a solution

$$\zeta^1_{(0)0}=t,\quad \zeta^1_{(1)0}=1,\quad \eta^{(0)0}_1=1,\quad \eta^{(1)0}_1=0.\tag{38}$$

From the condition (14) we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu_0=0\tag{39}$$

with a solution

$$\mu_0=p^{(0)}_1+q^1_{(1)}.\tag{40}$$

From the generators (38) and Eq.(40), we can obtain an exact invariant in the form of Hojman for the system from Theorem 1 as follows:

$$I_0=2\left(p^{(0)}_1+q^1_{(1)}\right)^{-1}=\text{const}.\tag{41}$$

Now let us study the adiabatic invariant of the system. Suppose that the system is disturbed by the following small forces of perturbation:

$$\varepsilon\tilde{Q}^{(0)}_1=-\varepsilon q^1_{(1)},\quad \varepsilon\tilde{Q}^{(1)}_1=-\varepsilon tq^1_{(1)}.\tag{42}$$

Let $m=1$, and the determining Eqs.(23) will give

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta^1_{(0)1}=\zeta^1_{(1)1},\quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\zeta^1_{(1)1}=\eta^{(1)1}_1,\\ &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^{(0)1}_1=-\eta^{(1)1}_1-\zeta^1_{(1)0},\quad \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta^{(1)1}_1=\zeta^1_{(1)1}-\eta^{(0)1}_1-t\zeta^1_{(1)0},\end{aligned}\tag{43}$$

with a following solution:

$$\zeta^1_{(0)1} = 1, \quad \zeta^1_{(1)1} = 0, \quad \eta_1^{(0)1} = -t, \quad \eta_1^{(1)1} = 0. \tag{44}$$

From the condition (24) we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0 \tag{45}$$

with a solution

$$\mu = p_1^{(0)} + q^1_{(1)} + \varepsilon q^1_{(0)}. \tag{46}$$

From the expression (41), the generators (44) and Eq.(46), according to Theorem 2, we obtain a first-order adiabatic invariant in the form of Hojman as follows:

$$I_1 = 2\left(p_1^{(0)} + q^1_{(1)}\right)^{-1} - t\left(p_1^{(0)} + q^1_{(1)}\right)^{-1} \varepsilon. \tag{47}$$

Furthermore we can find more higher order adiabatic invariants of the system.

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